OFF-DIAGONAL HYPERGRAPH RAMSEY NUMBERS

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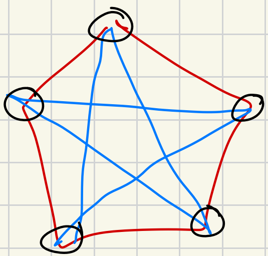
H — k-uniform hypergraph (k-graph)

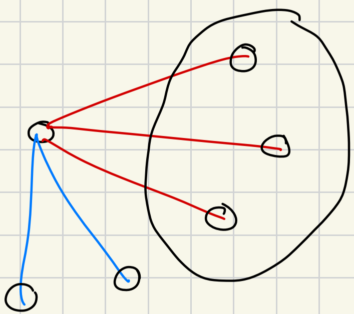
Ramsey number

$r(H, n) = \min N$ st every red/blue coloring of $K_N^{(k)} = \binom{[N]}{k}$ results in a red copy of H or a blue copy of $K_n^{(k)}$

EXAMPES

$r(K_3, 3) = 6$

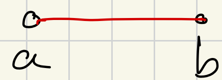




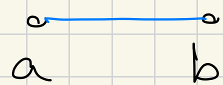
?

$r(K_4, 4) = 18$

$V(K_{17}) = \mathbb{Z}_{17}$



$a - b = x^2$



$a - b \neq x^2$

(Greenwood-Gleason 1955)

$r(K_4^{(3)}, 4) = 13$

Computer

(McKay-Radziszowski, 1991)

GRAPHS

Theorem (Spencer 1977, Campos-Griffiths, Morris, Sahasrabudhe 2023)

$$c \cdot n \cdot 2^{n/2} < r(K_{n,n}) < (4-\varepsilon)^n$$

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$$c_1 \frac{n^2}{\log n} \leq r(K_{3,n}) \leq c_2 \frac{n^2}{\log n}$$

Bohman-Keevash $\quad c_1 = \frac{1}{4} - o(1)$ $\qquad$ Shearer $c_2 = 1 + o(1)$

Fiz Pontiveros-Griffiths-Morris

Theorem (Mattheus–Verstraëte 2023)

$$r(K_4, n) = n^{3-o(1)}$$ (random graphs give $n^{5/2+o(1)}$)

Conjecture (M–Verstraëte ≈2019)

For fixed $s \geq 3$

$$r(K_s, n) = n^{s-1+o(1)}$$ (random graphs give $n^{\frac{s+1}{2}+o(1)}$)

TRIPLE SYSTEMS (DIAGONAL)

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r(K_n^{(3)}, n) < 2^{2^{4n}}$$

Conjecture (Erdős \$500)

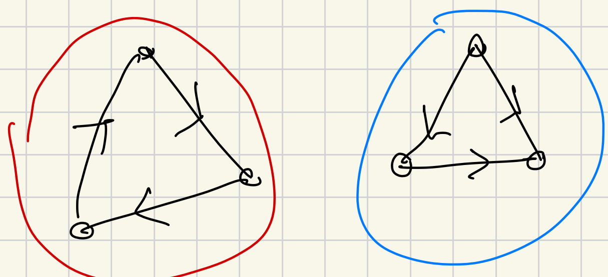
$$r(K_n^{(3)}, n) > 2^{2^{cn}}$$

Made with Goodnotes

TRIPLE SYSTEMS (OFF-DIAGONAL)

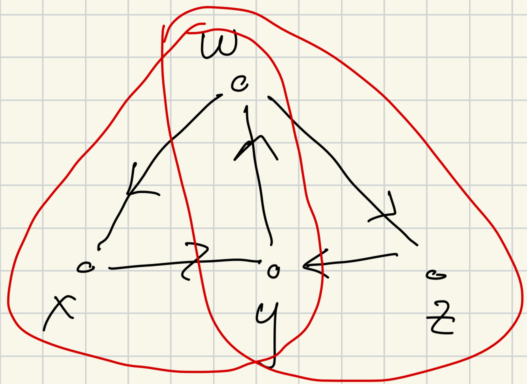
$$r(K_4^{(3)}, n) > 2^{cn} \qquad \text{(Erdős-Hajnal 1972)}$$

<u>Construction</u> T_N = random tournament on N vertices



Red iff directed triangle

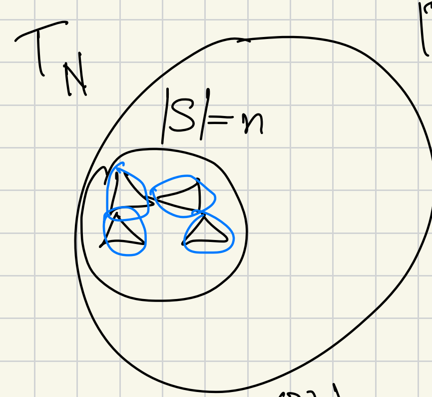
No red $K_4^{(3)}$-edge



wxz is BLUE!!

No blue $K_n^{(3)}$

$N = 2^{cn}$

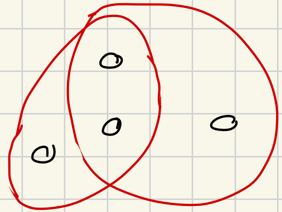


$\mathbb{P}(S \text{ is blue}) \leq 2^{-\frac{n^2}{7}}$

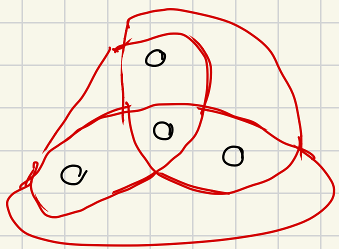
$$\mathbb{E}(\#\text{ blue } K_n^{(3)}) \leq \binom{N}{n} 2^{-\frac{n^2}{7}}$$

$$< 2^{cn^2 - \frac{n^2}{7}} < 1$$

POLY VERSUS EXP

$F_{4,2} =$  $r(F_{4,2}, n) = n^{\frac{1}{2} + o(1)}$

(Phelps-Rödl)

$F_{4,3} =$  $r(F_{4,3}, n) = 2^{cn \log n}$

(Erdős-Hajnal, Fox-He)

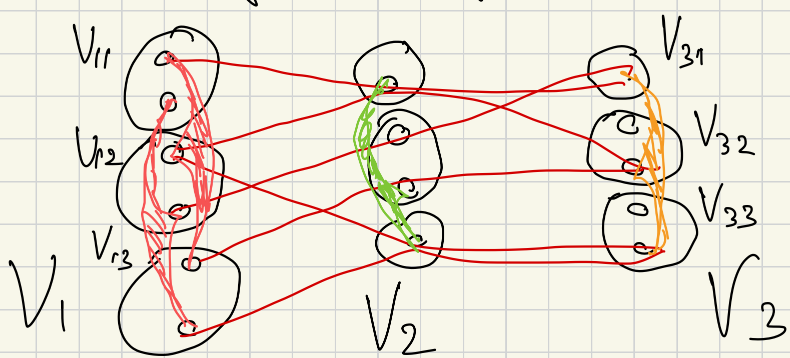
Problem For which F is $r(F, n)$

polynomial? superpolynomial? exponential?

ITERATIONS

<u>Definition</u> A k-graph H is iterated k-partite if H has a vertex partition $V_1 \cup \cdots \cup V_k$, each $H[V_i]$ is iterated k-partite, and all other edges are of the form $\{x_1, \dots, x_k\}$ $x_i \in V_i$

$k = 3$

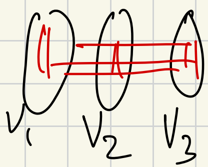


Theorem (Erdős-Hajnal 1972)

Suppose H is iterated 3-partite. Then

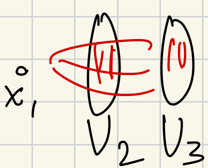
$$r(H, n) < n^{C_H} \quad \text{(i.e. polynomial)}$$

Proof (Sketch) • Shrink an "outer" part of H to form H'

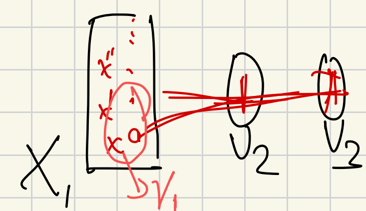
$H =$ 

• Find many copies of H' by induction

• Find one copy of H' where x_1 is blown up to polynomial size

$H' =$ 

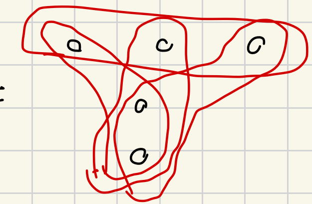
• Apply induction to X_1 to find V_1



Examples

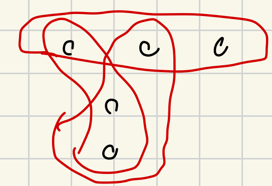
Theorem (Bohman - M - Picolleli 2016)

$$r(T_3, n) = \Theta\left(\frac{n^3}{\log n}\right) \qquad T_3 =$$



Theorem (Mattheus - M - Nie - Verstraëte 2024)

$$r(F, n) = n^{3-o(1)} \qquad F =$$

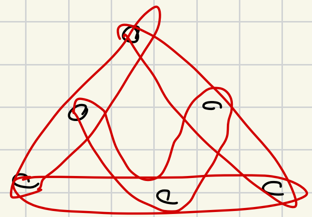


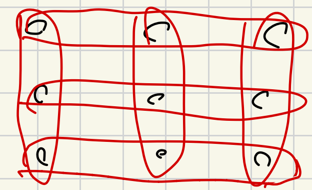
(random 3-graphs do not give exponent 3)

Definition

F is <u>linear</u> if every two edges share at most one vertex

E.g.



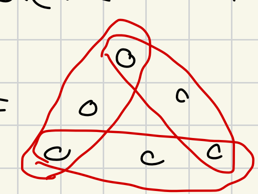


Conjecture (Folklore $\approx$ 90's onwards)

If F is linear, $r(F, n) < n^{c_F}$

Theorem (Kostochka - M - Verstraëte 2016)

$$r(C_3, n) = n^{3/2 + o(1)} \qquad C_3 =$$



Theorem (Conlon - Fox - Gunby - He - M - Suk - Verstraëte 2023)

For all sufficiently large k, there exists a linear 3-graph F on k-vertices such that

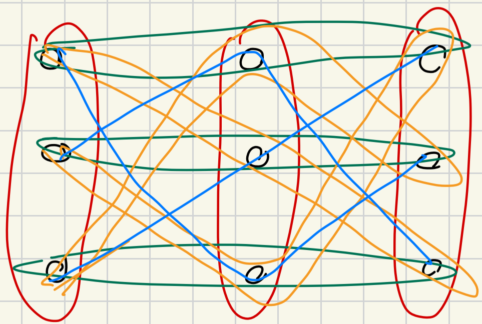
$$r(F, n) > 2^{c_F (\log n)^{k^{1/2 - o(1)}}}$$

F is a large (but fixed) random 3-graph on k-vertices with edge probability $p = \frac{1}{200k}$.

Note: $r(F, K^{(3)}_{n,n,n}) < n^c$ for F linear

Conjecture $r(\text{Fano}, n) < n^c$?

Note: $r(\text{Affine}(2), n) < n^C$



Affine(2) is iterated 3-partite

Question Does there exist a linear F s.t.

$$r(F, n) > 2^{n^c}, \quad c > 0 \quad ?$$

<u>Conjecture</u> (Conlon-Fox-Gunby-He-M-Suk-Verstraëte-Yu)

For all 3-graphs H, there exists $c = c(H)$ such that

$$r(H, n) < n^c \iff H \text{ is iterated 3-partite}$$

<u>Definition</u> H is <u>tightly connected</u> if, for any two edges $e, f \in H$ there is a tight path from e to f

Theorem (CFGHMSVY)

If H is tightly connected and not 3-partite, then

$$r(H, n) > 2^{c_H n^{2/3}}$$

Theorem (CFGHMSVY)

If H is not iterated 3-partite with at most two tight components, then

$$r(H, n) > 2^{c \log^2 n}$$

Proof

H — tightly connected and not 3-partite

Goal: $r(H, n) > 2^{cn^{2/3}} =: N$

V = r-trifference code in $[3]^\ell$,

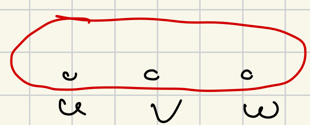
$\ell = C \log N, \quad r = \frac{\ell}{100}$

$V \subseteq [3]^\ell$

$\forall\ xyz \in V,\ \exists\ r$-coordinates s.t. $\{x_i, y_i, z_i\} = \{1,2,3\}$

$$\chi : \binom{V}{3} \longrightarrow \{\text{Red}, \text{Blue}\}$$

- $c(uv) = \{ i : u_i \neq v_i \}$
- $r \leq |c(uv)| \leq \ell$ $\qquad \ell = C \log N$
- $\phi(uv) \in c(uv)$ at random $\qquad r = \frac{\ell}{100}$
- $\chi(uvw) = \text{Red}$ iff $\phi(uv) = \phi(vw) = \phi(uw)$

Note  iff $\{u_i, v_i, w_i\} = \{1,2,3\}$

$\forall i \in \phi(uv) = \phi(uw) = \phi(vw)$

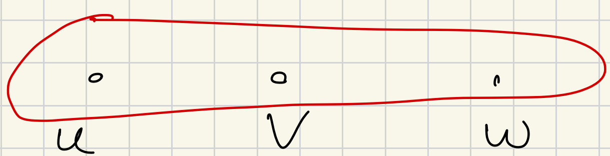
Example

	1	2	3	4	5	6	7	8	9	10	11
$u =$	1	2	1	1	3	1	2	3	1	2	3
$v =$	1	1	1	3	2	3	1	2	3	2	1
$w =$	2	2	1	3	1	2	1	3	1	1	2

$c(uv) = \{2, 4, 5, 6, 7, 8, 9, 11\}$ $\qquad \phi(uv) = 5$

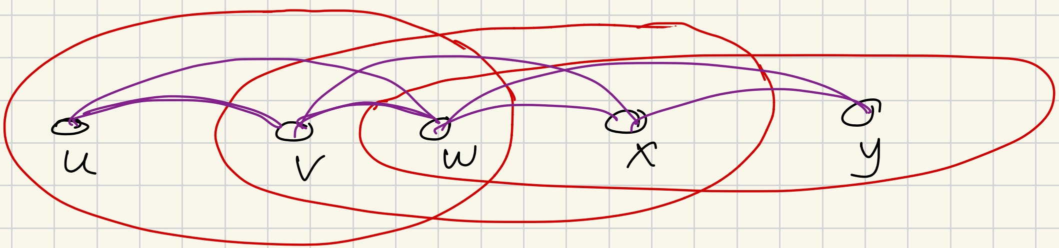
$c(uw) = \{1, 4, 5, 6, 7, 10, 11\}$ $\qquad \phi(uw) = 5$

$c(vw) = \{1, 2, 5, 6, 8, 9, 10, 11\}$ $\qquad \phi(vw) = 5$



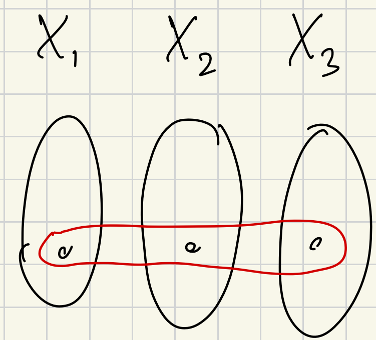
No Red $\mathcal{H}$

Let C be a red tight component



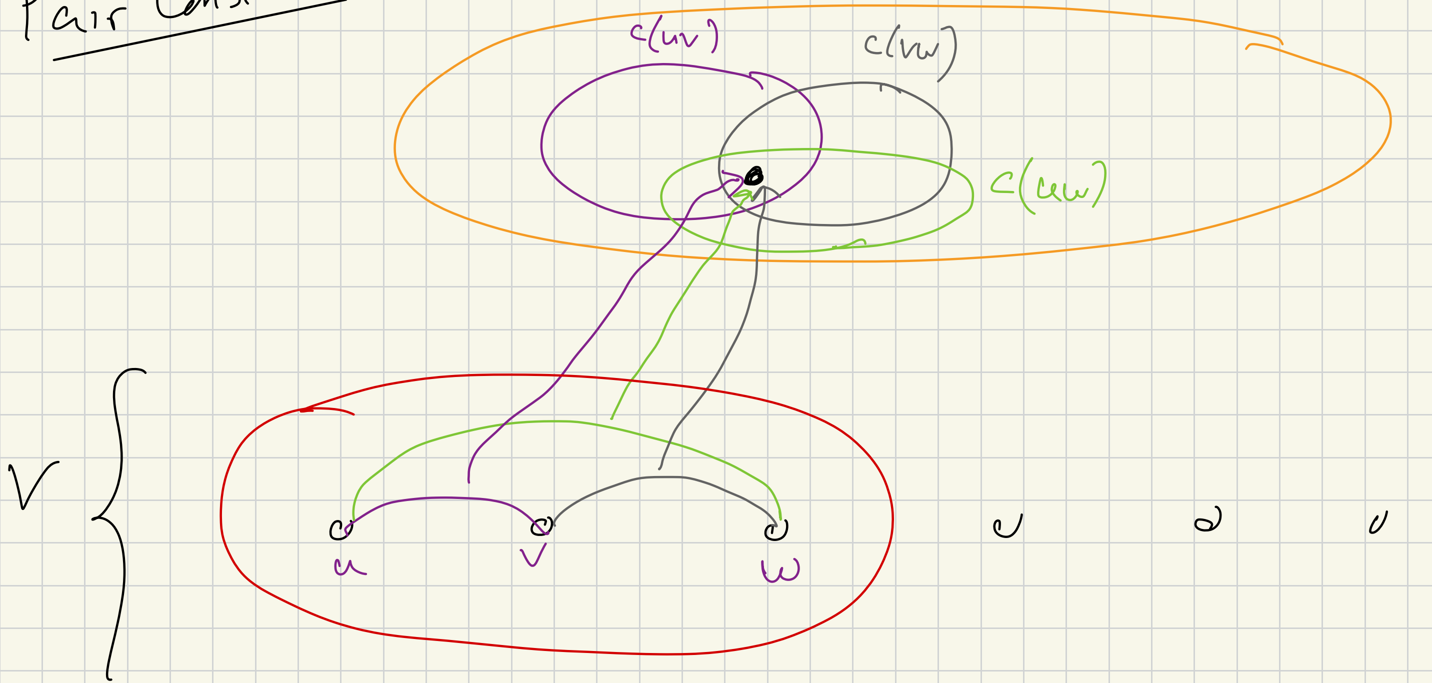
$\alpha\beta \in$ shadow: $\phi(\alpha\beta) = \phi(\alpha \; \beta) = i$

$\alpha \in X_j$ iff $\alpha_i = j$



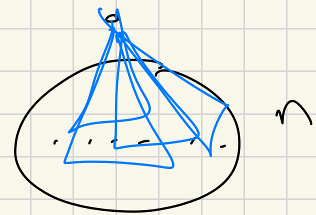
High Level Idea

Pair Constructions



An Intermediate Growth Rate

$S_n^{(3)}$ = n-star



$n+1$ vertices

$\binom{n}{2}$ edges

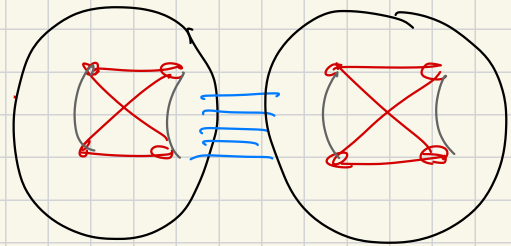
Theorem (Conlon - Fox - He - M - Suk - Verstraëte)

$$2^{c \log^2 n} < r\left(K_4^{(3)}, S_n^{(3)}\right) < 2^{c' n^{2/3} \log n}$$

Polynomial versus Exponential — Erdős-Hajnal

$g_k(s)$ = max # edges in s-vertex iterated k-partite k-graph

$g_2(s) = \binom{s}{2}$



complete

$g_k(s) = (1+o(1)) \frac{k!}{k^k - k} \binom{s}{k}$ k fixed $s \to \infty$

Achieved by recursively taking equitable partitions

Recall : H iterated k-partite k-graph

$$\Rightarrow \quad r(H, n) < n^c$$

$$\mathcal{H}_s^k = \{ H : |V(H)| = s, \ |E(H)| > g_k(s) \}$$

Conjecture (Erdös-Hajnal 1972 \$500)

$s \geq k \geq 3$

$$r(\mathcal{H}_s^k, n) > 2^{cn}$$

Theorem (M-Razborov 2021)

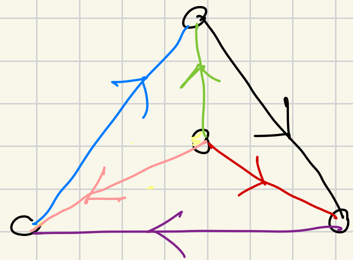
For $s \geq k \geq 4$,

$$r(\mathcal{H}_s^k, n) > 2^{cn}$$

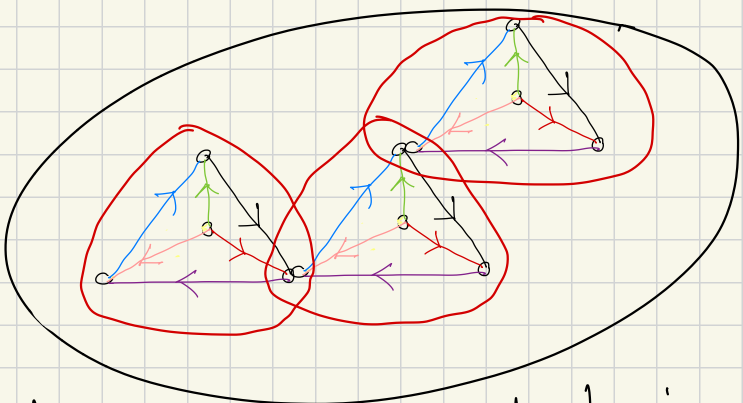
$k=3$ is still open for some values of s

True for $s = 3^t$ (Conlon-Fox-Sudakov)

$k=4$ Fix a 6-edge colored tournament T

E.g. $T =$ 

Construction of H:

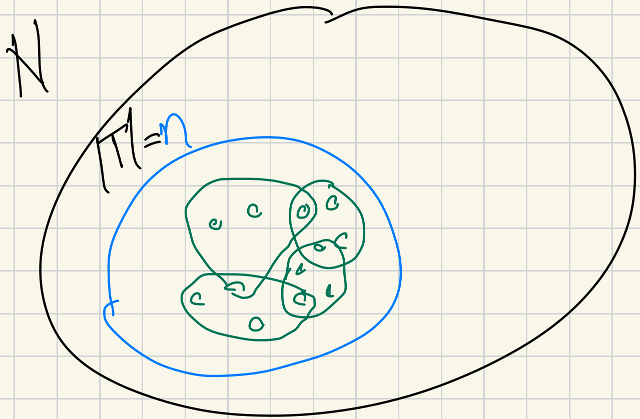


N vertices

$N = 2^{cn}$

random orientation and coloring

No Blue n-set: Fix Steiner 4-system $\mathcal{L}$ on T



$$\mathbb{P}(T \text{ is blue}) \le \mathbb{P}(\mathcal{L} \text{ is blue}) \le \left(\frac{1}{12}\right)^{5|\mathcal{L}|} < \left(\frac{1}{12}\right)^{5\frac{n^2}{12}}$$

$$\mathbb{E}(\text{Blue } n\text{-sets}) \le \binom{N}{n}\left(\frac{1}{12}\right)^{\frac{5n^2}{12}} < \left(N\left(\frac{1}{12}\right)^{\frac{5n}{12}}\right)^n < 1$$

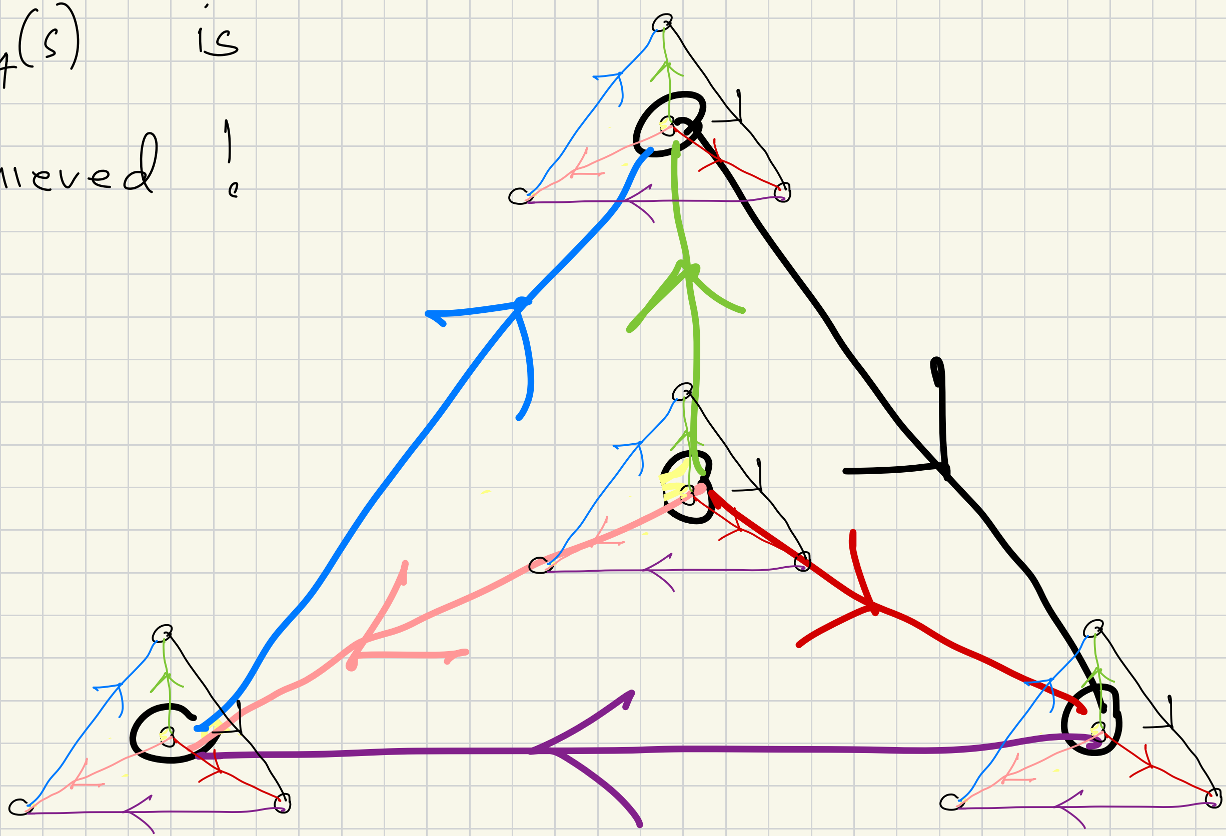
No Red graph $H \in \mathcal{H}_s^4$

<u>Theorem</u> (M-Razborov)

Given a 6-edge colored tournament with vertex set S, $|S| = s$.

$\#$ copies of $T \le g_4(s)$

$g_4(s)$ is achieved!



Erdős-Hajnal Problem

$r_k(s,t;n) = \min N$ s.t. every Red/blue coloring of $\binom{[N]}{k}$ results in a blue $K_n^{(3)}$ or a set of s vertices with at least t red edges

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<u>Problem</u> As t-grows from 1 to $\binom{s}{k}$ there is a well-defined value $t_1 = h_1^{(k)}(s)$ at which $r_k(s,t_1-1;n)$ is polynomial in n while $r_k(s,t_1;n)$ is exponential in a power of n, another well-defined value t_2 at which it changes from exponential to double-exponential and so on...

M-Razborov showed $t_1 = g_k(s)+1$

Theorem (M-Suk 2020)

For $4 \le t \le k-2$, $\exists\, c > 0$ s.t.

$$\mathrm{twr}_{t-1}\left(n^{k-t+1+o(1)}\right) > r_k(k+1, t; n) > \begin{cases} \mathrm{twr}_{t-1}\left(c n^{k-t+1}\right) & k-t \text{ even} \\ \mathrm{twr}_{t-1}\left(c n^{\frac{k-t+1}{2}}\right) & k-t \text{ odd} \end{cases}$$

I.e. as we increase t by one the tower height increases by one.

Thank You!