Sparse universal graphs for planarity

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Joint work with Vida Dujmović, Louis Esperet, Cyril Gavoille, Piotr Micek, and Pat Morin

Graph G is **universal** for a set $\mathcal{F}$ of graphs if G contains every member of $\mathcal{F}$ as a subgraph

Fix some class $\mathcal{C}$ of graphs, e.g.

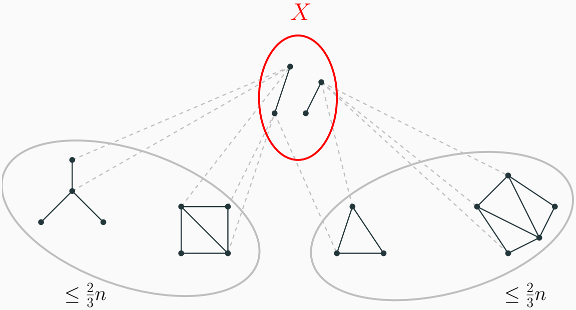
- trees
- graphs of maximum degree Δ
- planar graphs
- $\dots$

Want: universal graph $G(n)$ for n-vertex graphs in $\mathcal{C}$ having as few edges as possible

What is the minimum number of edges in an universal graph for n-vertex planar graphs?

Tool: Separators

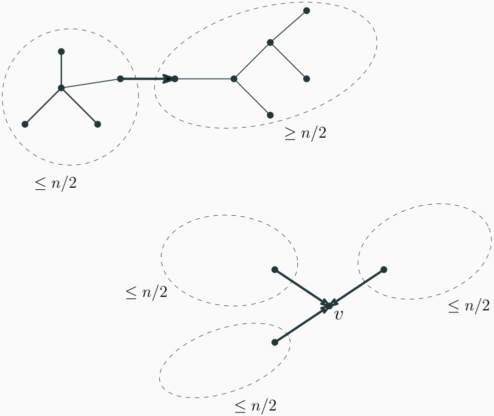
Separator of n-vertex graph G: vertex subset X s.t. components of $G - X$ can be grouped into 2 parts of size $\leq 2n/3$



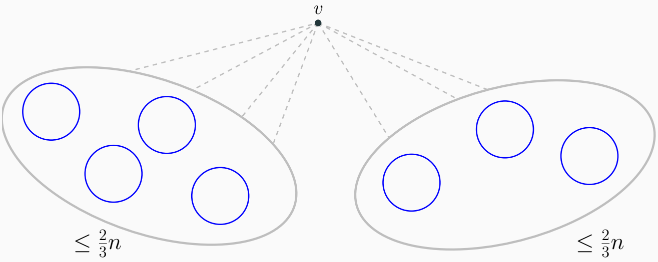
Universal graphs for trees

T n-vertex tree

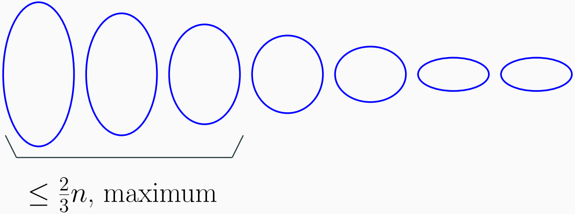
Fact: $\exists$ vertex v which is a separator of T



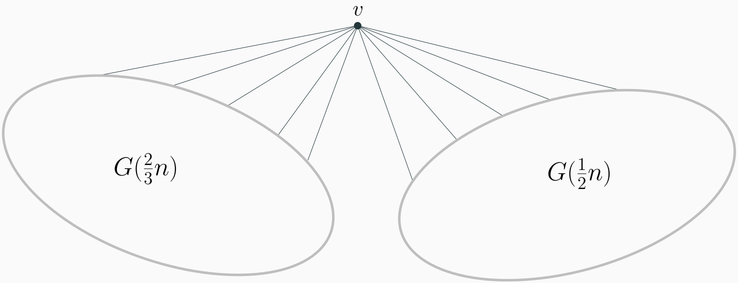
Can group components of $T - v$ into 2 parts of size $\leq 2n/3$:



Why?



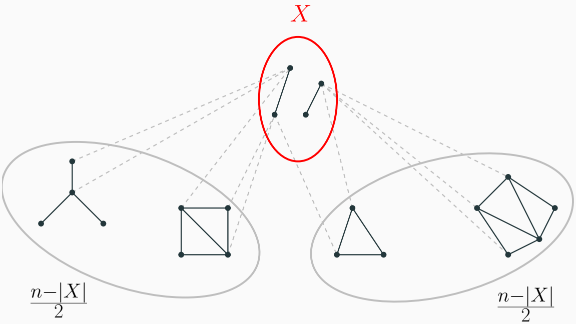
Universal graph $G(n)$



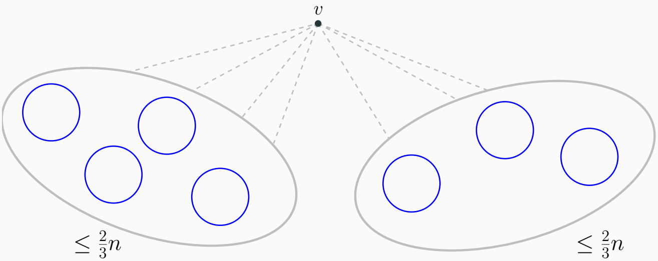
$|E(G(n))| = O(n^{1.3})$

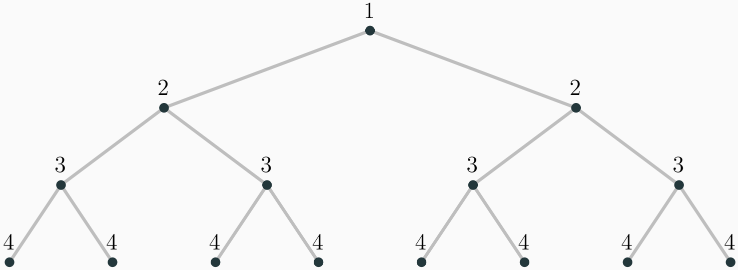
Better tool: Perfectly balanced separators

Perfectly balanced separator of graph G: vertex subset X s.t. components of $G - X$ can be grouped into 2 parts of *equal size*



Fact: Every n-vertex tree T has a perfectly balanced separator of size $O(\log n)$


v
$\leq \frac{2}{3}n$
$\leq \frac{2}{3}n$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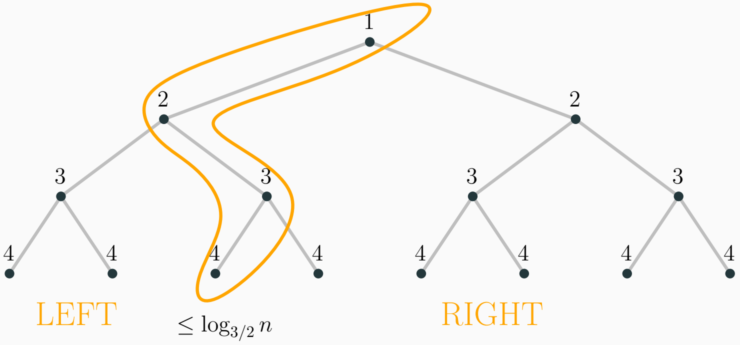
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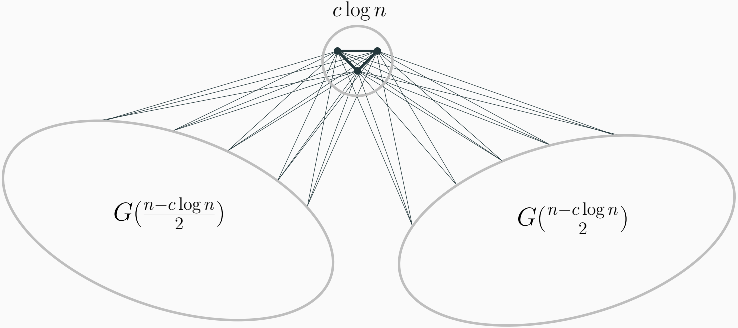
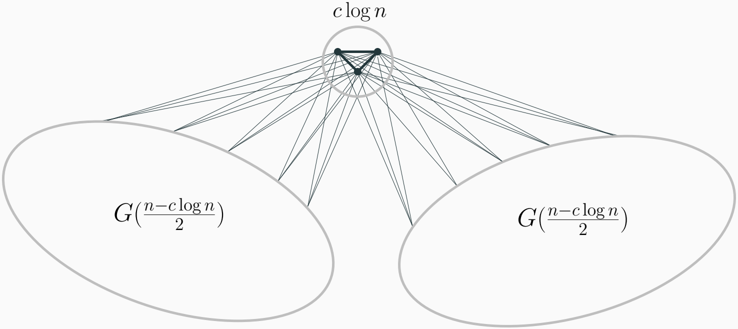
LEFT
$\leq \log_{3/2} n$
RIGHT

Improved universal graph $G(n)$



$|E(G(n))| = O(n \log^2 n)$

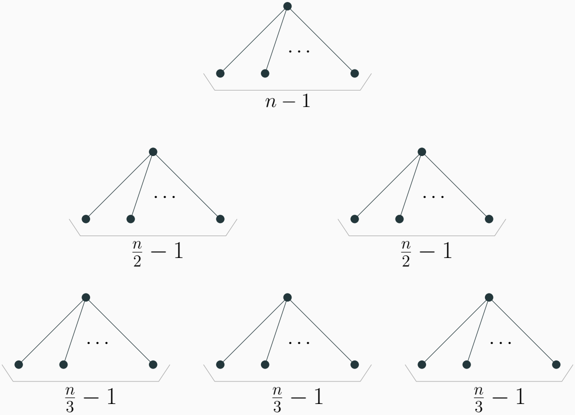
Improved universal graph $G(n)$



$|E(G(n))| = O(n \log^2 n)$

Theorem (Chung and Graham 1983) Universal graphs for n-vertex trees with $O(n \log n)$ edges

Lower bound for trees



Degree sequence of universal graph, sorted in non-increasing order, must dominate $(n-1, \frac{n}{2}-1, \frac{n}{3}-1, \frac{n}{4}-1, \dots)$

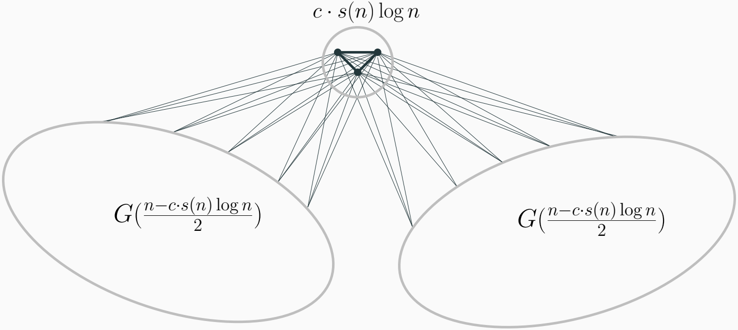
$\rightarrow \Omega(n \log n)$ edges

Class $\mathcal{C}$ of graphs, closed under subgraphs

Say every n-vertex graph $G \in \mathcal{C}$ has a separator of size $\leq s(n)$

$\Rightarrow$ perfectly balanced separator of size $O(s(n) \cdot \log n)$

Universal graph $G(n)$ for n-vertex graphs in $\mathcal{C}$:

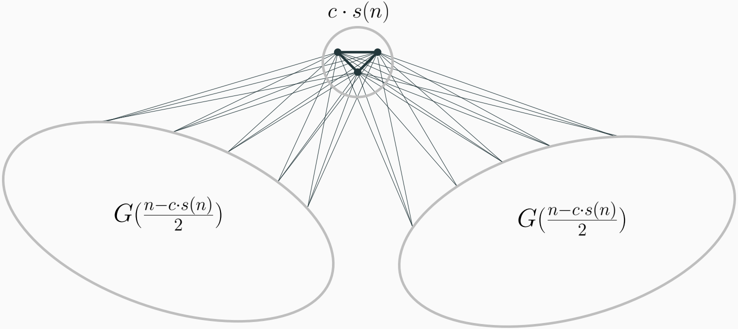


$O(s(n) \cdot n \log^2 n)$ edges

Remark: If $s(n) = n^\alpha$ with $0 < \alpha < 1$ then:

Separator of size $s(n) \Rightarrow$ perfectly balanced separator of size $O(s(n))$

Universal graph $G(n)$ for n-vertex graphs in $\mathcal{C}$:



$O(s(n) \cdot n)$ edges

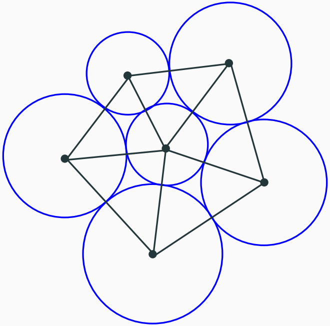
Theorem (Lipton & Tarjan 1979) Planar graphs have $O(\sqrt{n})$-size separators

$\Rightarrow O(\sqrt{n})$-size perfectly balanced separator

Theorem (Babai, Erdős, Chung, Graham, Spencer 1982) Universal graphs with $O(n^{3/2})$ edges for n-vertex planar graphs

Generalizations of planar graphs: 1. Higher dimension

Koebe's representation of planar graphs:

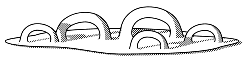


Theorem (Miller, Teng, Thurston, Vavasis '97) Intersection graphs of touching balls in $\mathbb{R}^d$ have $O(n^{1-1/d})$-size separators

$\rightarrow$ universal graphs with $O(n^{2-1/d})$ edges

Generalizations of planar graphs: 2. Excluding a minor

Graphs embedded in a fixed surface:



Theorem (Gilbert, Hutchinson, Tarjan 1984) $O(\sqrt{gn})$-size separators for graphs embedded in a surface of Euler genus g

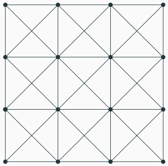
$\rightarrow$ universal graphs with $O(\sqrt{g} \cdot n^{3/2})$ edges

Theorem (Alon, Seymour, Thomas 1990) $O(h^{3/2}\sqrt{n})$-size separators for graphs excluding a minor H on h vertices

$\rightarrow$ universal graphs with $O(h^{3/2} \cdot n^{3/2})$ edges

Generalizations of planar graphs: 3. Allowing crossings

k-**planar graphs**: at most k crossings per edge



Theorem (Dujmović, Eppstein, Wood 2016) $O(\sqrt{kn})$-size separators for k-planar graphs

$\rightarrow$ universal graphs with $O(\sqrt{k} \cdot n^{3/2})$ edges

Back to planar graphs

Our main result:

Theorem (Esperet, J., Morin 2020) Universal graphs with $O(n^{1+o(1)})$ edges for n-vertex planar graphs

Remark: Proof builds on earlier work of Dujmović, Esperet, J., Gavoille, Micek, and Morin ('20) about induced-universal graphs for planar graphs

Treewidth

Measure of similarity with a tree (the lower the better)

Treewidth $k \Rightarrow \exists$ separator S of size $\leq k + 1$

Theorem (Dvořák & Norin '19) All subgraphs have separators of size $\leq k \Rightarrow$ treewidth $\leq 15k$

small treewidth $\Leftrightarrow$ **small separators**

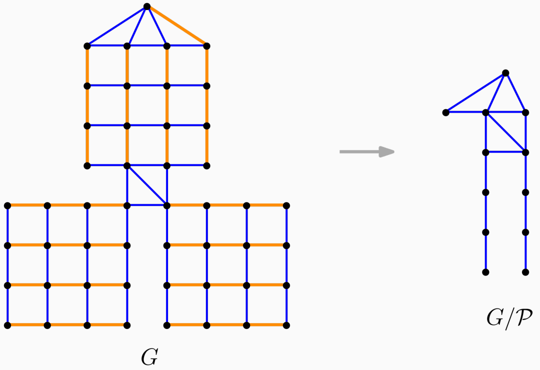
Many graph-theoretic problems become easier if the graph has bounded treewidth

A new way of decomposing planar graphs

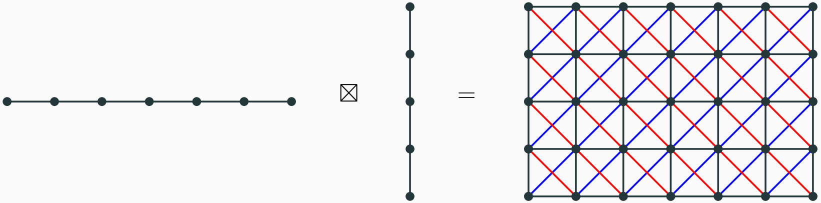
> **Theorem** (Mi. Pilipczuk & Siebertz '18) Every planar graph G has a vertex partition $\mathcal{P}$ into geodesics such that $G/\mathcal{P}$ has treewidth ≤ 8

geodesic = shortest path

$G/\mathcal{P}$ = graph obtained by contracting each path in $\mathcal{P}$ into a vertex

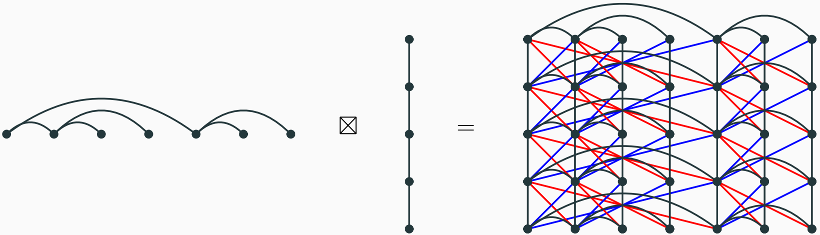


Strong product



Product structure of planar graphs

Theorem (Dujmović, J., Micek, Morin, Ueckerdt, Wood '19)
Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth ≤ 8 and some path P



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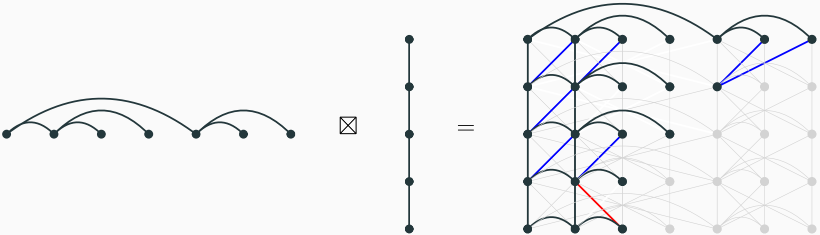
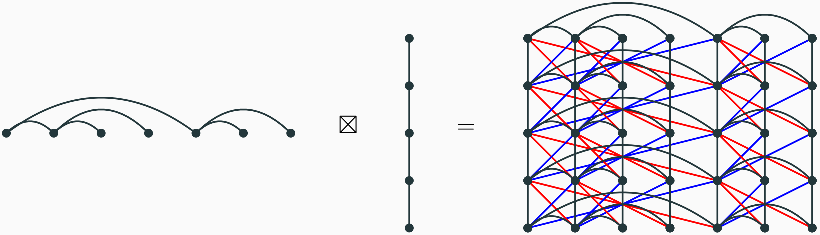


Main result

Theorem (Esperet, J., Morin '20) Universal graphs with $t^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges for n-vertex graphs that are subgraphs of $H \boxtimes P$ for some graph H with treewidth t and some path P

Cleaning up the problem



We may assume:

- H has n vertices
- P has n vertices

Subtlety: H not fixed, can be any n-vertex graph with treewidth t

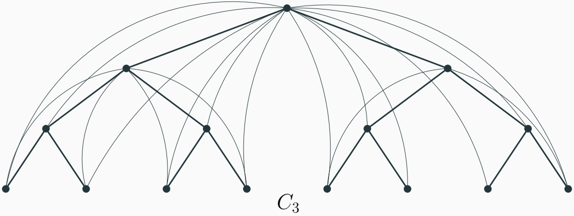
Solution: Replace H with universal graph for n-vertex graphs with treewidth t

Recall: Treewidth t implies

- separator of size $\leq t+1$
- perfectly balanced separator of size $t \cdot c \log n$
- universal graph with $O(t \cdot n \log^2 n)$ edges

Compact description of a universal graph: $\boxed{C_{\log n} \boxtimes K_\omega}$

- $C_d :=$ complete binary tree of height d + edges in transitive closure



- $\omega := t \cdot c \log n$

Main technical result

Wanted: universal graph for n-vertex subgraphs of $C_{\log n} \boxtimes K_\omega \boxtimes P_n$

Theorem (Esperet, J., Morin '20) Universal graph $G(n)$ with $n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges for n-vertex subgraphs of $C_{\log n} \boxtimes P_n$

This is enough:

- $G(n) \boxtimes K_\omega$ is universal for n-vertex subgraphs of $C_{\log n} \boxtimes P_n \boxtimes K_\omega = C_{\log n} \boxtimes K_\omega \boxtimes P_n$
- $G(n) \boxtimes K_\omega$ has
$$\omega^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})} = t^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$$
edges

Goal: Universal graph $G(n)$ with $n \cdot 2^{O(\sqrt{\log n \cdot \log\log n})}$ edges for n-vertex subgraphs of $C_{\log n} \boxtimes P_n$

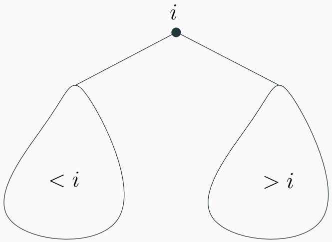
Vertices of $G(n)$: Triples

$$(x, y, z)$$

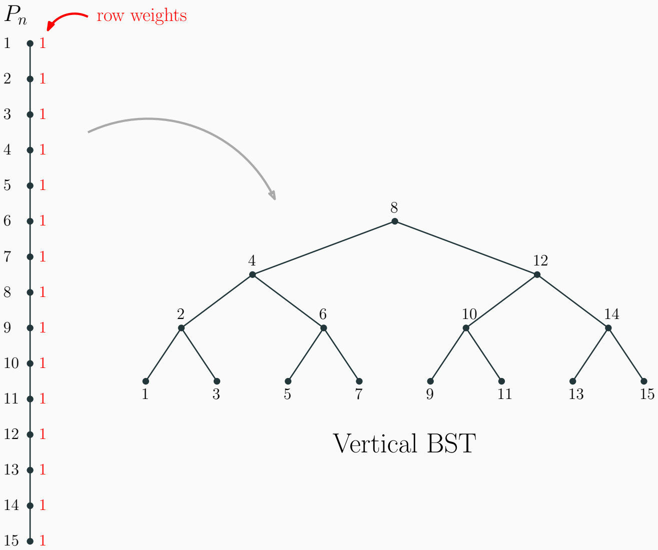
where

- x, y, z bitstrings
- $|x| + |y| \leq \log n + O(\sqrt{\log n \cdot \log\log n})$
- $|z| = \log\log n$
- x encodes position in **horizontal binary search tree** (one per row)
- y encodes position in **vertical binary search tree** (global)
- all binary search trees (almost) perfectly balanced

Binary search tree (BST)



Vertical BST: Stores rows $1, 2, \ldots, n$



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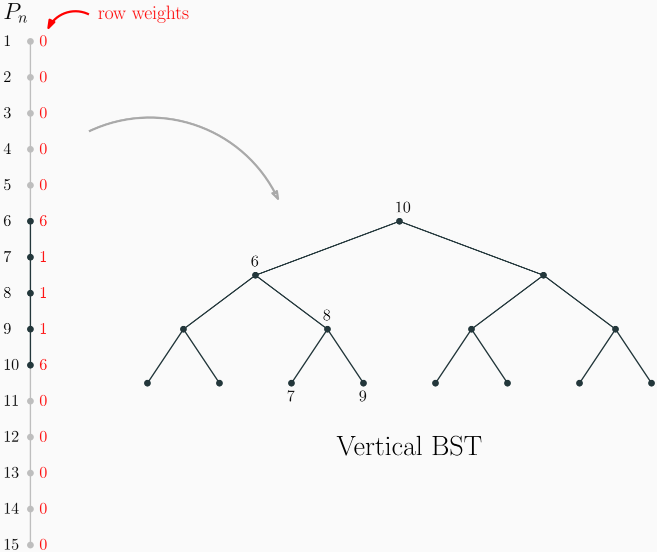
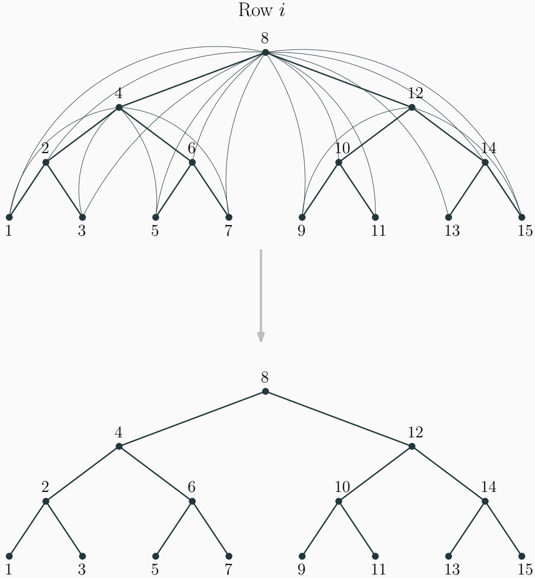
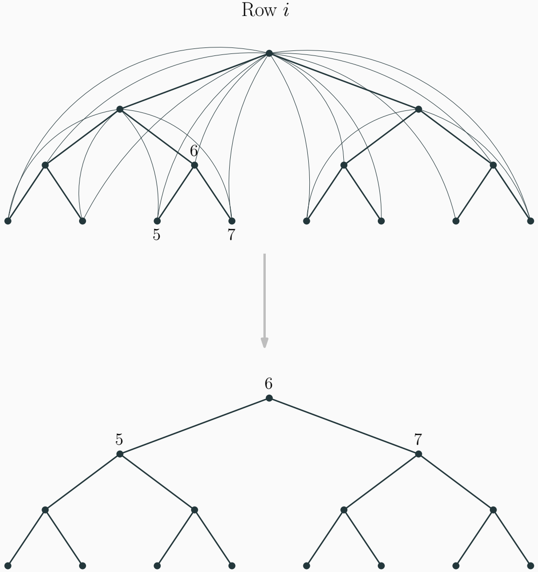


Horizontal BST for row i: Stores vertices of $C_{\log n}$ that are used



Horizontal BST for row i

Horizontal BST for row i: Stores vertices of $C_{\log n}$ that are used



Horizontal BST for row i

Engine of the proof: **Bulk tree sequences**

Horizontal BSTs are built sequentially starting with row 1

- Insertions
- Deletions
- Rebalancing

Height of i-th horizontal BST almost optimal:

$$\log r_i + O(\sqrt{\log n \cdot \log\log n})$$

if r_i vertices in row i

When rebalancing, a vertex moves to a new position among $O(\sqrt{\log n \cdot \log\log n})$ potential positions

Some consequences beyond planar graphs

Theorem (Dujmović, J., Micek, Morin, Ueckerdt, Wood '19) Every graph embeddable in a surface of Euler genus g is a subgraph of $H \boxtimes P$ for some graph H with treewidth $\leq 2g + 8$ and some path P

$\rightarrow$ universal graphs with $g^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges

Theorem (Dujmović, Morin, Wood '19) Every k-planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth $O(k^5)$ and some path P

$\rightarrow$ universal graphs with $k^{10} \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges

Induced-universal graphs

G is **induced-universal** for $\mathcal{F}$ if G contains every member of $\mathcal{F}$ as an *induced* subgraph

What is the minimum number of **vertices** in an induced-universal graph for n-vertex planar graphs?

A.k.a. **adjacency labeling schemes** for planar graphs

- $O(n^6)$ using 5-degenerate (Muller 1988)
- $O(n^{4+o(1)})$ using arboricity 3 (Kannan, Naor, Rudich 1988)
- $O(n^{2+o(1)})$ using vertex partition into two graphs of bounded treewidth (Gavoille & Labourel 2007)
- $O(n^{4/3+o(1)})$ using product structure (Bonamy, Gavoille, Pilipczuk 2019)

Theorem (Dujmović, Esperet, J., Gavoille, Micek, Morin '20)
Induced-universal graphs with $O(n^{1+o(1)})$ vertices for n-vertex planar graphs

Theorem (Dujmović, Esperet, J., Gavoille, Micek, Morin '20)
Induced-universal graphs with $n \cdot 2^{O(\sqrt{\log n \cdot \log\log n})}$ vertices for n-vertex graphs that are subgraphs of $H \boxtimes P$ for some graph H with treewidth $O(1)$ and some path P

Open problems

Universal graphs:

- Universal graphs for graphs excluding a fixed minor H? Best known bound on number of edges is still $O_H(n^{3/2})$
- Tight bound for planar graphs?
  - Upper bound: $n \cdot 2^{O(\sqrt{\log n \cdot \log\log n})}$ edges
  - Lower bound: $\Omega(n \log n)$ edges

Induced-universal graphs:

- Induced-universal graphs for graphs excluding a fixed minor H? Best known bound on number of vertices is still $O_H(n^{2+o(1)})$
- Tight bound for planar graphs?
  - Upper bound: $n \cdot 2^{O(\sqrt{\log n \cdot \log\log n})}$ vertices
  - Lower bound: $\Omega(n)$ vertices

THANK YOU!