Recent Progress in Ramsey Theory

Jacques Verstraete
University of California, San Diego
jacques@ucsd.edu
Outline

- Classical Ramsey Theory
- Random Graphs
- Pseudorandom Graphs
- Ramsey Numbers
- Random Blocks
- $r(4, t)$
- Erdős-Rogers Functions
- Open Problems
Classical Ramsey Theory

- In any sufficiently large “structure”, a relatively large “uniform” substructure exists. “Perfect disorder is mathematically impossible.”

- For integers $s, t \geq 2$, let $r(s,t)$ denote the minimum $n$ such that every red-blue edge-coloring of $K_n$ contains a red $K_s$ or a blue $K_t$.

- These are the classical Ramsey numbers.
Classical Ramsey Theory

- Example: \( r(2, t) = t \)
- Example: \( r(3, 3) = 6 \)

- The only other classical Ramsey numbers \( r(3, t) \) known are

  \[
  \begin{align*}
  r(3, 4) &= 9 & r(3, 5) &= 14 & r(3, 6) &= 18 \\
  r(3, 7) &= 23 & r(3, 8) &= 36 & r(3, 9) &= 39
  \end{align*}
  \]
Classical Ramsey Theory

- The only other known classical Ramsey numbers $r(3, t)$ for $t \geq 4$ are:

  $$
  r(3, 4) = 9 \quad r(3, 5) = 14 \quad r(3, 6) = 18 \\
  r(3, 7) = 23 \quad r(3, 8) = 36 \quad r(3, 9) = 39
  $$

- The only known classical Ramsey numbers $r(4, t)$ for $t \geq 4$ are:

  $$
  r(4, 4) = 18 \quad r(4, 5) = 25
  $$
Classical Ramsey Theory

\[ r(4, 4) = 18 \]

\[ r(4, 5) = 25 \]
Classical Ramsey Theory

• Erdős-Szekeres Theorem (1935)

For $s,t \geq 2$,

$$r(s,t) \leq r(s-1,t) + r(s,t-1) \leq \binom{s + t - 2}{s - 1} < t^{s-1}.$$
Classical Ramsey Theory

  
  For $s > 2$, as $t \to \infty$:

  $$r(s, t) \lesssim \frac{t^{s-1}}{(\log t)^{s-2}}.$$
Classical Ramsey Theory

- Maximum sets of points in the plane with no \( k \) in convex position.
- Permutations of \( n \) letters with no monotone subsequence of length \( k \).
- Unit distance graphs.
- Sets with no three-term arithmetic progressions.
- Embeddings of metric spaces with low distortion.
Classical Ramsey Theory

- Sets of points in the square with no triangles of small area.
- Grid points with no three on a line.
- Roth’s Theorem and arithmetic progressions of primes.
- Random graphs, percolation and cellular automata.
- Orchard planting problem.
Classical Ramsey Theory


  As $t \to \infty$:

  $$r(3, t) \gtrsim \frac{t^2}{4(\log t)}.$$
For $r(4, n)$, the best lower bound known is $c(n \log n)^{5/2}$ due to Spencer,\textsuperscript{33} again by using the Lovász local lemma. The best upper bound known is $c' n^3 / \log^2 n$, proved by Ajtai, Komlós and Szemerédi\textsuperscript{27}. So there is a nontrivial gap still remaining, as repeatedly pointed out in many problems papers \textsuperscript{34} of Erdős.

**Problem** (§250)
Prove or disprove that

$$r(4, n) > \frac{n^3}{\log^2 n}$$

for some $c$, provided $n$ is sufficiently large.

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\textsuperscript{31} P. Erdős, On the construction of certain graphs, *J. Comb. Theory* 17 (1966), 149-153


Classical Ramsey Theory

• Theorem (Mattheus-V, Ann. Math. 2024)

\[ r(4, t) \gtrsim \frac{t^3}{(64 \log t)^4}. \]

As \( t \to \infty \):

• The proof of this theorem illustrates the philosophy that good Ramsey graphs “hide” inside pseudorandom graphs.
Media

Quanta Magazine:

UC San Diego Today
• https://today.ucsd.edu/story/ramsey-problems

Carnegie Mellon MCS News
• https://www.cmu.edu/math/news-events/articles/20230712_random-algorithms-math-conference.html

SIAM News
• https://sinews.siam.org/Details-Page/off-diagonal-ramsey-numbers-from-pseudorandom-graphs

Belgian HLN Newspaper

The Brussels Times Newspaper:
Breakthrough in fiendishly hard puzzle has mathematicians partying

The key to a successful party is a good mix of people

fight off an alien invasion. Now mathematicians have made the first major advance in nearly a century
Random graphs

• Erdős and Rényi defined the Bernoulli or mean field model of random graphs $G_n$ where the edges of the complete $n$-vertex graph are sampled independently with probability $1/2$.

• An event $A_n$ occurs a.a.s (asymptotically almost surely) if

$$\Pr(A_n) \to 1 \quad \text{as} \quad n \to \infty.$$
Random graphs

- Let $A_n$ be the event that $G = G_n$ has no clique or independent set of size $t$.

- If $n > \sqrt{2}^t$

\[
\Pr(A_n) \leq 2 \cdot 2^{-\binom{t}{2}} \binom{n}{t} \\
\leq 2^{1 - \frac{1}{2} t(t-1)} \frac{n^t}{t!} \\
\leq 2^{1 - \frac{1}{2} t(t-1)} \frac{\sqrt{2}^t}{t!} < \frac{2^{1 + \frac{t}{2}}}{t!} \rightarrow 0.
\]
Random graphs

\[
\Pr(A_n) \leq \frac{2^{1+\frac{t}{2}}}{t!} \rightarrow 0.
\]
Random graphs

- Therefore there is a graph with $n > \sqrt{2^t}$ vertices and no clique or independent set of size $t$. 
Random graphs

• We can do better (deletion method): delete one vertex from each clique or independent set of size $t$ so that the average number of vertices left is at least

$$n - 2 \cdot 2^{t \choose 2} \cdot {n \choose t}$$

• Maximize this over $n$ to get an $N$-vertex graph with no clique or independent set of size $t$, where

$$N > \frac{t}{e} \cdot \sqrt{2^t}$$
Pseudorandom graphs

• Any graph \( G \) has an adjacency matrix

\[
A_{ij} = \begin{cases} 
1 & \text{if } ij \in E(G) \\
0 & \text{if } ij \notin E(G) 
\end{cases}
\]

• For example,
Pseudorandom graphs

- The adjacency matrix has eigenvalues that are real, since it is symmetric.

- In the case of random graphs, Wigner's semicircle law shows they look like:
Pseudorandom graphs

- If the eigenvalues are \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( G \) is \( d \)-regular and

\[
\lambda = \max \{|\lambda_i| : i > 1\}
\]

then \( G \) is \( \lambda \)-pseudorandom.

- **Alon-Boppana Theorem**

\[
\lambda > (2 - \frac{1}{\lceil \text{diam}(G)/2 \rceil}) \cdot \sqrt{d - 1}.
\]

- The infinite \( d \)-ary tree is the universal cover of \( d \)-regular graphs.
Pseudorandom graphs

• If $X$ is any set of in a $\lambda$-pseudorandom graph, then

$$2e(X) = \langle Ax, x \rangle$$

• If $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of eigenvectors let

$$x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$$
Pseudorandom graphs

• Then

\[ 2e(X) = \langle Ax, x \rangle = \sum_{i=1}^{n} \lambda_i x_i^2 \]

• Recalling \( \lambda_1 = d \)

\[ \left| 2e(X) - dx_1^2 \right| = \left| \sum_{i=2}^{n} \lambda_i x_i^2 \right| \]
Pseudorandom graphs

• Since the first eigenvector is constant,

\[ x_1 = \langle x, e_1 \rangle = \frac{|X|}{\sqrt{n}} \]

• Finally,

\[ 2e(X) - \frac{d}{n} |X|^2 \leq \lambda \left| \sum_{i=1}^{n} x_i^2 \right| \leq \lambda |X|. \]
Pseudorandom graphs

- If the eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $G$ is $d$-regular and

$$\lambda = \max\{|\lambda_i| : i \geq 1\}$$

then $G$ is $\lambda$-pseudorandom.
Pseudorandom graphs

- If the eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $G$ is $d$-regular and

$$\lambda = \max\{ |\lambda_i| : i > 1 \}$$

then $G$ is pseudorandom.
Pseudorandom graphs

- Let $n$ be a prime congruent to 1 mod 4.

- The vertex set of the **Paley graph** $P_n$ is $\{0, 1, 2, \ldots, n\}$.

- The edges are pairs $\{i, j\}$ such that $|i - j|$ is a **quadratic residue** mod $n$.

- Every edge is in $(n - 5)/4$ triangles and every pair of non-adjacent vertices has $(n - 1)/4$ common neighbors.
Pseudorandom graphs

• The vertex set of the Paley graph $P_{17}$ is $\{0,1,2,\ldots,17\}$.

• The edges are pairs $\{i,j\}$ such that $|i - j|$ is a quadratic residue mod 17.

• Paley graph $P_{17}$ is 8-regular and eigenvalues are

\[
8, \quad \frac{1+\sqrt{17}}{2}, \ldots, \frac{1+\sqrt{17}}{2}, \quad \frac{1-\sqrt{17}}{2}, \ldots, \frac{1-\sqrt{17}}{2}
\]

8 times \hspace{1cm} 8 times
Pseudorandom graphs

- Paley graph $P_{17}$ is 8-regular
Pseudorandom graphs

- Paley graphs
Pseudorandom graphs

• In a $d$-regular $n$-vertex graph, the number of independent sets of size $n/(d + 1)$ is at least $(d + 1)^t$.

• Alon and Rödl (2004) showed that one can do better in pseudorandom graphs.
Pseudorandom graphs

- Theorem (Mubayi-V, JEMS 2023)

\[ \text{The number of independent sets of size } t = n (\log n)^2 / d \text{ is at most} \]

\[ \left( \frac{4e^2 \lambda}{(\log n)^2} \right)^t \]
Pseudorandom graphs

- When $n = 2^k - 1$ the number of independent sets of size $t = kn/d$ in $P_n$ is at most

$$\left( \frac{4e^2 \lambda}{k} \right)^t = \left( \frac{2e^2 (1 + \sqrt{n})}{k} \right)^t$$

- Randomly sample vertices with probability

$$p = \frac{k}{2e^2 (1 + \sqrt{n})}$$
Pseudorandom graphs

- Let $X$ be the number of sampled vertices minus one vertex from each independent set of size $t = kn/d < 2k + 1$. Then

$$E(X) = pn - 1$$

- This gives a graph with no independent set of size $t$ and no clique of size $t$, where the number of vertices is

$$pn - 1 = \frac{kn}{2e^2(1 + \sqrt{n})} - 1 > \frac{k}{2e^2(\sqrt{n} - 2)} > \frac{t}{4e^2}$$
Ramsey Numbers

• In a triangle-free graph whose adjacency matrix has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

\[
tr(A^3) = \sum_{i=1}^{n} \lambda_i^3 = 0.
\]

• If the graph is $\lambda$-pseudorandom, then

\[
d^3 \geq (n-1)\lambda^3.
\]
Ramsey Numbers

- The Alon-Boppana Theorem shows

\[ \lambda \gtrsim 2\sqrt{d - 1}. \]

- We conclude an optimal pseudorandom triangle-free graph has

\[ d = \Omega(n^{\frac{2}{3}}) \quad \text{and} \quad \lambda = O(n^{\frac{1}{3}}). \]

- The first examples were constructed by Alon (1991) and later by Kopparty.
Ramsey Numbers

• Consider the **Cayley sum graph** with vertex set $\mathbb{F}_q^3$ and generators

$$S = \{(xy, xy^2, xy^3) : x \in A, y \in \mathbb{F}_q^*\}$$

where $A$ is a sum-free set closed under additive inverse.

• Eigenvalues are controlled by **character sums** / **Gauss sums**.
Ramsey Numbers

• Similarly, an optimal pseudorandom $K_s$-free graph has

$$d = \Omega(n^{1-\frac{1}{2s-3}}) \quad \text{and} \quad \lambda = O(d^2).$$

• Theorem (Mubayi-V, 2023)

If an optimal pseudorandom $K_s$-free graph exists, then

$$r(s, t) = \Omega\left(\frac{t^{s-1}}{\log^{2s-4} t}\right).$$
Random Blocks

• Let $G$ be a bipartite graph with parts $A$ and $B$ of sizes $n$ and $m$.

• The projection $G_\pi$ of $G$ onto is the graph consisting of edges between vertices of $A$ at distance two.

• This graph is a union of $m$ designated cliques, one for each vertex in $B$.

• If the graph is a point-line incidence graph, then the designated cliques are edge-disjoint.
Random Blocks

• In each designated clique of $G_{\pi}$, independently take a random complete bipartite graph to obtain a random graph $G_{\pi}^*$.

• We refer to this as a random block construction.
Random Blocks

\[ G \to G_\pi \to G^*_\pi \to G^{\circ}_\pi. \]
Random Blocks

- Suppose $G$ does not contain a 1-subdivision of $K_4$:

![Graph](image)

- Then every $K_4$ in $G^\pi$ contains a triangle in one of the designated cliques.

- Therefore $G^\pi$ is a random $K_8$-free graph.
$r(4, t) = \tilde{\Theta}(t^3)$

$G \rightarrow G_{\pi} \rightarrow G_{\pi}^* \rightarrow G_{\pi}^o$

- bipartite incidence graph
- Hermitian unital
- no subdivision of $K_4$
- O'Nan/Paasche configuration
- projection
- strongly regular graph
- pseudorandom
- designated clique structure
- counting independent sets
- random sampling
- Ramsey graph
- random blocks
- martingales
- pseudorandom
\( r(4, t) = \tilde{\Theta}(t^3) \)

- Starting point: the bipartite incidence graph \( G \) of a Hermitian unital in \( \text{PG}(2, q^2) \).

- Take all points \((x, y, z)\) satisfying \( x^{q+1} + y^{q+1} + z^{q+1} = 0 \).

- Then \( G \) is an \( m \) by \( n \) bipartite graph with \( m = q^3 + 1 \), \( n = q^2(q^2 - q + 1) \), and every vertex in the part of size \( m \) has degree \( q^2 \), and every vertex in the part of size \( n \) has degree \( q + 1 \).
\[ r(4, t) = \tilde{\Theta}(t^3) \]

- The key is that the Hermitian unital does not contain four lines in general position (the O’Nan/Paasche configuration), as proved by O’Nan (1972).
\[ r(4, t) = \tilde{\Theta}(t^3) \]

- As \{a, b, d\}, \{a, c, d\}, \{b, e, f\}, \{c, d, f\} are collinear triples, we may choose generators so
  \[ d = a + b \quad e = a + c \quad f = a + b + c. \]

- Let \( A \) be the matrix whose rows are \( a, b \) and \( c \) and let \( B \) be the matrix whose rows are \( a^q, b^q \) and \( c^q \). Then \( A \) and \( B \) are non-singular and so is \( AB \).
\( r(4, t) = \Theta(t^3) \)

- However

\[
\det(AB) = \det \begin{pmatrix}
\sigma(a, a) & \sigma(a, b) & \sigma(a, c) \\
\sigma(b, a) & \sigma(b, b) & \sigma(b, c) \\
\sigma(c, a) & \sigma(c, b) & \sigma(c, c)
\end{pmatrix}
\]

\[
= \sigma(a, b)\sigma(b, c)\sigma(c, a) - \sigma(b, a)\sigma(a, c)\sigma(c, b) = 0
\]
Erdős-Rogers Functions

• Given graphs $F$ and $G$, let $f_{F,G}(n)$ denote the maximum number of vertices in an $F$-free subgraph of every $G$-free $n$-vertex graph. (Erdős-Rogers 1962)

• Erdős-Rogers, Bollobás-Hind, Dudek-Rödl, Dudek-Retter-Rödl, Krivelevich, Alon-Krivelevich, Wolfowitz, Gowers-Janzer, Janzer-Sudakov, ...

• Theorem (Wolfowitz)

$$f_{K_3, K_4}(n) \leq n^2 (\log n)^{120}$$
Erdős-Rogers Functions

• It is easy to see

\[ f_{K_s, K_{s+1}}(n) > \sqrt{n} - 1. \]

• In fact (Shearer)

\[ f_{K_s, K_{s+1}}(n) > \sqrt{\frac{n \log n}{\log \log n}}. \]
Erdős-Rogers Functions

- Theorem (Mubayi-V, 2024)

Let $s > 3$. Then for any $K_4$-free graph $F$ containing a $k$-cycle,

$$bn^3 + \frac{1}{3k} < f_{F,K_4}(n) < cn^2 \log n.$$
Erdős-Rogers Functions

• Theorem (Mubayi-V, 2024)

For any triangle-free graph $F$,

$$f_{F,K_3}(n) = n^{1+o(1)}.$$
Open Problem I

• Conjecture (V)

For $s > 2$,

$$f_{K_s,K_{s+1}}(n) = \Theta(n^{\frac{1}{2}} \log n).$$

• Problem

Suppose the neighborhood of every vertex in an $n$-vertex graph $G$ of maximum degree $d$ induces a bipartite graph. Does $G$ contain an induced triangle-free subgraph of size

$$\omega(d) \cdot \frac{n \log d}{d}?$$
Open Problem I

- Theorem (Ajtai, Komlós, Pintz, Spencer, Szemerédi 1982, V-Wilson, 2024)

  Every locally sparse \(n\)-vertex triple system of maximum degree \(d\) has an independent set of size at least

  \[
  \frac{n \sqrt{\log d}}{4 \sqrt{d}}.
  \]
Open Problem II

- For some $C > 0$, is the number of independent sets of size

$$t = \left\lfloor \frac{Cn \log n}{d} \right\rfloor$$

in a $\lambda$-pseudorandom graph at most

$$\left( \frac{C \lambda}{\log n} \right)^t$$
Open Problem III

- In $\text{PG}(2, q^3)$ is there a set of roughly $q^4$ points and $q^6$ lines of size $q$ each such that no five lines are in general position?