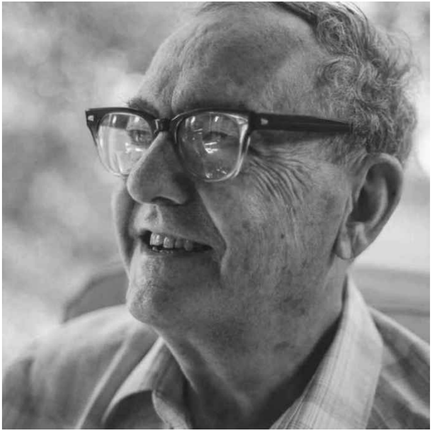
Scaling Exponents for Step Reinforced Random Walks

Jean BERTOIN

Institute of Mathematics, University of Zurich



Herbert A. Simon (1916-2001)

Simon's linear reinforcement algorithm

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$$\hat{X}_n = \begin{cases} \hat{X}_{U(n)} & \text{if } \varepsilon_n = 0, \\ X_{\sigma(n)} & \text{if } \varepsilon_n = 1, \end{cases}$$

where $U(n)$ is uniform on $\{1, \ldots, n-1\}$,

and $\sigma(n)$ is the number of innovations up to the n-step:

$$\sigma(n) = \sum_{j=1}^{n} \varepsilon_j.$$

Yule-Simon distribution

- Slow innovation regime: $\sigma(n) \approx n^{\rho}$ for some $\rho \in (0,1)$.
- Steady innovation regime: $\sigma(n) \sim qn$ for some $q \in (0,1)$. We rather use the parameter

$$\rho = 1/(1-q) > 1.$$

Theorem (Simon, 1955)

For every $k \geq 1$, the proportion of items that have appeared exactly k times at the n-th step converges as $n \to \infty$ towards

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Step reinforced random walks

Now $X_1, X_2, \ldots$ i.i.d. copies of some real r.v. X.

Our goal is to compare the asymptotic behavior of the random walk

$$S(n) = X_1 + \ldots + X_n$$

with that of the reinforced version

$$\hat{S}(n) = \hat{X}_1 + \ldots + \hat{X}_n.$$

Heyde (2004) first observed that when the ε_n are i.i.d. Bernoulli with $\mathbb{P}(\varepsilon = 1) = q$ and $X \sim \mathrm{Rademacher}$ (so $\rho > 1$ and $\alpha = 2$), then a phase transition occurs at $\rho = 2$:

- If $\rho > 2$, then

$$n^{-1/2}\hat{S}(n) \implies \mathcal{N}(0, s^2).$$

- If $1 < \rho < 2$, then

$$n^{-1/\rho}\hat{S}(n) \longrightarrow V.$$

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Case $\mathbb{E}(X^2) = \infty$, still for ε_n i.i.d. Bernoulli.
Businger (2018) observed a similar phase transition when X has the symmetric α-stable law :

- If $\rho > \alpha$, then

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Here, we are mainly interested in the case when the ε_n are general and S has a scaling exponent:

$$\lim_{n\to\infty} n^{-1/\alpha} S(n) = Y \quad \text{in law},$$

where $\alpha \in (0, 2]$ and Y denotes an α-stable variable.

What is the scaling exponent $\hat{\alpha} = \hat{\alpha}(\rho, \alpha)$ of $\hat{S}$?

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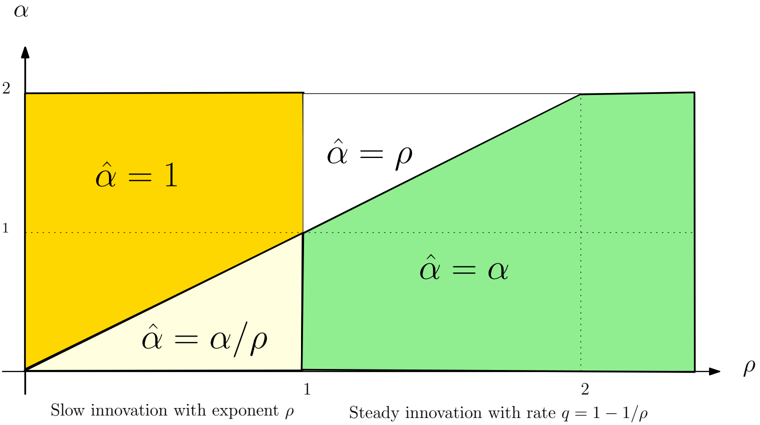
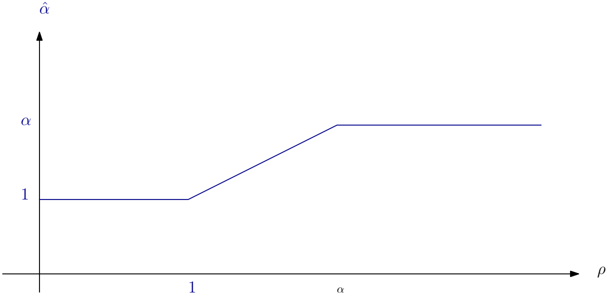


For $\alpha > 1$ fixed



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$\mathbb{E}(X) = 0$; repetitions perturb and finally disrupt the compensation between positive and negative steps.

Some comments

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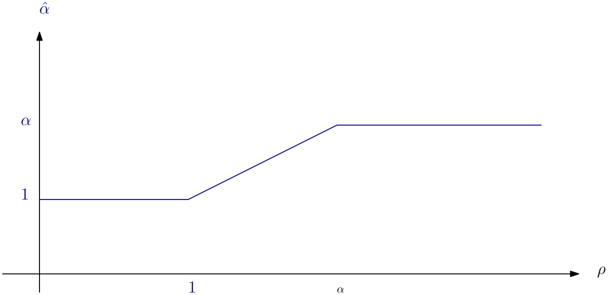

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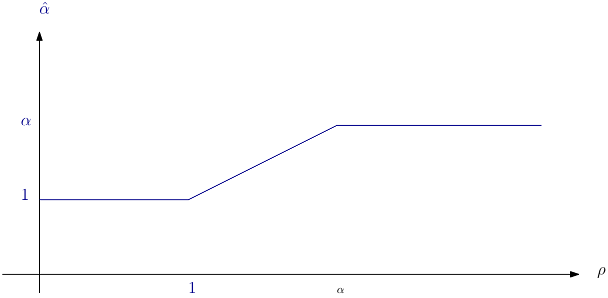

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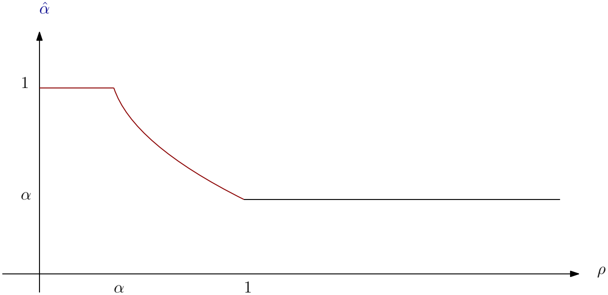

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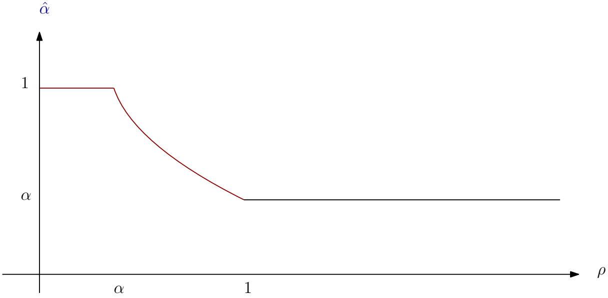

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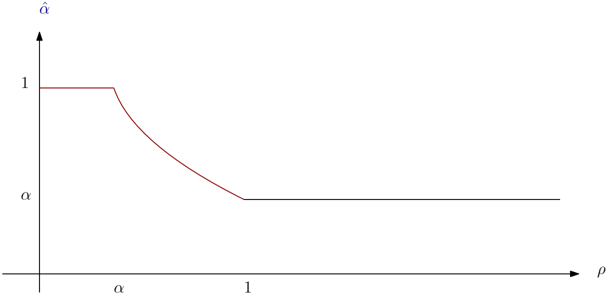

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Precise statements

Theorem (Ballistic behavior)

Let $\rho \in (0,1)$ *and* $\beta > \rho$, *and suppose that*

$$\sigma(n) = O(n^{\rho}) \qquad \text{as } n \to \infty,$$

and

$$\mathbb{P}(|X| > x) = O(x^{-\beta}) \qquad \text{as } x \to \infty,$$

Then

$$\lim_{n\to\infty} n^{-1}\hat{S}(n) = V' \qquad a.s.$$

where V' *is some non-degenerate random variable.*

Theorem (Sub-ballistic & Super-diffusive behavior)

Let $\rho = 1/(1-q) \in (1,2)$ *and suppose that*

$$\sum_{n=1}^{\infty} n^{-2} |\sigma(n) - qn| < \infty$$

and that for some $\beta > \rho$

$$\mathbb{E}(|X|^{\beta}) < \infty \quad \text{and} \quad \mathbb{E}(X) = 0.$$

Then

$$\lim_{n\to\infty} n^{-1/\rho} \hat{S}(n) = V' \qquad \text{in } L^{\beta}(\mathbb{P})$$

where V' *is some non-degenerate random variable.*

Theorem (Diffusive behavior)

Suppose that for some $q \in (0,1)$

$$\sum_{n=1}^{\infty} n^{-2} |\sigma(n) - qn| < \infty.$$

Assume also that X *belongs to the domain of normal attraction of a stable law with index* $\alpha \in (0,2]$

$$\lim_{n\to\infty} n^{-1/\alpha}(X_1 + \ldots + X_n) = Y \quad \text{in law.}$$

Suppose further that $\alpha < \rho$ *when* $\alpha > 1$. *Then*

$$\lim_{n\to\infty} n^{-1/\alpha} \hat{S}(n) = Y' \qquad \text{in law}$$

where Y' *is an* α*-stable random variable.*

Theorem (Super-ballistic & Sub-diffusive behavior)

Let $\alpha \in (0,1)$ and $\rho \in (\alpha, 1)$. Suppose that X belongs to the domain of normal attraction of an α-stable law:

$$\lim_{n\to\infty} n^{-1/\alpha}(X_1 + \ldots + X_n) = Y \quad \textit{in law.}$$

and that $\sigma(n)$ is regularly varying with exponent ρ:

$$\lim_{n\to\infty} \frac{\sigma(\lfloor cn \rfloor)}{\sigma(n)} = c^{\rho} \quad \textit{for all } c > 0.$$

Then

$$\lim_{n\to\infty} \sigma(n)^{-1/\alpha} \hat{S}(n) = Y' \qquad \textit{in law}$$

where Y' is an α-stable random variable.

The proofs rely on the analysis of the numbers of repetitions

$$N_j(n) = \#\{k \leq n : \hat{X}_k = X_j\};$$

one has to determine their asymptotic behaviors as $n \to \infty$ simultaneously for all $j \geq 1$.

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Recall that $\sigma(n)$ denotes the number of innovations and write

$$\begin{aligned}\tau(j) &= \inf\{n \in \mathbb{N} : \sigma(n) = j\} \\ &= \inf\{n \in \mathbb{N} : N_j(n) = 1\}\end{aligned}$$

for the first step of the algorithm at which X_j appears.

Introduce also

$$\pi(n) = \prod_{j=2}^{n} \left(1 + \frac{1 - \varepsilon_j}{j - 1}\right), \qquad n \in \mathbb{N}.$$

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One has

$$\pi(n) \approx \begin{cases} n^{1-q} = n^{1/\rho} & \text{in steady innovation regimes,} \\ n & \text{in slow innovation regimes.} \end{cases}$$

$a(n) \approx b(n)$ means $\lim_{n\to\infty} a(n)/b(n) \in (0, \infty)$.

A family of remarkable martingales

Lemma

Under mild assumptions on (ε_n), *for every* $j \geq 1$

$$\frac{N_j(n)}{\pi(n)}, \quad n \geq \tau(j),$$

is a square integrable martingale whose terminal value Γ_j *satisfies*

$$\mathbb{E}(\Gamma_j) = \frac{1}{\pi(\tau(j))} \quad \text{and} \quad \mathbb{E}(\Gamma_j^2) \asymp \frac{1}{\pi(\tau(j))^2}.$$

Sketch of proofs

For the strong limit theorems ($\alpha > \rho$), one writes first

$$\hat{S}(n) = \hat{X}_1 + \ldots + \hat{X}_n = \sum_{j=1}^{\infty} N_j(n) X_j .$$

Thus

$$\frac{\hat{S}(n)}{\pi(n)} = \sum_{j=1}^{\infty} \frac{N_j(n)}{\pi(n)} X_j ;$$

one has to check some uniform integrability property in order to exchange $\lim_{n\to\infty}$ and $\sum_{j=1}^{\infty}$ so that

$$\lim_{n\to\infty} \frac{\hat{S}(n)}{\pi(n)} = \sum_{j=1}^{\infty} \Gamma_j X_j .$$

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one has to check some uniform integrability property in order to exchange $\lim_{n\to\infty}$ and $\sum_{j=1}^{\infty}$ so that

$$\lim_{n\to\infty} \frac{\hat{S}(n)}{\pi(n)} = \sum_{j=1}^{\infty} \Gamma_j X_j.$$

For the weak limit theorems ($\alpha < \rho$), one considers the characteristic exponent φ of X

$$\mathbb{E}(\mathrm{e}^{i\theta X}) = \mathrm{e}^{-\varphi(\theta)}, \qquad \text{for } |\theta| \text{ small enough.}$$

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Then

$$\mathbb{E}(\mathrm{e}^{i\theta \hat{S}(n)} \mid (\varepsilon_\ell)) = \exp\left(-\sum_{k=1}^{\infty} R_k(n)\varphi(k\theta)\right),$$

where $R_k(n)$ denotes the total number of items that have occurred exactly k time at the n-th step of the reinforcement algorithm.

On the one hand, we knows from Simon's result that for each $k \geq 1$

$$\frac{R_k(n)}{\sigma(n)} \to \rho \mathrm{B}(k, \rho + 1).$$

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On the other hand, recall that X belongs to the normal domain of attraction of an α-stable distribution. If we write φ_α for the characteristic exponent of the latter, results of Ibragimov and Linnik show

$$\lim_{t \to \infty} t\varphi(\theta t^{-1/\alpha}) = \varphi_\alpha(\theta), \qquad \text{for all } \theta \in \mathbb{R}.$$

Another uniform integrability property is needed to exchange $\lim_{n\to\infty}$ and $\sum_{k=1}^{\infty}$ and conclude that

$$\lim_{n\to\infty} \mathbb{E}(\exp(i\theta\sigma(n)^{-1/\alpha}\hat{S}(n))) = \exp\left(-c\varphi_\alpha(\theta)\right),$$

with

$$c = \sum_{k=1}^{\infty} k^\alpha \rho B(k, \rho + 1).$$