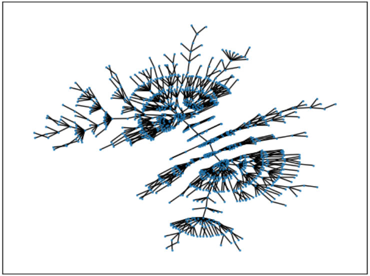
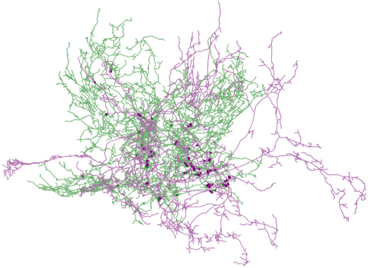
Preferential attachment trees built from random walks

Gábor Pete (Rényi Institute and TU Budapest)

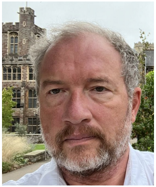


Tree Builder Random Walk



Physical networks from RWs

The Tree Builder Random Walk



János Engländer
(University of Colorado, Boulder)

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Rodrigo Ribeiro
(University of Denver)

The Tree Builder Random Walk

There are many models of RWs on graphs that change over time,

either independently of the walk (e.g., random walk on dynamical percolation clusters, by Peres, Sousi, Stauffer, Steif),

or the walk is changing the transition probabilities (e.g., reinforced random walk by Merkl-Rolles, Angel-Crawford-Kozma, . . . , true self-repelling motion by Tóth-Werner).

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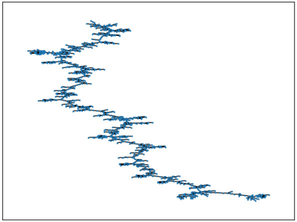
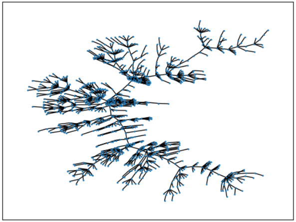
Introduced by Amorim, Figueiredo, Iacobelli, Neglia (2016).

Mixture of RW (recurrence, transience, ballisticity) and random graph questions (diameter, degree distribution).

The Tree Builder Random Walk



$\gamma = 0$



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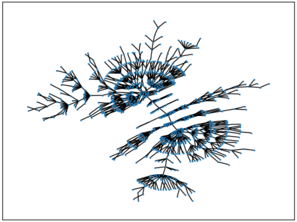
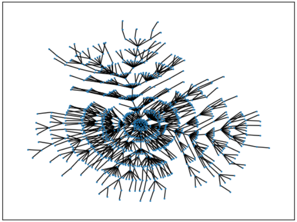


$\gamma = 0.55$



$\gamma = 0.9$

(Thanks to Ágnes Kúsz for the pictures.)

TBRW: the elliptic regime

$\gamma = 0$, always grow a leaf:

If X_n is a leaf, $\mathbb{E}[\,\mathrm{dist}(o, X_{n+1})\,] = \mathbb{E}[\,\mathrm{dist}(o, X_n)\,]$.

If X_n is not a leaf, $\mathbb{E}[\,\mathrm{dist}(o, X_{n+1})\,] \geq \mathbb{E}[\,\mathrm{dist}(o, X_n)\,] + 1/3$.

X_n spends at least a constant proportion of time at non-leaves.

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$\implies$ ballistic X_n, linearly growing T_n.

This and much more was proved by Figueiredo, Iacobelli, Oliveira, Reed, Ribeiro (2021), and Iacobelli, Ribeiro, Valle, Zuaznábar (2022) in the elliptic regime: $p_n > c > 0$, and versions of that.

TBRW: beyond ellipticity, $\gamma < 1/2$

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Mixing time of a tree:

$T_{\text{mix}} \leq \mathrm{vol}^{1+o(1)}\,\mathrm{diam}$, and *typically* $T_{\text{mix}} \geq c\,\mathrm{vol}$.

Also, *typical* hitting time of root from a distant leaf is $\tau_{\text{hit}} \geq c\,\mathrm{vol}$.

One stage: the walk between growth times.

By time n, around $n^{1-\gamma}$ growth times, so $\mathrm{vol}_n \approx n^{1-\gamma}$.

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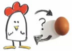
$\Longrightarrow$ transience should hold. Nothing is proved. 

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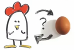
$\gamma > 1/2$: Hope to quite mix in a stage, so T_n is more balanced, hence diam is smaller, hence we get more easily mixed. 

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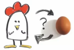
$\gamma > 1/2$: Hope to quite mix in a stage, so T_n is more balanced, hence diam is smaller, hence we get more easily mixed. 

Key proposition. *For $\gamma > 1/2$, have $\mathrm{diam}(T_n) = \mathrm{vol}_n^{o(1)}$ w.h.p.*

Thus $T_{\mathrm{mix}} = \mathrm{vol}_n^{1+o(1)}$, so

$\mathbb{P}[\, S_k < T_{\mathrm{mix}} \,] = \mathbb{P}[\, S_k < k^{1+o(1)} \,] = k^{\frac{-\gamma}{1-\gamma}} k^{1+o(1)} = k^{\frac{1-2\gamma}{1-\gamma}+o(1)}$.

For $\gamma \in (1/2, 2/3]$, this is $k^{-\varepsilon}$ with $\varepsilon \le 1$, small, but happens i.o.

For $\gamma \in (2/3, 1]$, this is summable in k, happens only fin. often.

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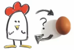
$\gamma > 1/2$: Hope to quite mix in a stage, so T_n is more balanced, hence diam is smaller, hence we get more easily mixed. 

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But actual proofs? What does “mixing has happened” mean?

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$\Longrightarrow$ recurrence.

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Key tool (Aldous-Diaconis 1987). *For any finite Markov chain $(X_t)_{t\geq 0}$ with stationary distribution π, for any starting state x, there is an optimal strong stationary stopping time η_x:*
$\mathbb{P}_x\big[\, X_{\eta_x} = y \,\big|\, \eta_x = t \,\big] = \pi(y)$, *and*
$\mathbb{P}_x[\, \eta_x > t \,] = s_x(t)$, *the separation distance at time t.*

With this, we can produce a coupling with BA-tree process from some random time on.

TBRW: structural corollaries $\gamma > 2/3$

All almost sure BA-tree limit results that do not depend on the starting tree (B. Pittel, T. Móri, Zs. Katona) can get transfered:

$$\frac{\mathrm{diam}(T_n)}{\log n} \to c$$

$$\frac{\mathrm{dist}(o, \mathrm{Unif}(T_n))}{\log n} \to 1/2$$

$$\frac{\max \deg(T_n)}{\sqrt{\mathrm{vol}_n}} \to \zeta \text{ with a non-trivial distribution}$$

$$\frac{|\{v \in T_n : \deg(v) = k\}|}{\mathrm{vol}_n} \to \frac{4}{k(k+1)(k+2)}$$

TBRW: open questions

1. Show transience for $\gamma < 1/2$.
   Diameter of T_n is $n^{1-\gamma-\delta(\gamma)}$ for what $\delta(\gamma)$?
   Degree distribution: exponential or polynomial or in between?
2. What happens at $\gamma = 1/2$?
3. For $\gamma \in (1/2, 2/3]$, the process is not the BA-tree process, but do we still have BA-like statistics?
4. What is the scaling limit of the height process of the random walker on a BA tree? It is typically at height $(1/2)\log n$, with Gaussian fluctuations $\asymp \sqrt{\log n}$, by results of Zsolt Katona. Do we see an Ornstein-Uhlenbeck process?

A network-of-networks model for physical networks

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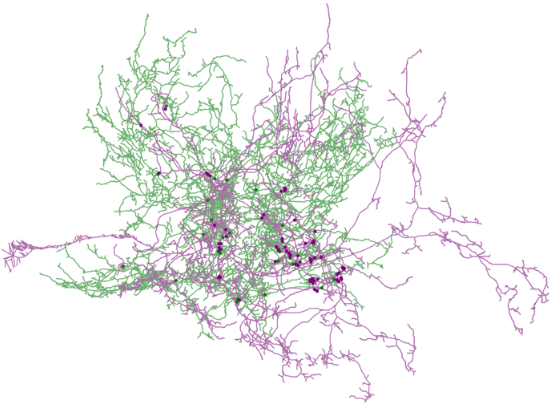
Representing physical networks

Physical network: not only is the graph embedded in space, but the vertices and edges are non-overlapping physical objects.

- Traditional: balls and tubes,
- More realistic: nodes are extended objects and links are the points of contact. E.g., neurons and synapses



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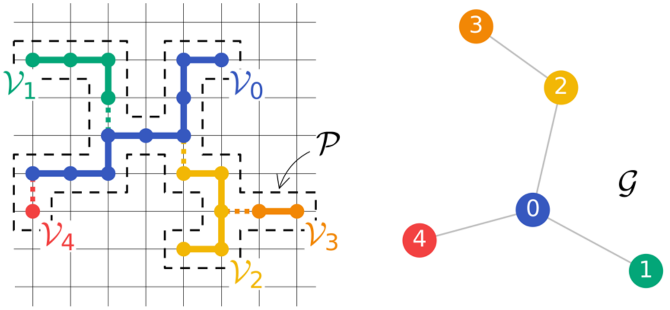
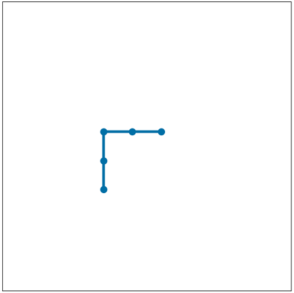


Nodes will be random walk paths (pieces) in some finite graph H. In principle one could choose simple random walk, self-avoiding walk, loop-erased walk...



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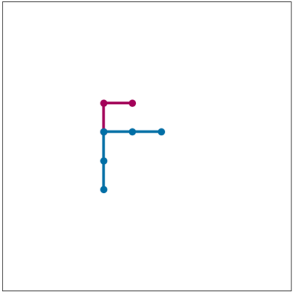

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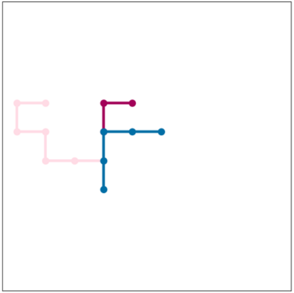

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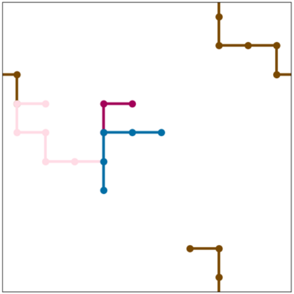

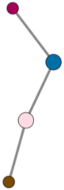

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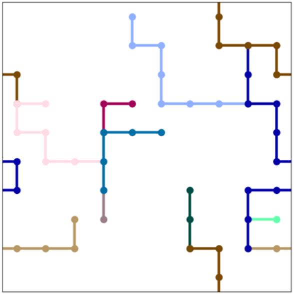
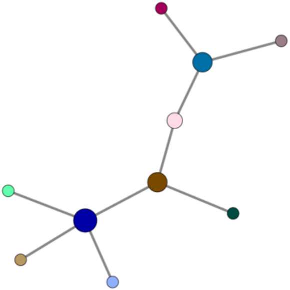
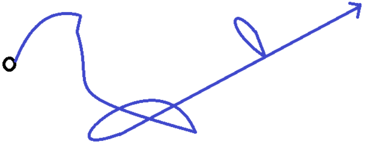




We will mainly focus on models generated by the loop-erased random walk (LERW).

LERW from x_1 to x_0: run simple random walk and erase all the loops as they are created.



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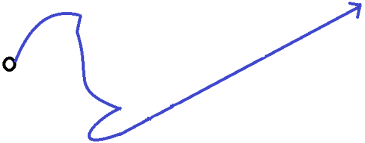
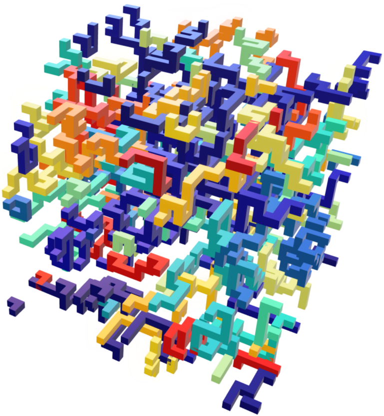
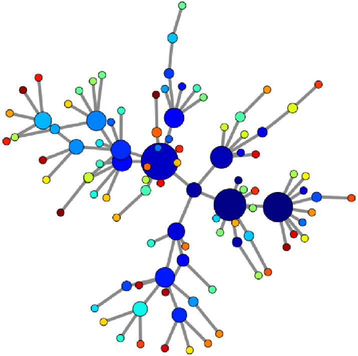
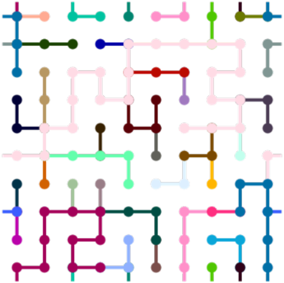


The resulting network-of-networks in 3 dimensions:





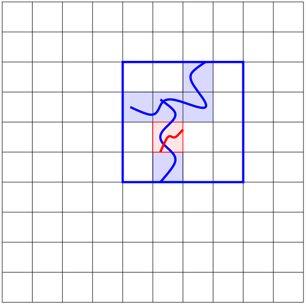
We will look at the fully packed model: every H-vertex is contained in some piece.



Heuristic calculation for degree distribution

Assume the pieces in $\mathbb{Z}^d$ have "fractal dimension" ζ, with $2\zeta \geq d$.

Box of t^{th} piece: side-length ℓ_t. Typically, $\ell_s \geq \ell_t$ for $s \leq t$.



If larger piece $\mathcal{V}_s$ intersects the box of the smaller piece $\mathcal{V}_t$, then, by $2\zeta \geq d$, they overlap with positive probability.

Tile the box $[L]^d$ with $(L/\ell_t)^d$ boxes of side-length ℓ_t. Number of boxes intersected by $\mathcal{V}_s$ is $\approx (\ell_s/\ell_t)^\zeta$.

Thus the intersection probability, randomly placed, using $v_s = |\mathcal{V}_s| \approx \ell_s^\zeta$:

$$p_{s,t} \approx \left(\frac{\ell_s}{\ell_t}\right)^\zeta \Big/ \left(\frac{L}{\ell_t}\right)^d = \frac{v_s v_t^{d/\zeta-1}}{L^d}.$$

Heuristic calculation for degree distribution

Probability that the piece added at time t intersects any existing piece $s < t$ is approximately

$$\sum_{s:s<t} p_{s,t} = \sum_{s:s<t} w_{t-1} v_t^{d/\zeta-1} / L^d,$$

where $w_s = v_1 + \cdots + v_s$.

In our growth process, $\mathcal{V}_t$ grows until it hits existing piece, until the above intersection probability becomes ≈ 1. I.e.:

$$v_t \approx \left(w_{t-1}/L^d \right)^{-\frac{\zeta}{d-\zeta}}.$$

This is an ODE, $v_t = w_t'$, with $w_1 \approx L^\zeta$. Get $w_t \approx L^d (t/L^d)^{1-\zeta/d}$ and $v_t \approx (t/L^d)^{-\zeta/d}$, and the degree of piece t at time T is

$$\deg_T(t) \approx 1 + \frac{v_t}{L^d} \sum_{s=t}^{T} v_s^{d/\zeta-1} \approx (T/t)^{\zeta/d}.$$

It follows that degree distribution is $\mathbb{P}\left[\deg_T(\sigma_T) > k \right] \approx k^{-d/\zeta}$.

Heuristic calculation for degree distribution

For $2\zeta < d$, get $\mathbb{P}\big[\deg_T(\sigma_T) > k\big] \approx k^{-2}$, mean-field, as in BA.

Example 1. LERW in $d \geq 5$.

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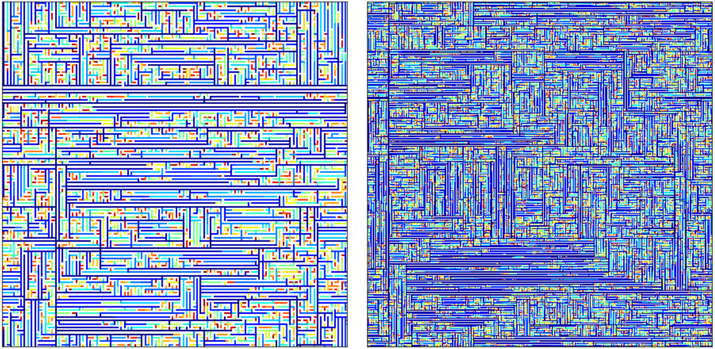
Example 2. $\zeta = 1$, $d = 2$, random ray model:



Ex1 is really a version of the BA-tree, not just in deg-distribution.

LERW physical networks

For the case of LERW, mathematically rigorous proofs can be given, using connection with the Uniform Spanning Tree:

Theorem

The degree distribution of the abstract network satisfies

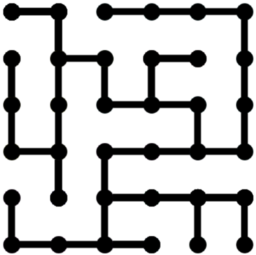
- $\mathbb{P}[\deg(\sigma) > t] = t^{-8/5+o(1)}$ for dim $= 2$;
- $\mathbb{P}[\deg(\sigma) > t] = t^{-2+o(1)}$ for dim ≥ 5.

Construction via the Uniform Spanning Tree

Consider a Uniform Spanning Tree (UST) T in G (finite). Take a uniform random ordering $x_0, x_1, \ldots, x_N$ of the vertices.

$P_1 :=$ the path from x_1 to x_0 in T
$P_k :=$ the path from x_k to $P_1 \cup \ldots \cup P_{k-1}$

The resulting physical network has the same distribution as our original model! We rely on Wilson's algorithm.

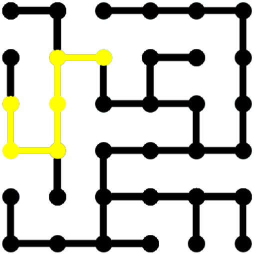


Construction via the Uniform Spanning Tree

Consider a Uniform Spanning Tree (UST) T in G (finite). Take a uniform random ordering $x_0, x_1, \ldots, x_N$ of the vertices.

$P_1 :=$ the path from x_1 to x_0 in T
$P_k :=$ the path from x_k to $P_1 \cup \ldots \cup P_{k-1}$

The resulting physical network has the same distribution as our original model! We rely on Wilson's algorithm.



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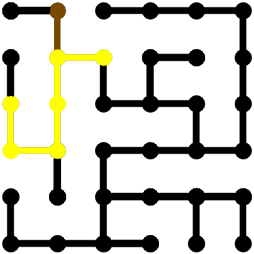

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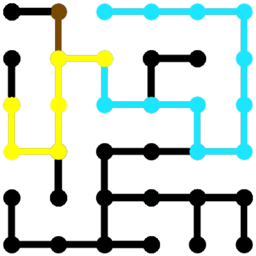

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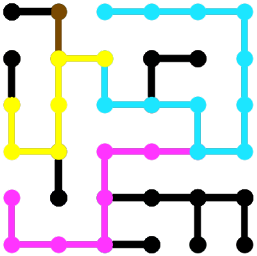

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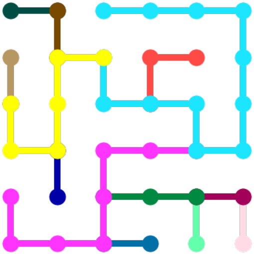

Limit networks

One consequence of the alternative description is the existence of the infinite-volume limit.

Theorem

Let G be an infinite transitive graph and G_n be an exhaustion by finite induced subgraphs. The LERW-generated physical network on G_n has a weak limit, invariant under the automorphisms of G.

The abstract network corresponding to the weak limit is a unimodular random forest consisting of one-ended trees whenever G has superlinear growth.

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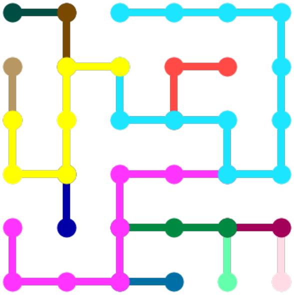
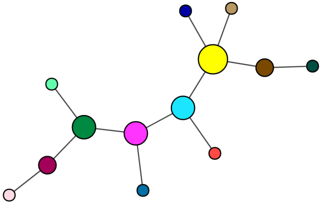
One can also construct a scaling limit, using work of Archer, Nachmias, Shalev (2021) for $d \geq 5$. (In progress.)

Gives answers to questions like: *how does the subtree induced by the first 100 pieces in the construction look like*?

Degree exponent, rough sketch of proof





Want: degree of a uniformly randomly selected node of the abstract graph.

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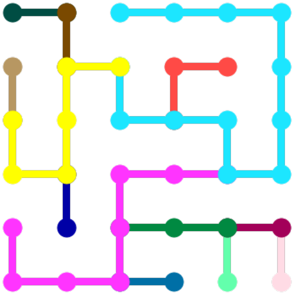

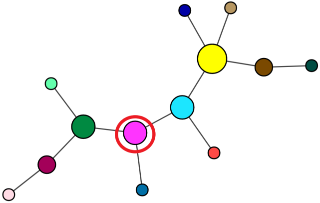

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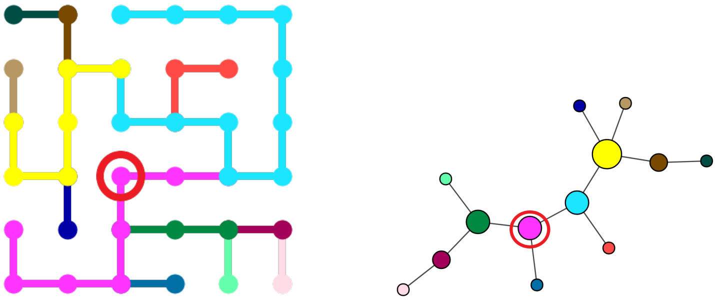
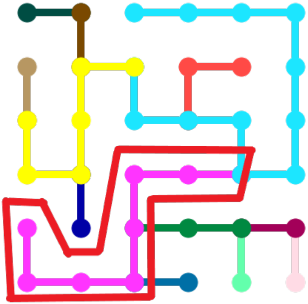
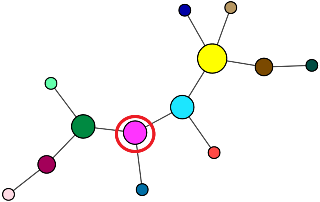


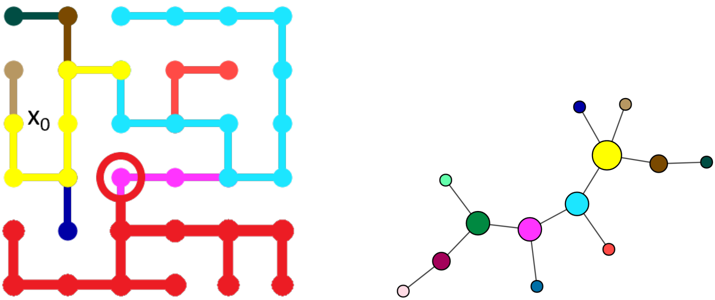
This is the same as the the "degree" of the piece of a random vertex x in H. We choose x with a bias $|P(x)|^{-1}$ where $P(x)$ is its piece.

Degree exponent, rough sketch of proof





Lemma 1: Degree of a piece is of the same magnitude as its length.



Lemma 2: Length of $P(x)$ is comparable to the diameter of the subtree of vertices in the UST that x separates from x_0 (the *past* of x).

So we need the distribution of this latter diameter, $\mathrm{diam}(\mathrm{past}_x)$.

Degree exponent, rough sketch of proof

Dimension 2:

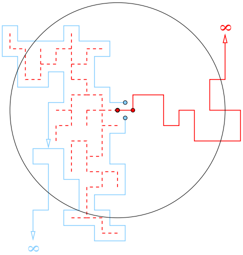
growth exponent is 5/4, Kenyon (2000), Barlow-Masson (2011)

Ignoring o(1) corrections in the exponents,

$$\mathbb{P}[\mathrm{diam}(\mathrm{past}_x) > r] \approx \mathbb{P}[\text{Eucl-diam}(\mathrm{past}_x)^{5/4} > r]$$

$$= \mathbb{P}[\text{Eucl-diam}(\mathrm{past}_x) > r^{4/5}] \approx (r^{4/5})^{-3/4} = r^{-3/5},$$

where the last $\approx$ is by a result of Masson about intersecting LERW and SRW:



Degree exponent, rough sketch of proof

Thus, for the uniformly selected point o of the *abstract network* we can summarize:

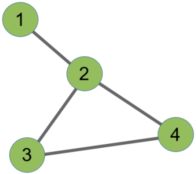
$$\mathbb{P}[\deg(o) = k] \approx \frac{1}{k}\mathbb{P}[|P(x)| = k] \approx \frac{1}{k}\mathbb{P}[\mathrm{diam}(\mathrm{past}_x) = k]$$

$$\approx \frac{1}{k} k^{-3/5-1} = k^{-2.6}.$$

Dimension ≥ 5:

Follows from the known distribution of $\mathrm{Eucl\text{-}diam}(\mathrm{past}_x)$, Bhupatiraju-Hanson-Járai (2017), and the concentration of $\mathrm{diam}(\mathrm{past}_x)$ around $\mathrm{Eucl\text{-}diam}(\mathrm{past}_x)^2$, by Lawler (1980).

$\mathcal{G}$

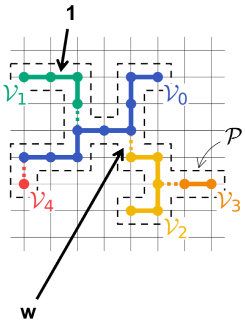


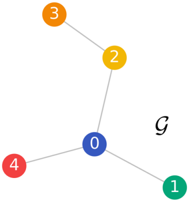
$$\mathbf{Q}_{\mathcal{G}} = \mathbf{D} - \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Spectrum: $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

Trivial eigenvector $\frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Physical Laplacian



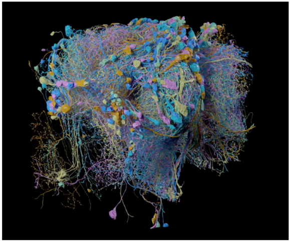


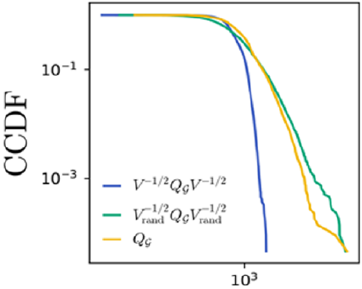
- Traditional approach of network science: look at $\mathbf{Q}_{\mathcal{G}}$
- What is the role of the physicality layout?

- Setup:
  - Coupling strength inside a physical node = 1
  - Between nodes = w
  - Small w limit

Fruitfly brain

- Shape of ~25,000 neurons + location of synapses
- Data published on several levels
  - Raw images: >20 TB
  - Skeletonized neurons: >117 million segments
  - **Connectome adjacency matrix + neuron volumes**





Open problems

1. Show $\gamma = 3$ degree distribution for random ray model in 2d.
2. Understand how exactly the Physical Laplacian feels physicality.

   For instance, prove that the spectrum does *not* have a fat tail.

Thank you for your attention!