Boundedness of discounted tree sums

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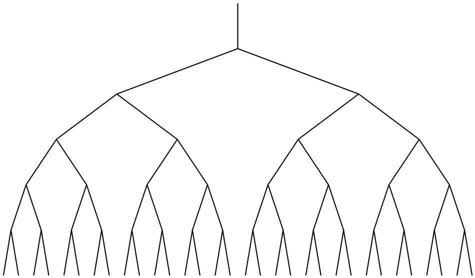
Title page

planted m-ary tree / $c \in (0,1)$ / $(\eta_e)_{\text{edges}}$ i.i.d.

discount rate — marks



at generation k, $\text{length}(e) = c^k \eta_e$

Question: Is the height of the tree finite?

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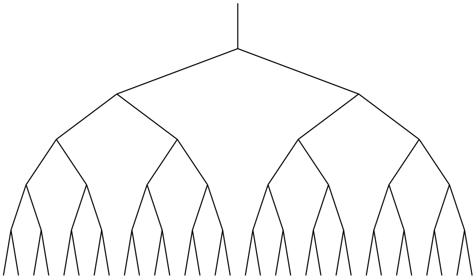

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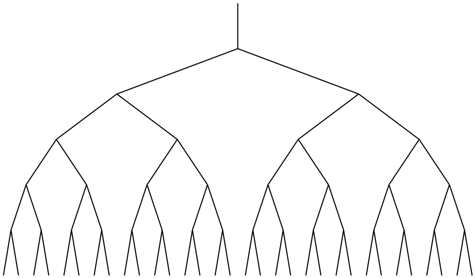

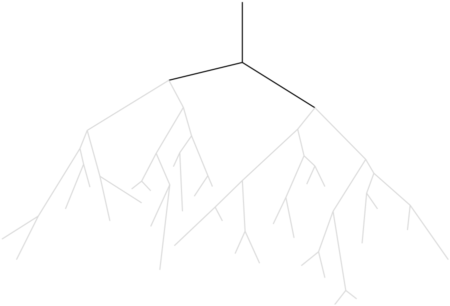

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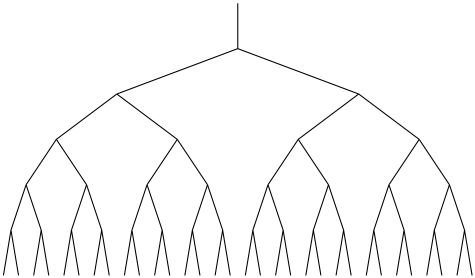

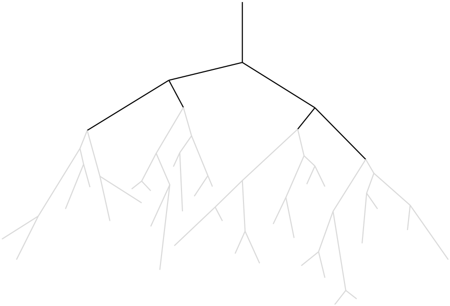

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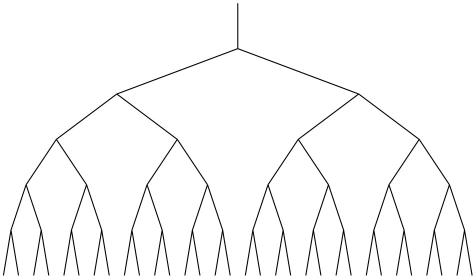

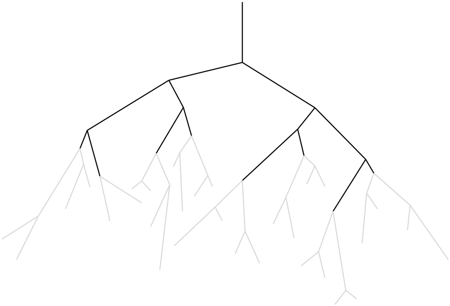

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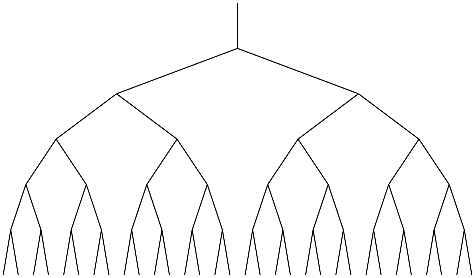

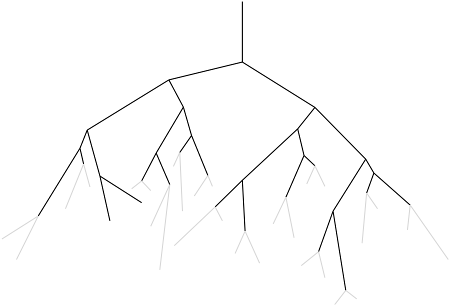

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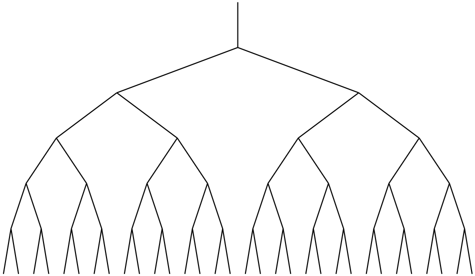

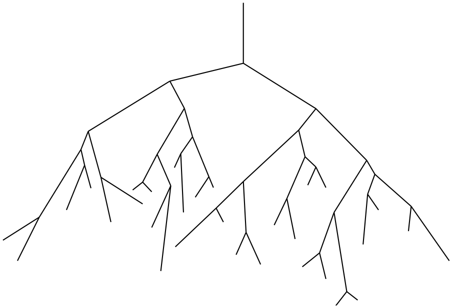

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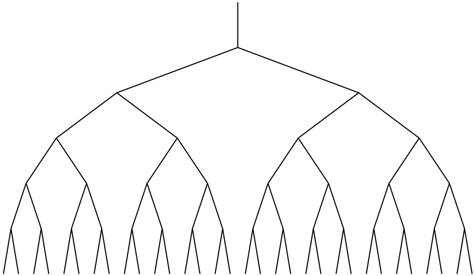

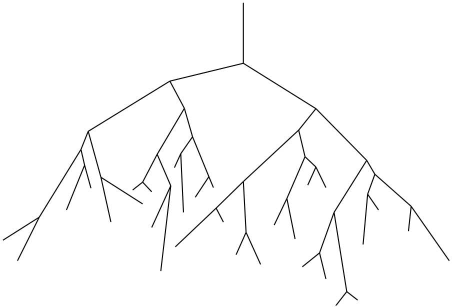

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Question: Is the height of the tree finite?

Suppose $\mathbb{P}(\eta > x) \sim x^{-\theta}$.

- If $m < c^\theta$, then $X < \infty$ a.s. $\max_{\text{generation } k} \ell(e)$ decreases exponentially
- If $m > c^\theta$, then $X = \infty$ a.s. $\max_{\text{generation } k} \ell(e)$ increases exponentially

Proof. $m^k \mathbb{P}(\ell(e) > x) = m^k \mathbb{P}(c^k \eta > x) \sim x^{-\theta}(mc^{-\theta})^k$.

Athreya (1985)
Endogenous solution of

$$X \overset{(d)}{=} \eta + \max_{1 \le i \le m} c\, X^{(i)}$$

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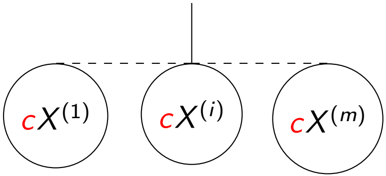
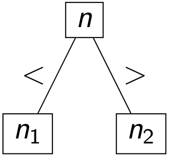


Goal: find the k-th smallest number (result) among n numbers.

FIND algorithm

Pick a random number (pivot).
Compare it with the other numbers.
If result=pivot, end.
If not, iterate.



Cost of the algorithm:

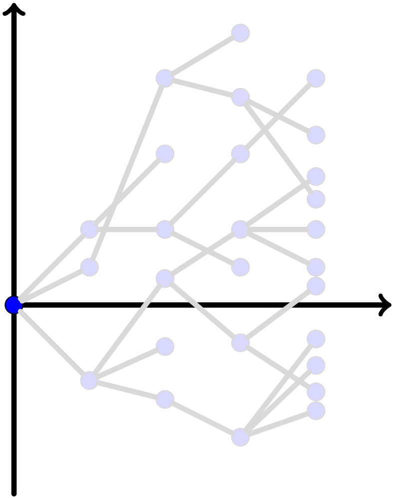
$$X_n = n + \max(X_{n_1}, X_{n_2})$$

$\frac{1}{n} X_n \overset{(d)}{\to} X$.

$$X \overset{(d)}{=} 1 + \max(U X^{(1)}, (1 - U) X^{(2)})$$

Endogenous solution $X < \infty$ (Grüber and Rösler, 1996).

Branching random walk $(V(u))_u$



$V(\emptyset) = 0$.

$(V(u), |u| = 1) \stackrel{(d)}{=} \mu$: point process on the real line.

At each generation, vertices have independently children with positions at distance a copy of μ from their parent.

$e^{V(u)}$: discount rates

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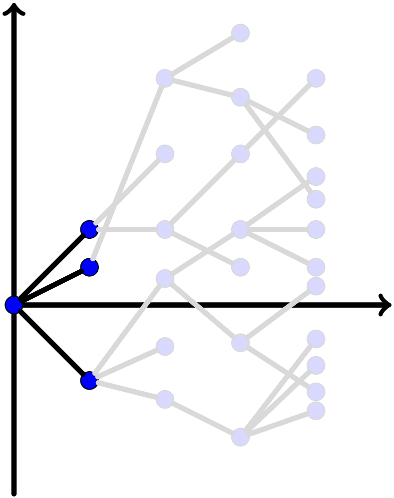

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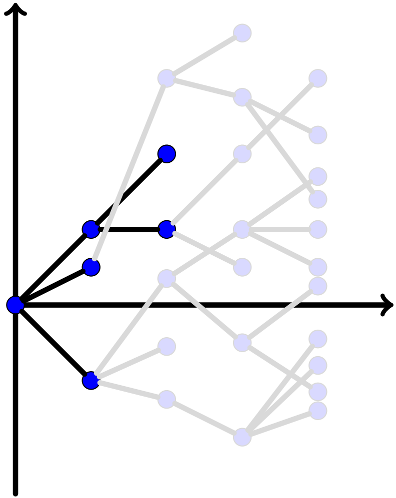

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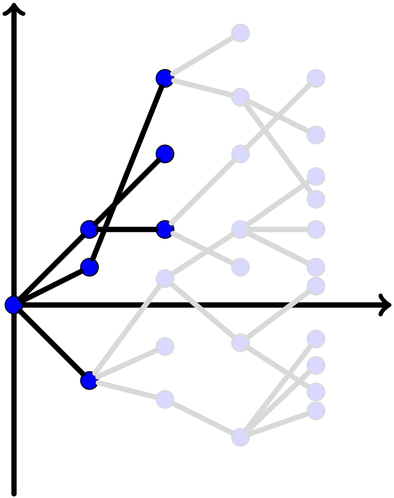

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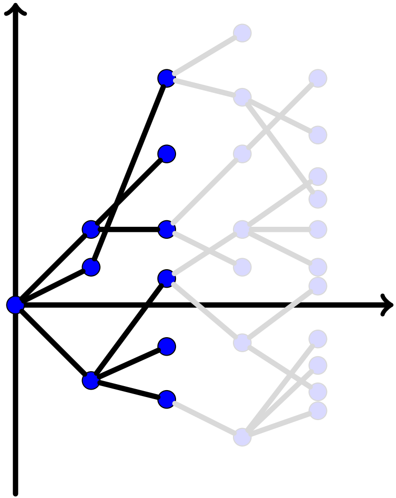

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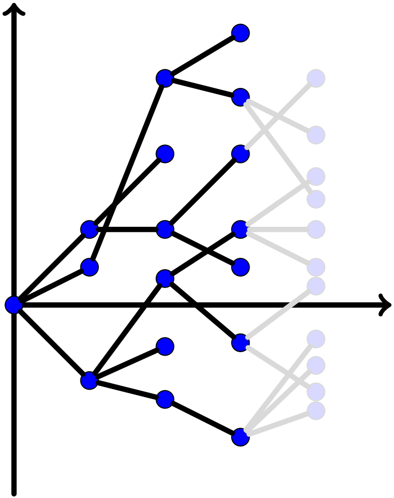

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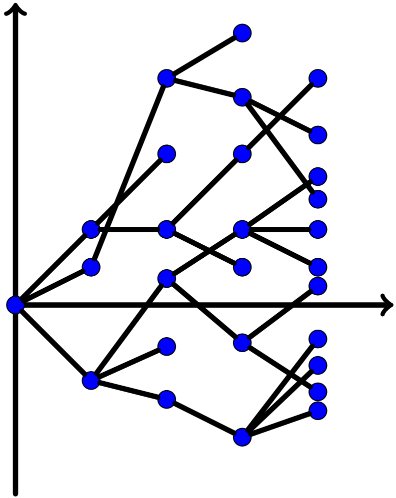

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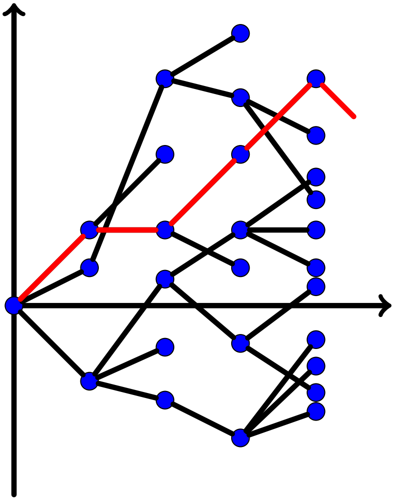


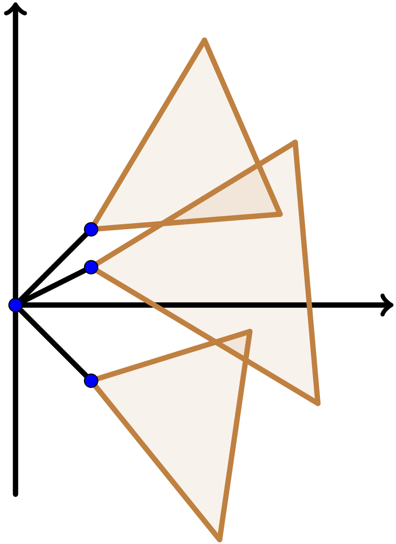
$(\eta_u)_u$: i.i.d. positive marks on the vertices.

$$D(\xi) := \sum_{u \in \xi} e^{V(u)} \eta_u \qquad \text{discounted sum}$$

$$X := \sup_{\xi \in \partial T} D(\xi)$$

Question: Is $X < \infty$?

(Aldous & Bandyopadhyay, 2005)



$$D(\xi) := \sum_{u \in \xi} e^{V(u)} \eta_u \qquad X := \sup_{\xi \in \partial T} D(\xi)$$

X is the endogenous solution of

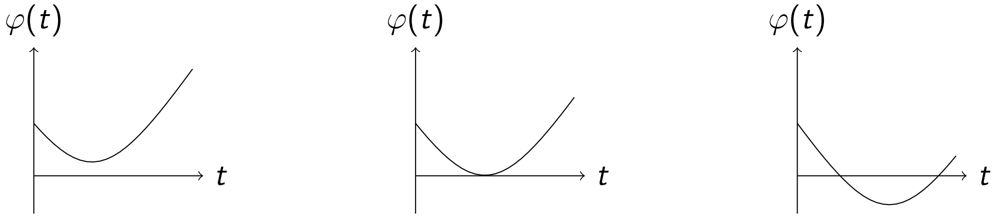
$$X \overset{(d)}{=} \eta + \sup_{|u|=1} e^{V(u)} X^{(u)}$$

Example I: step displacement is a constant

Example II: $\eta = 1$, step displacement is $-Exp(1)$.

$$\varphi(t) = \ln \mathbb{E}\left[\sum_{|u|=1} e^{tV(u)}\right]$$

$$M_n := \max_{|u|=n} V(u) \qquad \frac{1}{n} M_n \to \gamma := \inf_{t>0} \frac{\varphi(t)}{t}$$



$\gamma > 0$ $\gamma = 0$ $\gamma < 0$

$M_n - \gamma n - c \ln(n)$ converges in distribution, $c < 0$.

$$D(\xi) = \sum_{u \in \xi} e^{V(u)} = \sum_{n=0}^{\infty} e^{V(\xi_n)} \leq \sum_{n=0}^{\infty} e^{M_n}$$

- $\gamma < 0 \Rightarrow M_n \sim \gamma n \Rightarrow X < \infty$

$$X \geq e^{M_n}$$

- $\gamma > 0 \Rightarrow M_n \to \infty \Rightarrow X = \infty$
- What about $\gamma = 0$?

The upper bound $D(\xi) \leq \sum_{n=0}^{\infty} e^{M_n}$ is too rough. One cannot find a path which stays close to the maximum at all times.

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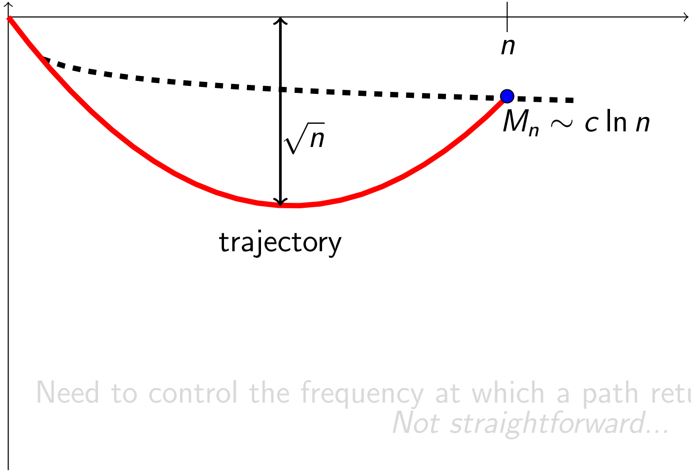
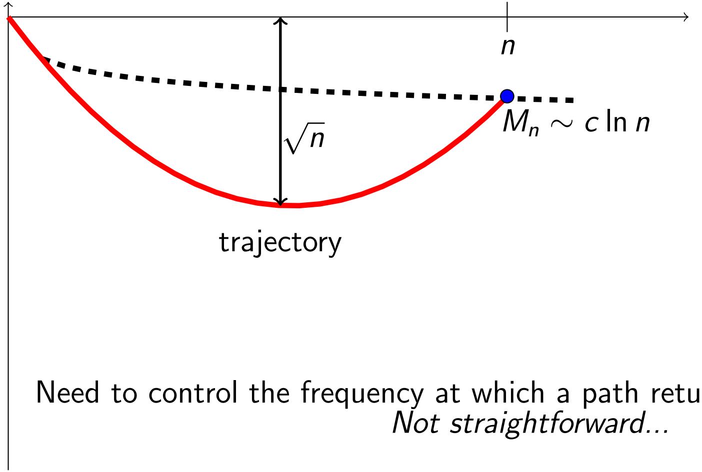


Need to control the frequency at which a path returns to levels of order $\ln n$

Not straightforward...



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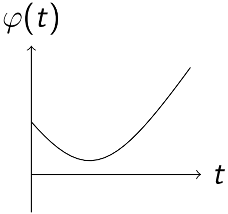

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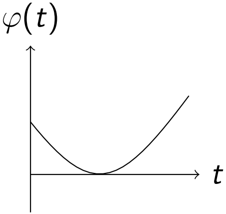

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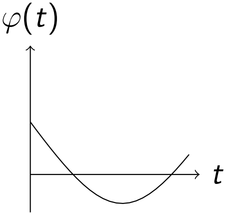

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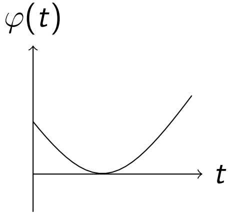

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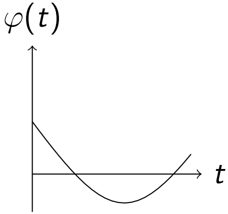

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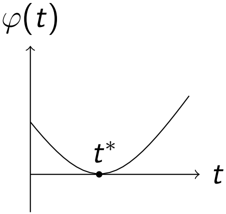

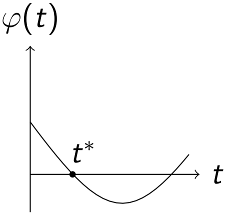

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$$\mathbb{E}\left[\sum_{|u|=1} e^{t^* V(u)}\right] = 1$$

Suppose that $\theta := \lim_{x\to\infty} \frac{-1}{\ln(x)} \ln \mathbb{P}(\eta > x) \in [0,\infty]$ exists.

Theorem (A.,Hu,Shi, 24+)

If $t^* < \theta$, then $X < \infty$ a.s. If $t^* > \theta$, then $X = \infty$ a.s. on non-extinction.

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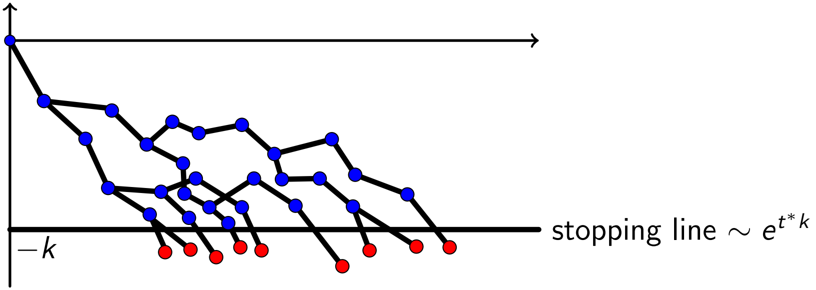
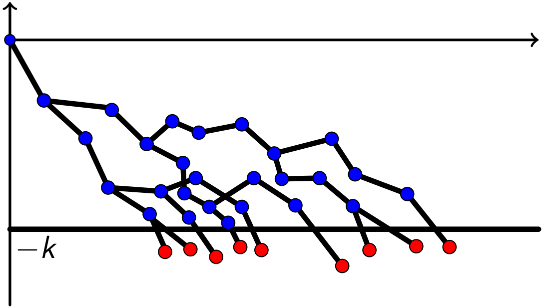
It suffices to prove $\sup_{|u|=n} e^{V(u)}\eta_u$ goes to ∞ exponentially fast.
Better idea: consider $e^{V(u)}\eta_u$ over BRW stopped at level $-k$ then show $\sup_{\text{stopping line}} e^{-k}\eta_u$ goes to infinity.



$$\mathbb{E}\left[\sum_{|u|=1} e^{t^* V(u)}\right] = 1 \Rightarrow 1 = \mathbb{E}\left[\sum_{\text{stopping line}} e^{t^* V(u)}\right] \approx e^{-t^* k}\,\mathbb{E}[\text{stopping line}]$$

If $t^* < \theta$, then $X < \infty$ a.s.



If $t^* < \theta$, then $X < \infty$ a.s.

It suffices to show that

- $\{u : V(u) \approx -k\}$ is of size $e^{t^* k}$.
- $\sum_{u\in\xi} \mathbf{1}_{\{V(u)\approx -k\}}$ grows at most polynomially uniformly in ξ.

$$D(\xi) = \sum_{u\in\xi} e^{V(u)}\eta_u \approx \sum_{k=0}^{\infty} e^{-k} \sum_{u\in\xi} \eta_u \mathbf{1}_{\{V(u)\approx -k\}}$$

$$\leq \sum_{k=0}^{\infty} \underbrace{e^{-k} \sup_{u:V(u)\approx -k} \eta_u}_{\text{exponentially small}} \underbrace{\sum_{u\in\xi} \mathbf{1}_{\{V(u)\approx -k\}}}_{\text{polynomial}}.$$

Stop each path ξ when it is $\approx -k$ for the ℓ-th time.

$\mathbb{E}[\text{stopping line}] \approx e^{t^* k}\mathbb{P}(S \approx -k \text{ for the } \ell\text{-th time}) = o(1)$ if $\ell \geq k^3$.

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$$D(\xi) = \sum_{u \in \xi} e^{V(u)} \eta_u \approx \sum_{k=0}^{\infty} e^{-k} \sum_{u \in \xi} \eta_u \mathbf{1}_{\{V(u) \approx -k\}}$$

$$\leq \sum_{k=0}^{\infty} \underbrace{e^{-k} \sup_{u : V(u) \approx -k} \eta_u}_{\text{exponentially small}} \underbrace{\sum_{u \in \xi} \mathbf{1}_{\{V(u) \approx -k\}}}_{\text{polynomial}}.$$

Stop each path ξ when it is $\approx -k$ for the ℓ-th time.

$\mathbb{E}[\text{stopping line}] \approx e^{t^* k} \mathbb{P}(S \approx -k \text{ for the } \ell\text{-th time}) = o(1)$ if $\ell \geq k^3$.

If $t^* < \theta$, then $X < \infty$ a.s.

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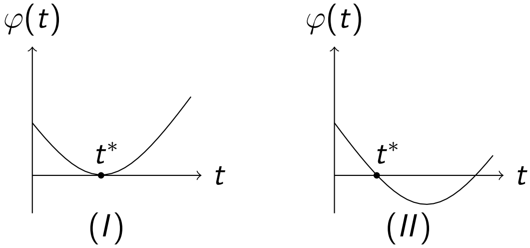
Suppose $V(u) \in \mathbb{Z}$ and $\mathbb{P}(V(u) \leq -2) = 0$ for $|u| = 1$.

$$N(\xi, k) := \sum_{u \in \xi} \mathbf{1}_{\{V(u)=-k\}}.$$



Theorem (A., Hu, Shi, 24+)

(I) $\sup_\xi \limsup_{k\to\infty} \frac{1}{k^2} N(\xi, k) = \frac{t^*}{2\theta}.$

(II) $\sup_\xi \limsup_{k\to\infty} \frac{1}{k} N(\xi, k) = -\frac{t^*}{\ln q}.$

What about lim inf?

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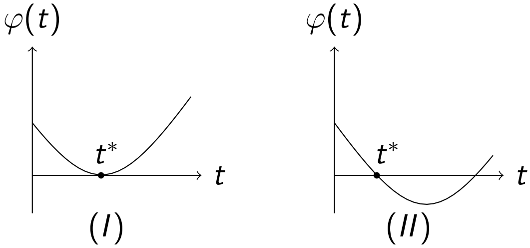

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Idea of the proof

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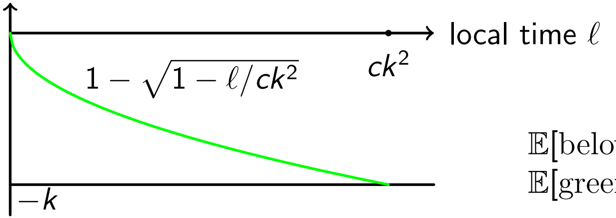

$\mathbb{E}[\text{below green line}] \to \infty$ if $c < \frac{t^*}{2\theta}$
$\mathbb{E}[\text{green stopping line}] \to 0$ if $c > \frac{t^*}{2\theta}$

- Hausdorff dimension of rays such that $N(\xi, k) \sim ak^2$?
- Weaker assumptions?
- Study of all solutions of the fixed point equation.

THANK YOU