Random determinants and the Elastic Manifold

Joint work with Paul Bourgade and Ben McKenna (Courant)

Gérard BEN AROUS

Courant Institute, New York University

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1. What is this talk about?

- I will report on recent joint works (arXiv 2105.05000, and arXiv 2105.05051) with Paul Bourgade and Ben McKenna, which constitute a part of Ben's PhD, defended in May 2021 at Courant (with his related work arXiv 2105.05043 on bipartite spin glasses).

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- We compute this topological complexity using the Kac-Rice formula, which brings us into the world of Random Matrix Theory.
- What RMT is contributing here is to give the (exponential) asymptotics of non-invariant determinants. We do this in arXiv 2105.05000 in rather broad settings. (Ben will speak about this is the second part of the OWPS seminar on Thursday)

2. A map of the talk

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1. Random determinants in the exponential scale
2. Complexity of random functions of many variables and Kac-Rice formula
3. The role of isotropy for the complexity of random functions of many variables
4. Disordered Elastic Media, or The Elastic manifold
5. A summary of our results on complexity for the Elastic Manifold

1. Random determinants in the exponential scale

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- Consider a large random real symmetric NxN matrix H_N. How can one compute the asymptotics of its determinant in the exponential scale?
- Our "annealed" question is to understand

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}(|\det H_N|) \tag{1}$$

2. Random determinants have a long history

- Exact formulas for small moments $\mathbb{E}[\det(H_N)^k]$ at finite N: Fortet '51, Forsythe-Tukey '52, Nyquist-Rice-Riordan '54, Prékopa '67, Dembo '89...
- Gaussian fluctuations: Goodman '63, Delannay-Le Caër '00, Tao-Vu '12, Nguyen-Vu '14, Bao-Pan-Zhou '15, Bourgade-Mody '19, Bourgade-Mody-Pain '21...
- Singularity probability for Bernoulli or other discrete matrices: Komlós '67, Kahn-Komlós-Szemerédi '95, Tao-Vu '06 + '07, Bourgain-Vu-Wood '10, K.Tikhomirov '20...

3. Why our question should be easy

- Let us look naively at our question. Obviously

$$\mathbb{E}[|\det(H_N)|] = \mathbb{E}[e^{N\psi(\hat{\mu}_{H_N})}]$$

where $\hat{\mu}_{H_N}$ is the empirical spectral measure

$$\hat{\mu}_{H_N} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(H_N)}$$

and

$$\psi(\mu) = \int \log|\lambda| d\mu(\lambda)$$

- Thus, if $\hat{\mu}_{H_N}$ converges to a deterministic limit μ_∞ and concentrates "fast enough", then it is temping to believe that we would get

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- There is a strong literature on concentration for the spectral measure, for instance to the semi-circle in the Wigner case. For instance Guionnet-Zeitouni 2000, Bobkov-Gotze-A.Tikhomirov in 2010, or Bordenave-Caputo-Chafai in 2011.

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- Also, what about the singularity of the logarithm at $0/\infty$? Can it be tamed easily??

5. A general Talagrand-type result

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- Consider the case where our random matrix is a nice function of independent random variables $H_N = \Phi(X_1, ..., X_M)$
- Theorem: Under the assumptions (I), (M), (E), (C), (S) , we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[|\det(H_N)|] = \int \log|\lambda| \mu_\infty(d\lambda)$$

where

- (I) $(X_1, ..., X_M)$ are independent, real r.v.'s
- (M) Φ is Lipschitz and pulls convex sets back to convex sets.
- (E) μ_∞ is a nice measure such that $d(\mathbb{E}[\hat{\mu}_{H_N}], \mu_\infty) \leq N^{-\epsilon}$.
- (C) Very coarse bounds on very large and small eigenvalues (scale $e^{\pm N^\epsilon}$).
- (S) The spectrum is stable when the X_i's are truncated at scale $N^{-\epsilon}/|\Phi|_{Lip}$.

6. Sketch of proof

- To prove this theorem, simply go back to basics and use Talagrand's results on concentration for product measures.
- Indeed, since Φ is Lipschitz and pulls convex sets back to convex sets, the log-potential is almost a Lipschitz, convex function of some bounded, independent random variables, namely the truncated X_i's.
- The benefit of writing in this generality is that it allows us to cover a wide variety of random-matrix models, in interesting cases (not linked to our problem today): Wigner, sample covariance, Erdos-Renyi, 1D band matrices etc...

7. Applying this theorem

- Theorem (BA-Bourgade-McKenna. '21)

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[|\det(H_N - E)|] = \int \log|\lambda - E|\mu_\infty(d\lambda)$$

when $E \in \mathbb{R}$ and H_N is one of the following:

1. a Wigner matrix with (near-optimal) $2+\epsilon$ finite moments
2. a sample covariance matrix with (near-optimal) $2+\epsilon$ finite moments
3. an Erdos-Renyi graph incidence matrix with parameter $p \geq N^\epsilon/N$
4. the free-addition model $A_N + O_N B_N O_N^T$ with O_N random orthogonal matrix (Haar)

- Of course the measures μ_∞ are the natural ones you can guess easily.

8. One example of application: the Wigner case

- This theorem applies to the Wigner case above. In this case $M = \frac{N(N+1)}{2}$ and the X_i's are the upper-triangular matrix entries, and Φ just copies the upper part below the diagonal.
- For Wigner with minimal moment assumptions, (E) is given by A.Tikhomirov 2009
- For Wigner, (C) is a consequence of Nguyen (2012), and (S) follows from arguments of Bordenave-Caputo-Chafai using Bennett's inequality.
- Note that our moment condition is almost optimal: If the distribution of the i.i.d entries has an infinite second moment, then actually $\mathbb{E}[|\det(H_N)|] = +\infty$ for every N.

9. A different approach

- The theorem above is based on product-measure-concentration results of Talagrand, and designed for minimal assumptions (i.e., Wigner matrices with $2 + \epsilon$ moments).
- We have another general theorem (with a different, and easier proof) for matrices H_N whose linear statistics are already known to concentrate at speed faster than N, for example under Gromov-Milman concentration on compact groups, or under a log-Sobolev inequality.
- This allows us to understand many other cases, for instance Gaussian matrices with a variance profile (or even with some correlations between the entries), or Gaussian block matrices.

10. The example we will need

- Theorem (BA-Bourgade-McKenna '21)

$$\lim_{N\to\infty}\left(\frac{1}{N}\log\mathbb{E}[|\det(H_N - E)|] - \int \log|\lambda - E|\mu_N(d\lambda)\right) = 0$$

when $E \in \mathbb{R}$ and H_N is

1. Gaussian with a (mean-field) (co)variance profile and/or a mean
2. Gaussian with good block structure (think $A_N + \begin{pmatrix} GOE & 0 \\ 0 & GOE \end{pmatrix}$)

- Here μ_N comes from the Matrix Dyson Equation, developed by Erdos and his collaborators: It is computed based only on information about H_N, so there is no a priori limit $\mu_N \to \mu_\infty$ (because no a priori connection between H_N and H_{N+1} in this generality), but in our applications we can find a limit μ_∞.

2. Complexity of random functions of many variables and Kac-Rice formula

11.The question in geometric terms

- Consider a smooth function $f(x)$ on a compact manifold M of large dimension N.
- Assume that every critical point is non-degenerate, i.e. that the function is a Morse function. Then the function has a finite number of critical points.
- We want to count the total number of critical points. We also want to count the number of critical points of given index. For instance the number of local minima (index=0). We may want also also fix a range for the critical values.
- In particular, in the context of statistical mechanics, f will be an energy (or a Hamiltonian), and we want to count the number of local minima of f, typically with low values.
- We might also be interested in the topology of the "sub-level" sets $A_u := \{x \in M, f(x) \leq u\}$, for instance through their Betti numbers.

12. Kac-Rice's formula

- Let $Crit_k(f)$ be the number of critical points $x \in M$ of the function f of index $k \in \{0...N\}$ and by $Crit_k(f; B)$ be the number of critical points $x \in M$ of the function f of index $0 \leq k \leq N$, such that $f(x) \in B$, for B a subset of the real line.

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- Assume that f is Gaussian and smooth enough then

$$E[Crit_k^f(B)] = \int_M a_k(x)\phi_x(0)dx. \tag{2}$$

where $\phi_x(u)$ is the density of the law of the gaussian vector $\nabla f(x)$ and

$$a_k(x) = E\left[|\det Hess(f)(x)| 1_{f(x)\in B}, 1_{i(x)=k} \middle| \nabla f(x) = 0\right] \tag{3}$$

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- The important step is then to understand the behavior of its determinant, conditioned by the fact that x is critical i.e. $\nabla f(x) = 0$.
- One class of examples where this task is understood is when the Gaussian distribution of the function is isotropic (or more recently with isotropic increments)

3. The role of isotropy for the complexity of random functions of many variables

14. The natural models of Isotropic Gaussian Smooth Functions

- If M is a Riemannian manifold, consider the centered Gaussian process $f(x, \omega)$, indexed by $x \in M$, such that the covariance is a function of the distance.

$$cov(f(x), f(y)) = g(d_M(x, y)) \tag{4}$$

- Obviously, the function $g(d_M(x, y))$ has to be positive definite.
- The variance is constant: $Var(f(x) = g(0)$.
- wlog we assume that $Var(f(x) = g(0) = 1$
- The metric induced by the Gaussian process is topologically equivalent to the Riemannian metric.

$$E[(f(x) - f(y))^2] = 2(1 - g)(d_M(x, y)) \tag{5}$$

15. Isotropic Gaussian Functions on the Sphere

- If the manifold M is the unit sphere S^{N-1}, the functions g such that $g(d_M(x, y))$ is positive definite have been characterized by Schoenberg in 1942.
- If we assume that g is independent of the dimension N, then there exists a sequence $a_p \geq 0$ such that

$$g(d) = \sum_{p=1}^{\infty} a_p (\cos d)^p \tag{6}$$

- Another way to write this is to introduce the function $\nu(r) = \sum_{p=1}^{\infty} a_p r^p$ and

$$cov(f(x), f(y)) = \nu(< x, y >) \tag{7}$$

16. Isotropic Gaussian Functions on the Sphere

- In fact this class of examples are exactly the class of mixed spherical spin glasses, where $x \in S_{N-1}(\sqrt{N})$

$$H(x) = \sum_{p=2}^{\infty} \sqrt{a_p} H_p(x) \tag{8}$$

and $H_p(x)$ is the random homogeneous polynomial

$$H_p(x) = \sum_{i_1 \ldots i_p = 1}^{N} J_{i_1 \ldots i_p} x_{i_1} \ldots x_{i_p} \tag{9}$$

where the J's are i.i.d N(0,1)

17. Isotropic Gaussian Functions on the Sphere

- There is now a large literature on the complexity of these models and on the behavior of the Gibbs measure at low temperature. (Fyodorov et al, Auffinger-BA-Cerny, Subag, Subag-Zeitouni, Auffinger-Gold, BA-Jagganath...)

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- The basic first step is that in this case the random matrix to be studied is a simple modification of the GOE! (with a shift by a scalar matrix). So the whole arsenal of RMT is accessible, including Large Deviations!

18. Isotropic Gaussian Functions on $\mathbb{R}^N$

- Consider an isotropic centered Gaussian function $V_N(u)$ on $\mathbb{R}^N$

$$\mathbb{E}[V_N(u)V_N(v)] = NB(\frac{1}{N}||u-v||^2) \tag{10}$$

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- Note also that the full classification of covariances of isotropic processes on $\mathbb{R}^N$ goes back to Schonberg 1938 (after Bochner, see also Yaglom in 1957). The function B has to be of the form

$$B(r) = c_0 + \int_0^\infty e^{-t^2 r} d\nu(t) \tag{11}$$

  where $c_0 \geq 0$, and ν is a finite positive measure.
- We will always assume that $B(0)$, $B'(0)$, $B''(0)$ are non zero to avoid degeneracies

19. Isotropic Gaussian Functions on $\mathbb{R}^N$

- Now consider the random function above in a harmonic well, in order to confine it
$$H_N(u) = \frac{\mu}{2}||u||^2 + V_N(u) \tag{12}$$

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- Here the Random Matrix is also a very simple modification of the GOE.

20. Soft Spin Glass in an anisotropic well

- Now consider the random function above in an anisotropic well

$$H_n(u) = \frac{\mu}{2} < u, D_N u > + V_N(u) \tag{13}$$

where D_N is a real symmetric positive matrix

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- This is a "soft spin-glass in an anisotropic well". We will see that this is of real importance to understand the Elastic Manifold in d=0!! One can ask the same questions about the topological complexity of this model.
- Here the Random Matrix is obviously the Hessian of V_N which is isotropic, so related to the GOE, plus the matrix D_N. We are thus entering the realm of free convolution!

21. Soft Spin Glass in an anisotropic well

- Assuming that the eigenvalues of D_N are bounded away from 0 and ∞ and that the spectral measure of D_N converges say to μ_D, we can compute the total annealed complexity. Let $\mathcal{N}$ be the total number of critical points, then
- Theorem (BA-Bourgade-McKenna):

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\mathcal{N}] = \Sigma_{tot}(\mu_D, B''(0)) \tag{14}$$

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22. Soft Spin Glass in an anisotropic well

- The complexity function is explicitly given by a variational principle.

$$\Sigma_{tot}(\mu_D, b) = -\int \log \lambda \mu_D(d\lambda) \tag{15}$$

$$+ \sup_{u\in\mathbb{R}} \left(\int \log|\lambda - u| \mu_D \boxplus \sigma_b(d\lambda) - \frac{u^2}{2b} \right) \tag{16}$$

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- And the transition can also be understood when $b \to b_c$ from above

$$\Sigma_{tot}(\mu_D, b) = c_{tot}(b - b_c)^2 + O(b - b_c)^3 \tag{17}$$

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- The same can be done for the complexity of minima (if D_N has no outliers). The topological transition is at the same value but is now cubic.
- Understanding this variational principle is a delicate step using Burger's equation for the semi-circle and a recent inequality by Guionnet-Maida (2020) for "free convolution at the edge".

4. Disordered Elastic Media, or The Elastic manifold

24. The Model

"Many seemingly different systems ranging from magnets to superconductors, with extremely different microscopic physics share the same essential ingredients, and can be described under the unifying concept of disordered elastic media. In all these systems an internal elastic structure, such as an interface between regions of opposite magnetizations in the magnetic systems, is subjected to the effects of disorder existing in the material. A specially interesting feature of all these systems is that these disordered elastic structures can be set in motion by applying an external force on them, and that the motion will be drastically affected by the presence of disorder. What properties result from the competition between elasticity and disorder is an extremely complicated problem which constitutes the essence of the physics of disordered elastic media."
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- One could ask (very classically) about the minimization problem: find the u's minimizing $H(u)$? (under a decent boundary condition)

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$$H(u) = \int_\Omega ||\nabla u||^2(x)dx + \int_\Omega V(x, u(x))dx \tag{18}$$

where V is a smooth potential on $\Omega x \mathbb{R}^N$

- One could ask (very classically) about the minimization problem: find the u's minimizing $H(u)$? (under a decent boundary condition)
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- One typically start with a "confining" potential like the harmonic potential, and then add disorder, in the form of a random potential depending on the position x and on the value of the field $u(x)$

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- So that we get

$$H(u) = \int_\Omega ||\nabla u||^2(x)dx + \int_\Omega ||u(x)||^2 dx + \int_\Omega V_N(x, u(x))dx \tag{20}$$

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$$H(u) = \sum_{x,y\in[1,L]^d} 1_{x\sim y}||u(x) - u(y)||^2 + \sum_{x\in[1,L]^d} ||u(x)||^2 + V_N(x, u(x)) \tag{22}$$

where V_N is an isotropic smooth Gaussian centered function on $\mathbb{R}^N$ with covariance

$$\mathbb{E}[V_N(x, u)V_N(y, v)] = \delta_{x,y} NB(\frac{1}{N}||x - y||^2) \tag{23}$$

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- Note that, when $L = 1$ (or $d = 0$), we have only one site and no interaction. The model is now back to the soft spin glass in a harmonic potential mentioned above! The general model is thus a system of L^d such disordered models with an "elastic" interaction.

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- More generally this model covers many models of random interfaces when $N = d + 1$ and $L \to \infty$
- We will not look at these cases but rather at another important limit, i.e. when d and L are fixed and $N \to \infty$.
- This problem was studied massively in physics back to Fischer 1986, Mezard- Parisi (1991 and 1992), Le Doussal-Mueller-Wiese 2007, and more recently Fyodorov-Le Doussal (2020) for a result that started this work.

- Summarizing, we look at the Hamiltonian

$$H(u) = a \sum_{x,y \in [1,L]^d} 1_{x \sim y} ||u(x) - u(y)||^2 + b \sum_{x \in [1,L]^d} ||u(x)||^2 + V_N(x, u(x)) \tag{26}$$

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where a, and b are two free parameters, and we impose the periodic boundary condition.

- This Hamiltonian can also be written as

$$H(u) = \sum_{x,y \in [1,L]^d} (\mu_0 Id - t_0 \Delta)_{x,y} < u(x), u(y) > + \sum_{x \in [1,L]^d} V_N(x, u(x)) \tag{27}$$

32. The Model, as we study it

- Here μ_0 and t_0 are two free parameters (mass and elasticity constants), and Δ is the periodic lattice Laplacian :

$$\Delta_{x,y} = \delta_{x\sim y} - 2d\delta_{x=y} \tag{28}$$

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- And as before V_N is a centered smooth Gaussian field with

$$\mathbb{E}[V_N(x,u)V_N(y,v)] = \delta_{x,y} NB(\frac{1}{N}||u-v||^2) \tag{29}$$

- We assume that B is 4 times differentiable, which ensures that V_N is C^2, and that $B^{(i)}(0) \neq 0$ for $i = 0,1,2$ to avoid degeneracies.

5. A summary of our results on complexity for the Elastic Manifold

33. What we do

1. We compute the annealed topological complexity of this Hamiltonian (in the limit $N \to \infty$, and d and L are fixed). That is: we compute the logarithmic behavior of the number of critical points and of local minima of this Hamiltonian.

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3. We thus show a sharp transition between a region of positive exponential complexity and a region of vanishing complexity, i.e. a form of topological trivialization at high enough mass.
4. We understand the transition at the critical "Larkin" mass
These results confirm fully the recent work by Fyodorov and Le Doussal (2020).

34. What we have not done (yet?)

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- Even if we believe our physicist friends, we do not see yet how this transition might be used to prove a transition for pinning and de-pinning for this random manifold.
- And even less how de-pinning would happen dynamically at high enough force.

5. Stating our results

35. Annealed Complexity

- Let $\mathcal{N}_{tot}$ the (random) number of all critical points of the Elastic Manifold Hamiltonian $H(u)$, then
- Theorem (BA-Bourgade, McKenna 2021) The annealed total complexity is given by

$$\lim_{N\to\infty} \frac{1}{NL^d} \log \mathbb{E}[\mathcal{N}_{tot}] = \Sigma(\mu_0, t_0, 4B''(0)) \tag{30}$$

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$$\lim_{N\to\infty} \frac{1}{NL^d} \log \mathbb{E}[\mathcal{N}_{tot}] = \Sigma(\mu_0, t_0, 4B''(0)) \tag{30}$$

- Similarly let $\mathcal{N}_m$ the (random) number of all local minima of the Elastic Manifold Hamiltonian

$$\lim_{N\to\infty} \frac{1}{NL^d} \log \mathbb{E}[\mathcal{N}_{min}] = \Sigma_{min}(\mu_0, t_0, 4B''(0)) \tag{31}$$

where the functions Σ and Σ_{min} are explicit.

36. The complexity functions Σ and Σ_{min}

- We give here an explicit formula for Σ and Σ_{min}.

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- Consider the simple (and deterministic) real-symmetric matrix of size L^d given by

$$D(\mu_0, t_0) = \mu_0 Id - t_0 \Delta \tag{32}$$

and its spectral measure

$$\mu(t_0, \mu_0) = \frac{1}{L^d} \sum_{i=1}^{L^d} \delta_{\lambda_i} \tag{33}$$

and finally denote by σ_b the semi-circle measure of radius $2\sqrt{b} > 0$

37. The complexity functions Σ and Σ_{min}

- Then we have the variational formulae

$$\Sigma(\mu_0, t_0, b) = -\frac{1}{L^d} \log\det D(\mu_0, t_0) + \tag{34}$$

$$\sup_{u\in\mathbb{R}}\left(\int \log|\lambda - u|(\sigma_b \boxplus \mu(t_0, \mu_0))(d\lambda) - \frac{u^2}{2b}\right) \tag{35}$$

and

$$\Sigma_{min}(\mu_0, t_0, b) = -\frac{1}{L^d} \log\det D(\mu_0, t_0) + \tag{36}$$

$$\sup_{u\leq\ell}\left(\int \log|\lambda - u|(\sigma_b \boxplus \mu(t_0, \mu_0))(d\lambda) - \frac{u^2}{2b}\right) \tag{37}$$

where $\ell = \ell(t_0, \mu_0)$ is the left end of the support of the free convolution $\sigma_b \boxplus \mu(t_0, \mu_0)$.

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- For t_0 and b given, define the "Larkin mass" as the unique solution $\mu_c = \mu_c(t_0, b)$ of

$$\int \frac{1}{(\mu_c + \lambda)^2} \hat{\mu}_{-t_0\Delta}(d\lambda) = \frac{1}{b} \tag{38}$$

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- Then, when $\mu \geq \mu_c$, both the total and the minima complexities $\Sigma(\mu_0, t_0, b)$ and $\Sigma_{min}(\mu_0, t_0, b)$ vanish! i.e. " a large enough mass kills the exponential complexity of the Landscape " !!

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- We could also phrase this by saying that the complexities vanish when the noise level $b = 4B''(0)$ is lower than the critical noise level

$$b_c = b_c(t_0, \mu_0) = \left(\int \frac{1}{(\mu_0 + \lambda)^2} \hat{\mu}_{-t_0\Delta}(d\lambda)\right)^{-1} \tag{39}$$

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- Indeed, the supremum in the formula giving the total complexity is achieved at an explicit $v \in \mathbb{R}$
- The supremum in the formula giving the minima complexity is achieved at $\ell(t_0, \mu_0)$, i.e. the left end of the support of the free convolution $\sigma_b \boxplus \mu(t_0, \mu_0)$

40. The phase transition at the Larkin mass or at the critical noise level

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- We express it here in terms of the noise level $b = 4B''(0)$.
- When b approaches the critical level b_c from above, the total annealed complexity vanishes quadratically, and the minima annealed complexity vanishes cubically as a function of the noise level

$$\Sigma(\mu_0, t_0, b) = c_{tot}(b - b_c)^2 + O((b - b_c)^3) \tag{40}$$

$$\Sigma_{min}(\mu_0, t_0, b) = c_{min}(b - b_c)^3 + O((b - b_c)^4) \tag{41}$$

41. The proof in a nutshell

1. Apply Kac-Rice (easy, since our function is Gaussian and smooth)
2. Compute the distribution of the Hessian at a point conditioned by the fact that the point is critical (easy, since our function is Gaussian).
3. Apply the result of our companion paper on the random determinant of this matrix (this is the case of a block structured Gaussian matrix).
4. Apply Laplace's formula, and get a (very heavy) variational formula
5. Simplify the variational problem to the one mentioned above, through a miracle (an unexpected convexity)
6. Recognize this variational problem as related to the problem in $d = 0$, i.e,. for one point. This problem is a "spin glass" type model: a soft spin in an anisotropic random potential due to the discrete Laplacian.
7. Use our understanding of this spin glass problem, as mentioned above, to deduce the results about the topological complexity and the topological transition for the Elastic manifold.