Recent progress on random graph matching problems

Jian Ding
Peking University

based on joint works with
Hang Du (PKU)
Shuyang Gong (PKU)
Zhangsong Li (PKU)

Application 1: Network de-anonymization

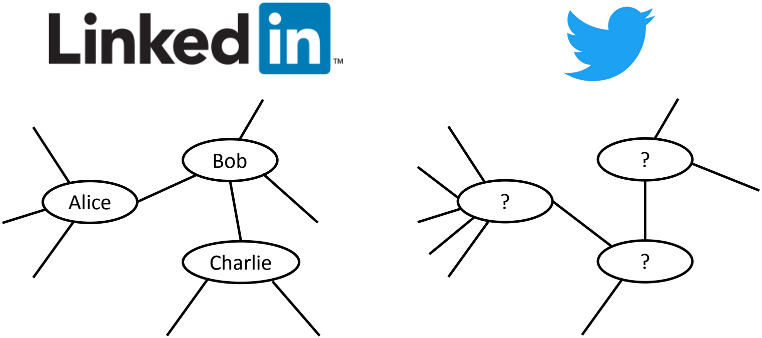

Figure: Picture courtesy of R. Srikant

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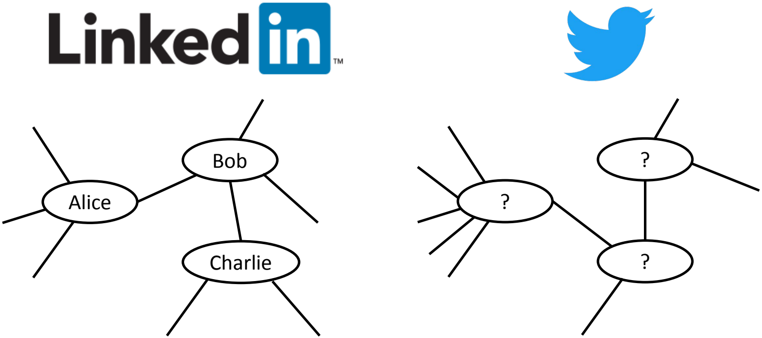

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- Successfully de-anonymize Netflix by matching it to IMDB [Narayanan-Shmatikov '08]
- Correctly identified 30.8% of node mappings between Twitter and Flickr [Narayanan-Shmatikov '09]

Application 2: Protein-Protein Interaction network

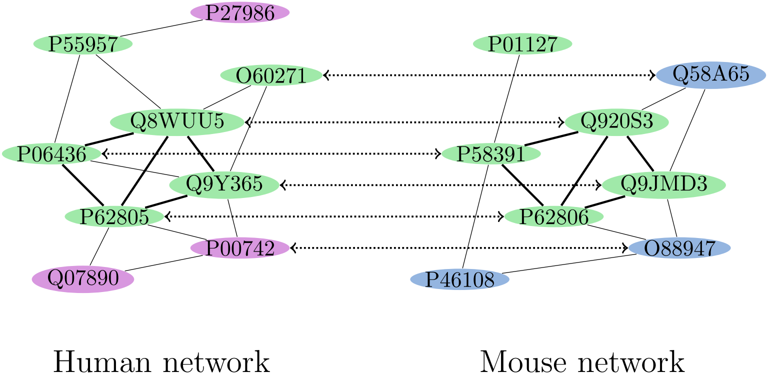

Figure: [Kazemi-Hassani-Grossglauser-Modarres '16]

Ontology: Discover proteins with similar functions across different species based on interaction network topology

Application 3: Computer vision

A fundamental problem in computer vision: Detect similar objects that undergo different deformations

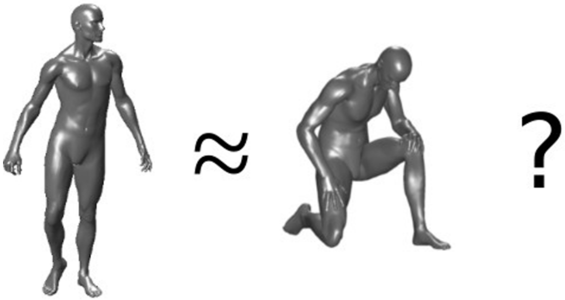

Figure: Shape REtrieval Contest (SHREC) dataset [Lähner et al '16]

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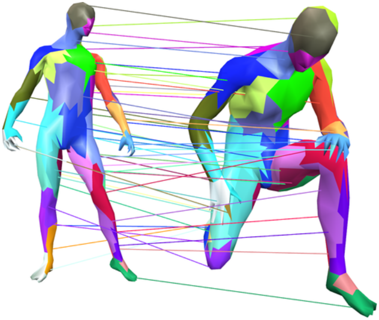

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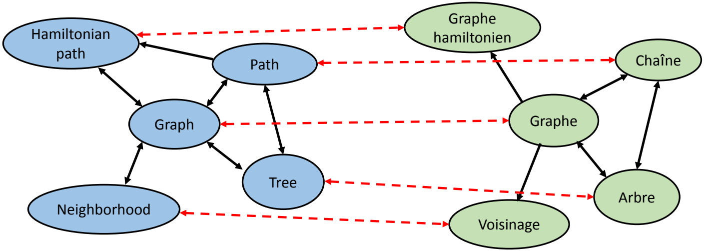
3-D shapes $\rightarrow$ geometric graphs (features $\rightarrow$ nodes, distances $\rightarrow$ edges)

Determine whether two graphs are topologically similar

Application 4: Machine Translation



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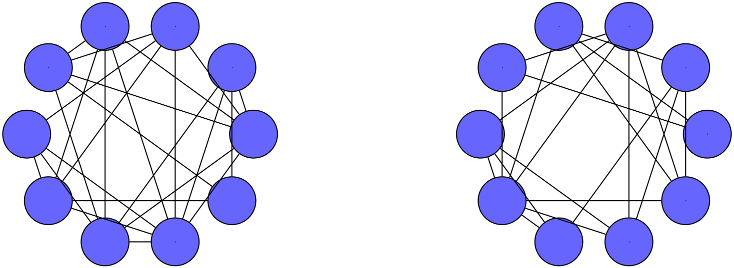
Automatically find/correct corresp. wiki articles in different languages [Fishkind-Adali-Patsolic-Meng-Lyzinski-Priebe '12]

Graph matching (network alignment)



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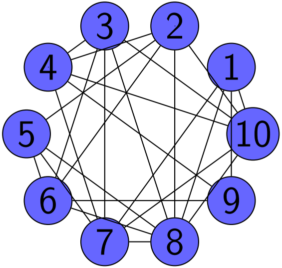

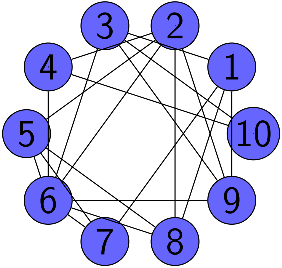

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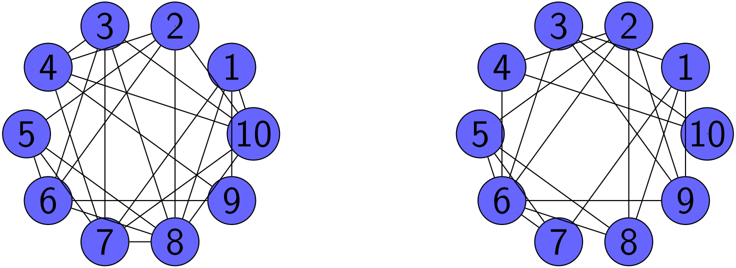

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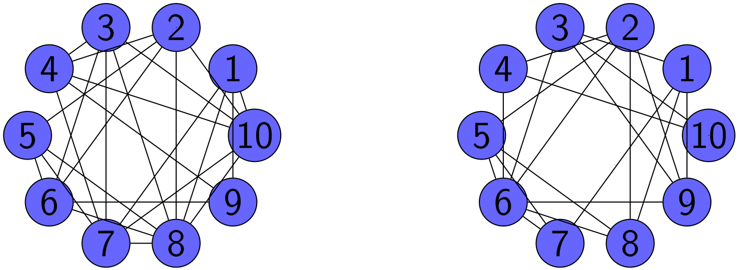

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Graph matching is a hard optimization problem (NP-hard), and we seek help from randomness.

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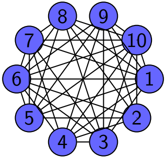

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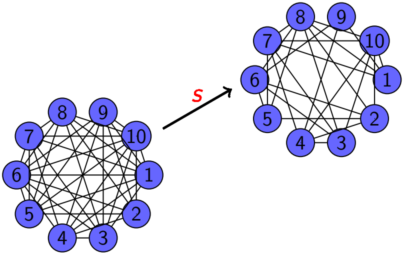

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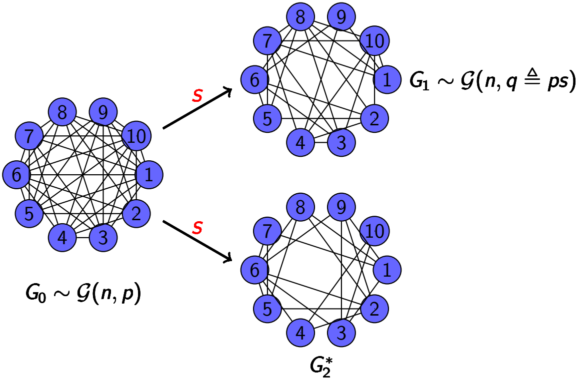

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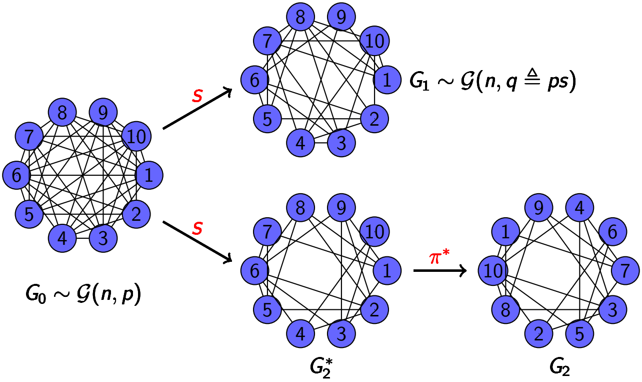

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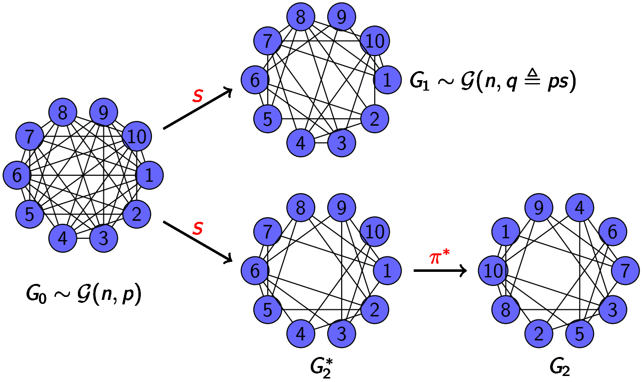

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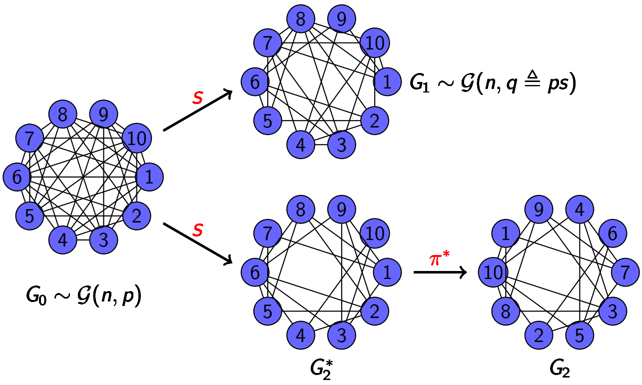

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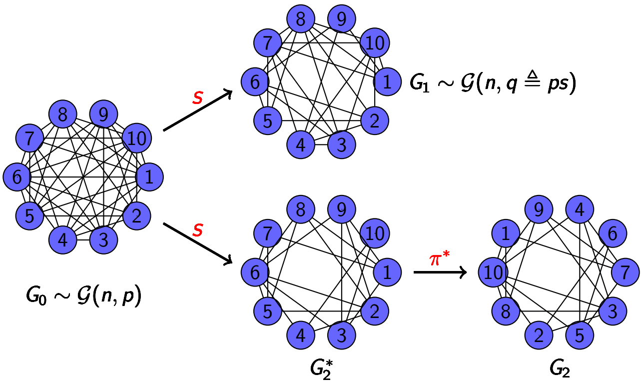

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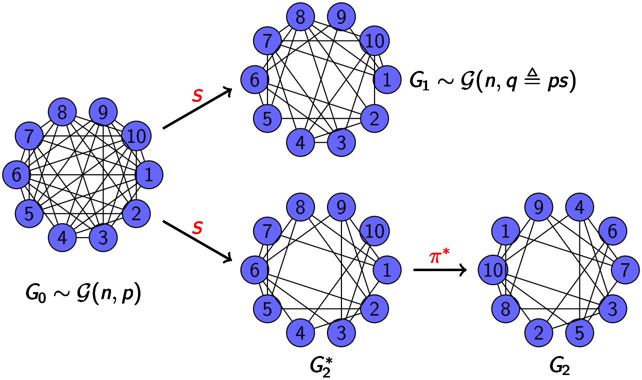

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Disadvantage: almost all realistic networks are not Erdős-Rényi.

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- Analysis: when two graphs are correlated it is hard to analyze $\hat{\pi}$ and WXY used "$\hat{\pi} \geq \pi^*$" to lower-bound the maximal common graph.

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- Wu-Xu: maybe related to overlap gap property for correlated model.
- Computation for correlated model seems much more difficult.

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  - Succeeds when noise decays in $\mathrm{polylog}(n)$.

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  - Shed lights on many matching problems too.

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- **Information-computation gap**: a **major** challenge in many random combinatorial optimization and constraint satisfaction problems!

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- Bridging what is wanted with what is possible.

Reference: all mentioned works available on arXiv.