Tail asymptotics for extinction times of self-similar fragmentations

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Goal

Goal: obtain some information on the distribution of some positive random variables, which, depending on the point of view, can be seen as:

- the extinction times of some fragmentation processes
- the heights of continuous compact rooted random trees
- the scaling limits of the heights of sequences of discrete trees (e.g. the scaling limit of the height of a uniform rooted random tree with n nodes)

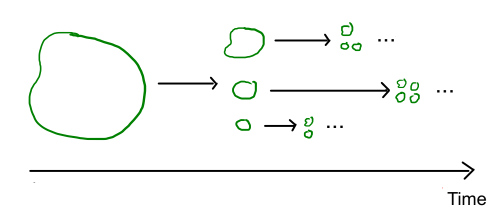
Outline of the talk

Three parts:

1. Self-similar fragmentations, extinction times and connections with random trees
2. Large time asymptotics of the distribution tails of the extinction times; examples
3. Two main steps of the proof

Fragmentation models

Fragmentation models: describe the evolution of objects that split repeatedly as time goes on



Extensive study in Mathematics since the mid-1900s (both from deterministic and random points of view) explained by:

- many motivations coming from biology and population genetics, computer science, polymerization, but also random trees and graphs
- the setting of fairly general models that are relatively easy to study

Self-similar fragmentations

We focus on random models where objects are only characterized by their mass and the dynamic is governed by:

- a branching property: different objects evolve independently
- a self-similarity property: an object splits at a rate proportional to a power of its mass

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▶ Starting at time 0 with a unique object of mass 1, we let $F(t)$ denotes the sequence of masses present at time $t \geq 0$:

$$F(t) \in \mathcal{S} := \left\{ (s_i)_{i \geq 1} : s_1 \geq s_2 \geq s_3 \ldots ; \sum_{i=1}^{\infty} s_i \leq 1 \right\}$$

$(F(0) = (1, 0, 0, \ldots))$

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▶ The splitting rule depends on two parameters: $\alpha \in \mathbb{R}$ (the index of self-similarity) and a measure ν on $\mathcal{S}$ such that

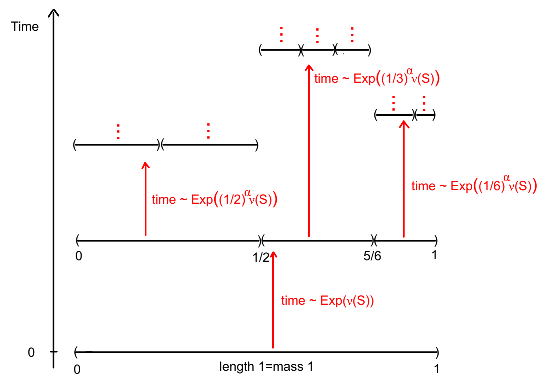
a mass m splits in masses $(ms_1, ms_2, \ldots)$ at rate $m^\alpha \mathrm{d}\nu(s_1, s_2, \ldots)$

First ref.: Kolmogorov 41, Filippov 61, Brennan and Durrett 86-87, Bertoin 01-02

Many studies on those models since 2000+.

Self-similar fragmentations

- The (α, ν)-model when ν is finite:



a mass m:

- splits after a time $\sim \text{Exp}(m^\alpha \nu(\mathcal{S}))$
- in masses $(mS_1, mS_2, \ldots)$ where $(S_1, S_2, \ldots) \sim \nu(\cdot)/\nu(\mathcal{S})$

Remark. Mean time of splitting of a fragment with mass m: $m^{-\alpha}/\nu(\mathcal{S})$:

- when $\alpha > 0$ small fragments splits slower than the large ones

- when $\alpha < 0$, small fragments splits faster than the large ones.

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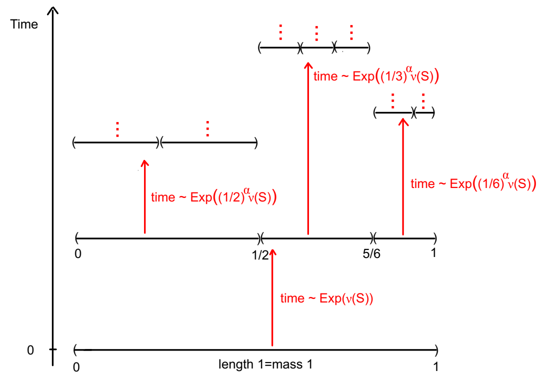

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- When ν is infinite: infinitely many fragmentations in any strictly positive interval of times. Necessity that $\int_{\mathcal{S}}(1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$ to prevent the system to explose entirely at time 0+.

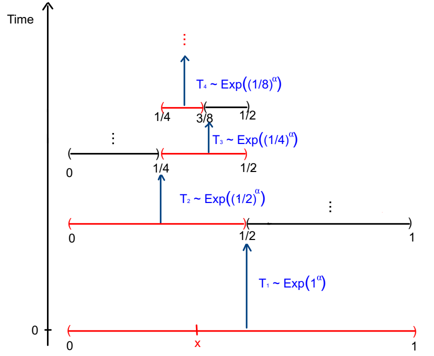
Extinction time

Hypotheses: $\alpha < 0$ and $\nu(\mathcal{S}) > 0 \Rightarrow$ very small objects split very quickly!

Ex.: $\nu = \delta_{(1/2,1/2,0\dots)}$

for any $x \in (0,1)$ non-dyadic, the fragment containing x reaches mass 2^{-n} at time $\sum_{i=1}^{n} T_i$, with $T_i \sim \mathrm{Exp}(2^{-\alpha(\mathrm{i}-1)})$

hence reaches 0 at time $\sum_{i=1}^{\infty} T_i < \infty$ a.s.



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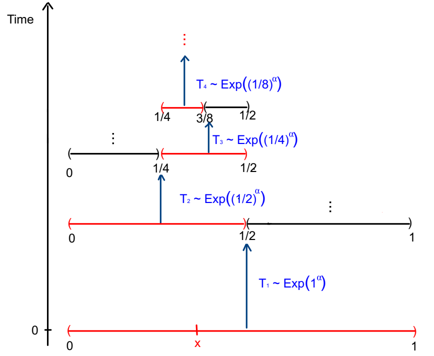

In general: For any (α, ν), $\alpha < 0$ and any (α, ν) fragmentation F:

$$\zeta := \inf\{t \geq 0 : F(t) = (0, 0, \ldots)\},$$

the first time at which the entire initial mass is reduced to dust.

Extinction time

For any (α, ν), $\alpha < 0$:

Proposition (Filippov 61, McGrady & Ziff 87, Bertoin 02)

The extinction time ζ is finite almost surely.

Proposition (H. 03)

The tail of ζ is exponential or even lighter:

$$\exists \theta \geq 1 : \mathbb{P}(\zeta > t) \leq \exp(-\mathrm{cst} \cdot t^{\theta}) \text{ for all } t \text{ large enough.}$$

Connection with random trees

The r.v. ζ may also be seen as the height of a random tree which is the scaling limit of models of discrete trees.

- Ex.1: H_n: height of a Galton-Watson tree with offspring distribution with mean 1 and variance $0 < \sigma^2 < \infty$ conditioned on having total progeny n.

  Aldous 93: This GW tree, appropriately normalized, converges to the *Brownian continuum tree*. In particular,

$$\frac{H_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{(law)}} \frac{2}{\sqrt{\sigma^2}} \cdot \zeta_{\text{Br}}$$

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  Bertoin 02: Aldous' Brownian tree is the genealogical tree of a self-similar fragmentation with parameters

  $$\alpha = -1/2, \qquad \nu(s_1 + s_2 < 1) = 0 \text{ and } \nu(s_1 \in \mathrm{d}x) = \frac{\mathbf{1}_{\{x>1/2\}}}{\sqrt{\pi}(x(1-x))^{3/2}} \mathrm{d}x$$

  The r.v. ζ_{Br} is its extinction time.

Connection with random trees

- Ex.2: When the offspring distribution of the GW tree has a tail $\mathbb{P}(\text{offspring} \geq k) \sim ck^{-\beta}$ for some $\beta \in (1,2)$, then (**Duquesne 03**)

$$\frac{H_n}{n^{1-\frac{1}{\beta}}} \xrightarrow[n\to\infty]{\text{(law)}} C(c,\beta) \cdot \zeta_\beta$$

  where ζ_β is the height of the *β-stable Lévy tree* of Duquesne, Le Gall, Le Jan

  Miermont 03: the β-stable Lévy tree is the genealogical tree of a self-similar fragmentation with parameters $\beta^{-1} - 1$.

- More generally: models of random discrete trees satisfying a *Markov-Branching property*, were proved to converge in the scaling limit to continuous trees describing the genealogy of (α, ν)-fragmentations

  (**H.-Miermont-Pitman-Winkel 08, H.-Miermont 12**)

  $\Rightarrow$ their rescaled heights converge to the r.v. ζ.

Connection with random trees

Kennedy 76 and **Duquesne** & **Wang 17**: asymptotic expansions at all orders of ζ_{Br} and ζ_β

Theorem (Kennedy 76, Duquesne & Wang 17)

$$\mathbb{P}(\zeta_{\mathrm{Br}} > t) \underset{t\to\infty}{\sim} 2t^2 \exp(-t^2) \quad \text{and} \quad \mathbb{P}(\zeta_\beta > t) \underset{t\to\infty}{\sim} C(\beta) t^{1+\frac{\beta}{2}} \exp(-(\beta-1)^{\beta-1} t^\beta)$$

for some explicit $C(\beta)$

Goal: obtain similar results for general (α, ν) random variables ζ

Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

The parameters $\alpha < 0$ and ν are fixed; ζ denotes the corresponding extinction time.

Two functions: we let for x large enough

$$\phi(x) = \int_{\mathcal{S}} (1 - s_1^{x+1}) \nu(\mathrm{d}\mathbf{s}) \quad \text{and} \quad \psi : \quad \frac{\psi(x)}{\phi(\psi(x))} = x$$

Ex.: if $\nu(s_1 \leq u) \underset{u \to 1}{\sim} c(1-u)^{-\gamma}, \gamma \in [0,1)$ then:

$$\phi(x) \underset{x \to \infty}{\sim} c\Gamma(1-\gamma) x^{\gamma} \quad \text{and} \quad \psi(x) \underset{x \to \infty}{\sim} (c\Gamma(1-\gamma)x)^{\frac{1}{1-\gamma}}$$

Brownian frag.: $\phi(x) \underset{x \to \infty}{\sim} 2\sqrt{x}, \ \ \psi(x) \underset{x \to \infty}{\sim} 4x^2$

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<ins>Notation:</ins> For positive functions f, g,

$$f(t) \asymp g(t)$$

means there exists $a, b > 0$ such that $a \cdot g(t) \leq f(t) \leq b \cdot g(t)$ for t large enough.

Proposition (H. 03)

If ϕ is regularly varying at ∞,

$$\ln(\mathbb{P}(\zeta > t)) \asymp -\psi(t).$$

We want to sharpen this estimate by removing the logarithm

Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

Main hypothesis:

$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} < 1 \tag{H}$$

Not restrictive at all!

Theorem (H. 21)

Assume (**H**). Then

$$\mathbb{P}(\zeta > t) \asymp \left(\frac{\psi(|\alpha|t)}{t}\right)^{\frac{1}{|\alpha|}-1} (\psi'(|\alpha|t))^{\frac{1}{2}} \exp\left(-\int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r}\mathrm{d}r\right)$$

Corollary

If ϕ is regularly varying at ∞,

$$\mathbb{P}(\zeta > t) \asymp \left(\frac{\psi(|\alpha|t)}{t}\right)^{\frac{1}{|\alpha|}-\frac{1}{2}} \exp\left(-\int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r}\mathrm{d}r\right).$$

Examples with finite splitting rate

Here $\psi(x) \underset{x\to\infty}{\sim} |\nu(\mathcal{S})|x$, hence $\int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r}\mathrm{d}r = |\nu(\mathcal{S})|t + o(t)$.

Ex.1: **Fragmentations into k identical pieces**: a fragment of size m splits into k fragments of same sizes m/k. For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \underset{t\to\infty}{\sim} \mathrm{c}\exp(-t)$$

for some $\mathrm{c} \in (0, \infty)$.

Ex.2: **Uniform fragmentation**: a fragment of size m splits into two fragments of sizes $mU, m(1-U)$, where U is uniform on $[0,1]$. For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \asymp t^{\frac{2}{|\alpha|}} \exp(-t).$$

Examples with finite splitting rate

Ex.3: **Beta fragmentations:** a fragment of size m splits into two fragments of sizes mB, $m(1-B)$, where $B \sim \mathrm{Beta}(a,b)$, $b \geq a > 0$ (density on $(0,1)$ proportional to $x^{a-1}(1-x)^{b-1}$). For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \asymp \begin{cases} \exp(-t) & \text{if } b \geq a > 1 \\ t^{\frac{1}{|\alpha|}} \exp(-t) & \text{if } b > a = 1 \\ t^{\frac{2}{|\alpha|}} \exp(-t) & \text{if } b = a = 1 \\ \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a}\right) & \text{if } b > 1 > a > 1/2 \\ t^{\frac{1}{|\alpha|}} \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a}\right) & \text{if } 1 = b \geq a > 1/2 \\ \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a} + \frac{\Gamma(b)}{(1-b)|\alpha|^b} t^{1-b}\right) & \text{if } 1 > b \geq a > 1/2. \end{cases}$$

If a (and possibly b) is smaller than 1/2, there will be additional terms.

Examples with infinie splitting rates

Ex.4: **Aldous' beta-splitting models:** scaling limits of discrete models introduced by **Aldous 96** to interpolate between some phylogenetic trees.

Parametrized by $\beta \in (-2, -1)$; binary splitting ($\nu(s_1 + s_2 < 1) = 0$) and

$$\nu(s_1 \in \mathrm{d}u) = \frac{-\beta - 1}{\Gamma(2+\beta)}(u(1-u))^{\beta}, u \in (1/2, 1) \quad \text{and} \quad \alpha = 1 + \beta.$$

Then for $\beta \in (-2, -3/2]$:

$$\mathbb{P}(\zeta > t) \asymp t^{\frac{-2\beta-1}{2(\beta+2)}} \exp\left(-a_\beta t^{\frac{1}{\beta+2}} + b_\beta t\right)$$

where $a_\beta = (-\beta - 1)^{\frac{-\beta-1}{\beta+2}}(\beta + 2)$ and $b_\beta = \frac{(2\beta+3)\Gamma(\beta+2)}{(\beta+2)\Gamma(2\beta+4)}$.

For $\beta \in (-3/2, 1)$: additional power terms in the exponential.

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For $\beta \in (-3/2,1)$: additional power terms in the exponential.

Ex.5: Height of stable Lévy trees. Then $\phi(x) = \beta x^{1-\frac{1}{\beta}}\left(1 - \frac{\beta-1}{2\beta^2}x^{-1} + O(x^{-2})\right)$

So we retrieve, for all $\beta \in (1,2]$:

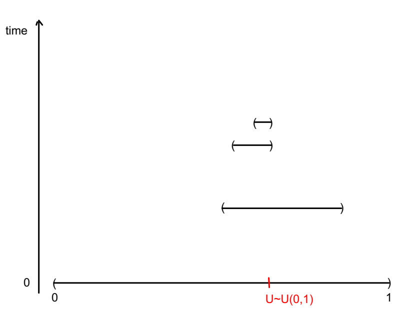
$$\mathbb{P}(\zeta > t) \asymp t^{1+\frac{\beta}{2}} \exp\left(-(\beta-1)^{\beta-1}t^{\beta}\right).$$

Outline of the proof of the theorem

An intermediate tool: the extinction time of a typical point

$U \sim \mathrm{Unif}(0,1)$

I: time at which U is reduced to dust



Proposition (Bertoin 02)

$$I = \int_0^\infty \exp(\alpha \xi_t)\mathrm{d}t$$

where ξ is a subordinator (increasing Lévy process) with Laplace exponent $\bar{\phi}$ (i.e. $\mathbb{E}[\exp(-x\xi_t)] = \exp(-t\bar{\phi}(x)), \forall x, t \geq 0$) where $\bar{\phi}(x) = \int_{\mathcal{S}}(1 - \sum_i s_i^{x+1})\nu(\mathrm{d}\mathbf{s})$.

Rk.: $\bar{\phi}(x) = \phi(x) + O(2^{-x})$ as $x \to \infty$.

Two main steps

Step 1. Link between the tails of ζ and I

Proposition 1 (H. 21)

Assume (**H**). Then,

$$\mathbb{P}(\zeta > t) \asymp \left(\frac{\psi(|\alpha| t)}{t}\right)^{\frac{1}{|\alpha|}} \cdot \mathbb{P}(I > t)$$

Step 2. Asymptotics of the tail of I

Proposition 2 (H. 21)

Assume (**H**). Then there exists $\mathrm{c} \in (0, \infty)$ such that

$$\mathbb{P}(I > t) \underset{t \to \infty}{\sim} \mathrm{c} \cdot \frac{t(\psi'(|\alpha| t))^{1/2}}{\psi(|\alpha| t)} \cdot \exp\left(-\int_1^t \frac{\psi(|\alpha| r)}{|\alpha| r} \mathrm{d}r\right).$$

Some hints for Step 1

Remark: $I < \zeta$ and it is a priori not obvious how to compare their tails

Step 1. a) Connections with moments of typical fragments.

U_1, U_2 uniformly distributed on $(0,1)$, independent

$\Lambda_{(i)}(t)$: mass of the fragment containing U_i at time t, $i = 1, 2$

Proposition (H. 21)

There exists $c \in (0, \infty)$ such that for all t large enough

$$\frac{\mathbb{E}\left[\Lambda_{(1)}(t)\right]^2}{\mathbb{E}\left[\Lambda_{(1)}(t)\Lambda_{(2)}(t)\right]} \leq \mathbb{P}(\zeta > t) \leq c\left(\frac{\psi(|\alpha| t)}{t}\right)^{\frac{2}{|\alpha|}} \mathbb{E}\left[\Lambda_{(1)}(t)\right]$$

Idea: Introduce $S(t) := \sum_{i \geq 1}(F_i(t))^2$ and use the first and second moments methods.

Some hints for Step 1

Step 1. b) Asymptotics of moments of 1 and 2 typical fragments

Proposition (H.- Rivero 12)

Assume (**H**). Then for all $a > 0$ there exists a constant $\mathrm{c} \in (0, \infty)$ such that

$$\mathbb{E}\left[\Lambda^{a}_{(1)}(t)\right] \underset{t\to\infty}{\sim} \mathrm{c}\left(\frac{t}{\psi(|\alpha|t)}\right)^{\frac{a}{|\alpha|}} \mathbb{P}(I > t)$$

Proposition (H. 21)

For all $a, b > 0$,

$$\mathbb{E}\left[\Lambda^{a}_{(1)}(t)\Lambda^{b}_{(2)}(t)\right] \asymp \left(\frac{t}{\psi(|\alpha|t)}\right)^{\frac{a+b+1}{|\alpha|}} \mathbb{P}(I > t).$$

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