Bootstrap Percolation and Kinetically Constrained Models: critical time scales

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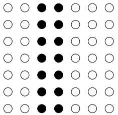
Bootstrap percolation

First example: 2-neighbour bootstrap on $\mathbb{Z}^2$

- At time $t = 0$ sites are i.i.d., empty with probability q, occupied with probability $1 - q$
- At time $t = 1$
  - each empty site remains empty
  - each occupied site is emptied iff it has at least 2 empty nearest neighbours
- Iterate

$\Rightarrow$ *deterministic monotone dynamics*

$\Rightarrow$ $\exists$ *blocked clusters*



Critical density and Infection time

- *Will the whole lattice become empty eventually?*

- $q_c := \inf\{q \in [0,1] : \mu_q(\text{origin is emptied eventually}) = 1\}$

- *How many steps do we "typically" need to empty the origin?*

- $\tau^{\text{BP}}(q) := \mu_q(\text{first time at which origin is empty})$

Critical density and Infection time

- *Will the whole lattice become empty eventually?*
  $\rightarrow$ Yes (Van Enter '87)

- $q_c := \inf\{q \in [0,1] : \mu_q(\text{origin is emptied eventually}) = 1\}$
  $\rightarrow q_c = 0$

- *How many steps do we "typically" need to empty the origin?*

- $\tau^{\text{BP}}(q) := \mu_q(\text{first time at which origin is empty})$

$$\rightarrow \tau^{\text{BP}}(q) \sim \exp\left(\frac{\pi^2}{18q}(1+o(1))\right) \quad \text{for} \quad q \rightarrow 0$$

[Aizenmann-Lebowitz '88, Holroyd '02, ...]

The general framework: $\mathcal{U}$-bootstrap percolation

- Choose the update family, a finite collection $\mathcal{U} = \{U_1, \ldots, U_m\}$ of local neighbourhoods of the origin, i.e. $U_i \subset \mathbb{Z}^2 \setminus 0, \quad |U_i| < \infty$
- At time $t = 1$ site x is emptied iff at least one of the translated neighborhoods $U_i + x$ is completely empty
- Iterate

Ex.: 2-neighbour bootstrap percolation has

$\mathcal{U} =$ collection of the sets containing 2 nearest neighb. of origin

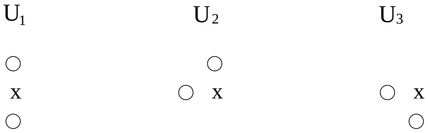
Some other examples

- r-neighbour bootstrap percolation:
  $\mathcal{U}$ = all the sets containing r nearest neighb. of origin

- East model $\mathcal{U} = \{U_1, U_2\}$ with $U_1 = (0, -1)$, $U_2 = (-1, 0)$

- North-East model $\mathcal{U} = \{U_1\}$ with $U_1 = \{(0, 1), (1, 0)\}$

- Duarte model $\mathcal{U} = \{U_1, U_2, U_3\}$



Universality classes

- q_c?
- Scaling of $\tau^{\mathrm{BP}}(q)$ for $q \downarrow q_c$?

Three universality classes

- Supercritical models: $q_c = 0, \quad \tau^{\mathrm{BP}}(q) = 1/q^{\Theta(1)}$
- Critical models: $q_c = 0, \quad \tau^{\mathrm{BP}}(q) = \exp(1/q^{\Theta(1)})$
- Subcritical models: $q_c > 0$

There is a very easy-to-use recipe to determine the class of any given $\mathcal{U}$

[Bollobas, Smith, Uzzell '15, Balister, Bollobas, Przykucki, Smith '16]

Kinetically Constrained Models, a.k.a. KCM

Configurations : $\eta \in \{0,1\}^{\mathbb{Z}^2}$

Dynamics: continuous time Markov process of Glauber type, i.e. birth / death of particles

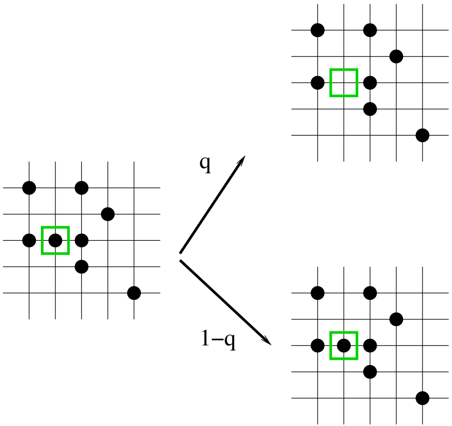
Fix an update family $\mathcal{U}$ and $q \in [0,1]$.

Each site for which the $\mathcal{U}$ bootstrap constraint is satisfied is updated to empty at rate q and to occupied at rate $1-q$.

KCM are a stochastic version version of BP:

- $\Rightarrow$ non monotone dynamics ;
- $\Rightarrow$ reversible w.r.t. product measure at density $1 - q$;
- $\Rightarrow$ blocked clusters for BP $\leftrightarrow$ blocked clusters for KCM;
- $\Rightarrow$ empty sites needed to update $\rightarrow$ slowing down when $q \downarrow 0$

2-neighbour KCM



Origins of KCM

KCM introduced by physicists in the '80's to model the liquid/glass transition

- understanding this transition is a major open problem in condensed matter physics;
- sharp divergence of timescales;
- no significant structural changes.

KCM:

$\Rightarrow$ constraints mimic *cage effect*:
if temperature is lowered free volume shrinks, $q \leftrightarrow e^{-1/T}$

$\Rightarrow$ trivial equilibrium, sharp divergence of timescales when $q \downarrow 0$, glassy dynamics (aging, heterogeneities, ...)

Why are KCM mathematically challenging?

- KCM dynamics is not attractive
  - $\rightarrow$ more empty sites can have unpredictable consequences
  - $\rightarrow$ coupling and censoring arguments fail
- Blocked clusters
  - $\rightarrow$ relaxation is not uniform on the initial condition
  - $\rightarrow$ worst case analysis is too rough
  - $\rightarrow$ $\exists$ several invariant measures
- Coercive inequalities (e.g. Log-Sobolev) behave anomalously

$\rightarrow$ most standard IPS tools fail for KCM $\rightarrow$ we need new tools

KCM: time scales

$$\tau^{\text{KCM}}(q) := \mathbb{E}_{\mu_q}(\text{ first time at which origin is emptied })$$

- How does τ^{KCM} diverge when $q \downarrow q_c$?
- How does it compare with τ^{BP}, the infection time of the corresponding bootstrap process?

An (easy) lower bound:

$$\tau^{\text{KCM}}(q) \geq c\,\tau^{\text{BP}}(q) \qquad \text{(for the same choice of } \mathcal{U})$$

General, but it does not always capture the correct behavior

Supercritical KCM : a refined classification

We identify 2 subclasses: supercritical rooted and unrooted

Theorem 1. [Martinelli, Morris, C.T. '17 + Marêché, Martinelli, C.T. '18]

(i) for all supercritical unrooted models $\tau^{\mathrm{KCM}} = 1/q^{\Theta(1)}$

(ii) for all supercritical rooted models $\tau^{\mathrm{KCM}} = 1/q^{\Theta(\log(1/q))}$

Recall: $\tau^{\mathrm{BP}}(q) = 1/q^{\Theta(1)}$ for all supercritical models

$\rightarrow$ for supercritical rooted $\tau^{\mathrm{KCM}}(q) \gg \tau^{\mathrm{BP}}(q)$

- 1-neighbour model is supercritical unrooted
- East model is supercritical rooted

Heuristic for 1-neighbour model

- a single empty site creates an empty site nearby at rate q
- at rate $1-q$ two nearby empty sites coalesce
- nearest empty site is at distance $L = 1/q^{1/2}$ from the origin

$\rightarrow$ $\tau^{\text{BP}} = L = 1/q^{1/2}$

$\rightarrow$ $\tau^{\text{KCM}} \sim L^2/q = 1/q^2$ (log corrections)

Heuristic for East model

- a single empty site can empty only its right or top neighbour $\rightarrow$ it can infect only its upper right quadrant

$\rightarrow$ $\tau^{\text{BP}} = L = 1/q^{1/2}$

- which trajectory is best for the KCM to empty the origin? the one that avoids creating too many simultaneous zeros!

- a deterministic combinatorial result: maximum number of simultaneous zeros on best trajectory is $\Delta = c \log L$

$\rightarrow$ $\tau^{\text{KCM}} \sim 1/q^{\Delta} \sim 1/q^{\Theta(\log(1/q))}$

- N.B. super rough heuristics: we neglect entropy, that matters for the value of c in $\tau^{\text{KCM}} = e^{c \log q^2}$

The East game

N tokens can be placed or removed from the integer sites $\{1, 2, \dots\}$ according to the following rules:

- each site has at most one token;
- a token can *always* be placed or removed on site 1;
- on each site $x \geq 2$ a token can be placed or removed *only if* there is a token on site $x - 1$

Q. Which is the maximum site that can be occupied by a token?

site $2^N - 1$ [Sollich Evans '99, Chung Diaconis, Graham '01]

$\rightarrow$ Logarithmic energy barrier for the East model in $d = 1$

Heuristic for supercritical unrooted and rooted KCM

- General supercritical unrooted models:

  same behavior as 1-neighbour with

  - single empty site $\leftrightarrow$ finite empty droplet

  $\rightarrow \tau^{\mathrm{BP}}$ and τ^{KCM} diverge as $1/q^{\Theta(1)}$

- General supercritical rooted models:

  same behavior as East with:

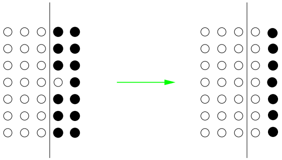
  - single empty site $\leftrightarrow$ finite empty droplet
  - upper right quadrant $\leftrightarrow$ cone

  a deterministic combinatorial result (much tougher game!): logarithmic energy barriers [L.Marêche '19]

  $\rightarrow \tau^{\mathrm{BP}} = 1/q^{\Theta(1)} \ll \tau^{\mathrm{KCM}} = 1/q^{\Theta(\log(1/q))}$

2-neighbour model

- all finite clusters of zeros cannot move
- a vertical (horizontal) segment of zeros can empty the next segment if this contains at least one empty site



$\rightarrow$ an empty segment of length $\ell = 1/q|\log q|$ can (typically) empty the next one

$\rightarrow$ same role as droplet for supercritical unrooted, but 2 key differences: ℓ depends on q + droplets need external help

2-neighbour KCM: Results and heuristics

- Renormalize on $\ell(q) \times \ell(q)$ boxes
- at $t = 0$ w.h.p. the origin belongs to a cluster of good boxes containing a droplet at distance $\sim 1/q^{\ell}$
- droplets move on the good cluster as 1-neighbour KCM
- in time poly$(1/q^{\ell})$ the droplet moves near origin and we can empty the origin

$$\downarrow$$

Theorem 2. [Martinelli, C.T. '17]

$$e^{\frac{c}{q}} \le \tau^{\text{KCM}} \le e^{\frac{(\log q)^{\Theta(1)}}{q}}$$

Upcoming work sharp threshold for 2-neighbour KCM

Theorem 3. [Martinelli, I.Hartarsky, C.T. '20$^+$]

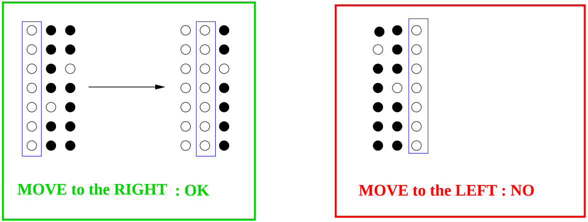
$$e^{\frac{\pi^2}{9q}(1+o(1))} \leq \tau^{\mathrm{KCM}} \leq e^{\frac{\pi^2}{9q}(1+o(1))}$$

As some you might have noticed for 2-neighbour KCM ...

$$\tau^{\mathrm{KCM}} = \left(\tau^{\mathrm{BP}}\right)^2$$

Duarte model

Constraint: ≥ 2 empty among N, W and S neighbours



An empty segment of length $\ell = 1/q\,|\log q|$ can (typically) create an empty segment to its right, but never to its left!

$\rightarrow$ it is a droplet that performs an East dynamics

Duarte model: heuristics

- the nearest empty droplet to the origin is typically at distance $L = 1/q^{\ell}$

$\rightarrow$ $\tau^{\text{BP}} \sim L = \exp\left(\frac{c|\log q|^2}{q}\right)$ [T.Mountford '95, B. Bollobas, H. Duminil-Copin, R. Morris, and P. Smith '17]

- Duarte droplets move East like $\rightarrow$ to empty the origin Duarte KCM has to to create $\log(L)$ simultaneous droplets
- to create a single droplet we pay $1/q^{\ell}$

$\rightarrow$ $\tau^{\text{KCM}} \sim \frac{1}{q}^{\ell \log L} \sim \exp\left(\frac{c|\log q|^4}{q^2}\right)$

Duarte model: results

Theorem 4. [Marêché, Martinelli, C.T. '18 + Martinelli, Morris, C.T. '18]

$$\exp\left(\frac{c_1|\log q|^4}{q^2}\right) \leq \tau^{\mathrm{KCM}} \leq \exp\left(\frac{c_2|\log q|^4}{q^2}\right)$$

$$\tau^{\mathrm{BP}} \sim \exp\left(\frac{c|\log q|^2}{q}\right) \ll \tau^{\mathrm{KCM}}$$

Critical KCM: a refined classification

α = critical exponent for BP $\sim$ minimal number of empty sites to move the droplet , e.g. $\alpha = 1$ for 2-neighbour and Duarte

Theorem 5. [Hartarsky, Martinelli, C.T. '19 + Martinelli, Morris, C.T. '18 + Hartarsky, Marêché, C.T. '19]

For critical KCM it holds

$$\exp\left(\frac{c}{q^{\nu}}\right) \leq \tau^{\mathrm{KCM}} \leq \exp\left(\frac{c(\log q)^{\Theta(1)}}{q^{\nu}}\right)$$

- $\nu = \alpha$ for models with finite number of stable directions;
- $\nu = 2\alpha$ for models with infinite number of stable directions

Upper bound: Main obstacles

- droplets move only on a "good environment"
- the environment evolves and can "loose its goodness"
- the motion of droplets is not random walk like
  $\rightarrow$ it is very difficult to apply canonical path arguments!
- the droplet is not a "rigid object", it can be destroyed
- no monotonicity, no coupling arguments

Upper bound: Main tools and ideas

- we upper bound τ^{KCM} with T_{rel} (= inverse spectral gap)
- we define an auxiliary KCM dynamics with long range and very likely constraints $\simeq$ existence of long good paths with at least one droplet;
- we prove that, under very flexible conditions, $T_{\text{rel}}^{\text{aux}} = O(1)$
- use variational formula of T_{rel} to compare the auxiliary dynamics with a 1-neighbour or East dynamics of droplets
- we recover the original KCM dynamics via canonical paths

Lower bound

How do we construct an efficient bottleneck?

- we provide an algorithm identifying "droplets" that
  - occur independently
  - have each probability $q^{1/q^{\alpha}}$
  - evolve East-like
- we identify a likely event on which to empty the origin we should "move" one such droplet at distance $L = q^{-1/q^{\alpha}}$

  $\rightarrow$ we need to create $\log L$ simultaneous droplets

  $\rightarrow$ this requires a time $\geq q^{1/q^{\alpha} \log L} = e^{c/q^{2\alpha}}$

Summary

- KCM are the stochastic counterpart of BP
- time scales for KCM may diverge very differently from those of BP due to the occurrence of *energy barriers*
  - τ^{BP} = length of the optimal path to empty origin
  - $\tau^{\mathrm{KCM}} \simeq$ length of optimal path $\times$ time to go through it
- we establish the universality picture for KCM in $d = 2$

Thanks for your e-attention !