Thresholds

Jinyoung Park

Stanford University

University of Oxford

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Conjecture [Kahn-Kalai ’06]; proved by P.-Pham (’22).

There exists a universal $K > 0$ such that for every finite set $X$ and increasing property $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq K p_E(\mathcal{F}) \log |X|$$

- $p_c(\mathcal{F})$: threshold for $\mathcal{F}$
- $p_E(\mathcal{F})$: expectation threshold for $\mathcal{F}$
Basic definitions

- \( X \): finite set; \( 2^X = \{ \text{subsets of} \ X \} \)
- \( \mu_p \): \( p \)-biased product probability measure on \( 2^X \)
  \[ \mu_p(A) = p^{|A|}(1 - p)^{|X \setminus A|} \quad A \subseteq X \]
- \( X_p \sim \mu_p \) "\( p \)-random" subset of \( X \)
  - e.g. 1. \( X = (\binom{n}{2}) = E(K_n) \)
    \( \rightarrow X_p = G_{n,p} \) Erdős-Rényi random graph
  - e.g. 2. \( X = \{ k \text{-clauses from} \ \{ x_1, \ldots, x_n \} \} \)
    \( \rightarrow X_p : \text{random CNF formula} \)
- \( \mathcal{F} \subseteq 2^X \) is an **increasing property** if
  \[ B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F} \]
  - e.g. 1. \( \mathcal{F} = \{ \text{connected} \} \); \( \mathcal{F} = \{ \text{contain a triangle} \} \)
  - e.g. 2. \( \mathcal{F} = \{ \text{not satisfiable} \} \)

* \( K_n \): the complete graph on \( n \) vertices
Theses

Fact.
For any increasing property \( \mathcal{F} (\neq \emptyset, 2^X) \), \( \mu_p(\mathcal{F}) \) is continuous and strictly increasing in \( p \).

\[ \mu_p(\mathcal{F}) \]

\[ \begin{array}{c|c|c}
\hline
p_c(\mathcal{F}) & 0 & 1/2 & 1 \\
\hline
\end{array} \]

- \( p_c(\mathcal{F}) \) is called the threshold for \( \mathcal{F} \).

- cf. Erdős-Rényi: \( p_0 = p_0(n) \) is a threshold function for \( \mathcal{F}_n \) if

\[ \mu_p(\mathcal{F}_n) \rightarrow \begin{cases} 
0 & \text{if } p \ll p_0 \\
1 & \text{if } p \gg p_0
\end{cases} \]

* \( p_c(\mathcal{F}_n) \) is always an Erdős-Rényi threshold (Bollobás-Thomason '87).
The Kahn–Kalai Conjecture

"It would probably be more sensible to conjecture that it is not true."

- Kahn and Kalai (2006)

Question.

What drives $p_c(\mathcal{F})$?
Example 1. Containing a copy of $H$

- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}_H$: contain a copy of $H$

Example 1.

What's the threshold for $G_{n,p}$ to contain a copy of $H$?

- Usual suspect: expectation calculation

$$\mathbb{E}[\text{# } H's \text{ in } G_{n,p}] \asymp n^4 p^5 \rightarrow \begin{cases} 
0 & \text{if } p \ll n^{-4/5} \\
\infty & \text{if } p \gg n^{-4/5}
\end{cases}$$

"threshold for $\mathbb{E}$" $\asymp n^{-4/5}$

- triv. $p_c(\mathcal{F}_H) \gtrsim n^{-4/5}$ (∵ $\mathbb{E}X \rightarrow 0 \Rightarrow X = 0$ with high probability)

- truth: $p_c(\mathcal{F}_H) \asymp n^{-4/5}$
Example 2. Containing a copy of $K$

- $X = \binom{[n]}{2}$ (so $X_p = G_{n,p}$); \(\mathcal{F}_K\): contain a copy of $K$

Example 2.
What’s the threshold for $G_{n,p}$ to contain a copy of $K$?

\[
\mathbb{E}[\# \text{K's in } G_{n,p}] \asymp n^5 p^6 \rightarrow \begin{cases} 
0 & \text{if } p \ll n^{-5/6} \\
\infty & \text{if } p \gg n^{-5/6}
\end{cases}
\]

“threshold for $\mathbb{E}$” \(\asymp n^{-5/6}\)

- Q. $p_c(\mathcal{F}_K) \asymp n^{-5/6}$? (triv. $p_c(\mathcal{F}_K) \gtrsim n^{-5/6}$)

- truth: $p_c(\mathcal{F}_K) \asymp n^{-4/5}$

Erdős-Rényi ('60), Bollobás ('81)

(Rough:) For fixed graph $K$,

\[p_c(\mathcal{F}_K) \asymp \text{”threshold for $\mathbb{E}$” of the ”densest” subgraph of $K”} \]
Example 3. Containing a perfect matching

$X = \binom{n}{2}$ (so $X_p = G_{n,p}$); $\mathcal{F}$: contain a perfect matching

Example 3.

What’s the threshold for $G_{n,p}$ to contain a perfect matching? $(2 \mid n)$

$\mathbb{E}[^{\# \text{Perfect matchings in } G_{n,p}}] \approx \left(\frac{np}{e}\right)^{n/2} \rightarrow \begin{cases} 0 & \text{if } p \ll 1/n \\ \infty & \text{if } p \gg 1/n \end{cases}$

“threshold for $\mathbb{E}$” $\asymp 1/n$

Q. $p_c(\mathcal{F}) \asymp 1/n$? (triv. $p_c(\mathcal{F}) \gtrsim 1/n$)

truth: $p_c(\mathcal{F}) \asymp \log n/n$

Fact. $p \ll \log n/n \Rightarrow G_{n,p}$ has an isolated vertex w.h.p.
One more example: perfect hypergraph matchings

- Now, $X = \binom{[n]}{r}$
- $X_p = \text{random } r\text{-uniform hypergraph } \mathcal{H}_{n,p}^r$

Example 3'. (Shamir’s Problem (‘80s))
For $r \geq 3$, what’s the threshold for $\mathcal{H}_{n,p}^r$ to contain a perfect matching? ($r|n$)

cf. $r = 2$: Erdős-Rényi (‘66) $r \geq 3$ much harder

e.g. $r = 3$:
- $\mathbb{E}[\# \text{ perfect matchings in } \mathcal{H}_{n,p}^r] \asymp (n^2 p)^{n/3} \rightarrow \text{“threshold for } \mathbb{E}^{\prime} \asymp n^{-2}$
- Lower bound from coupon-collector:
  $$p_c(\mathcal{F}) \gtrsim \log n / n^2$$
- $p_c(\mathcal{F}) \ll \log n / n^2$ (Johansson-Kahn-Vu ‘08) * log $n$ gap again
What drives $p_c(F)$?

- We have some **trivial lower bounds** on $p_c$:
  - Ex 1, 2 (contain H/K): "threshold for $E$"
  - Ex 3, 3' (contain a PM): coupon collector-ish behavior ($\log n$ gap)
- Historically, in many interesting cases, the main task is to find a matching upper bound.

**The Kahn-Kalai Conjecture ('06): rough statement**

For any increasing property, the threshold is at most $\log |X|$ times the "expectation threshold".

- This is a VERY strong conjecture: immediately implies (e.g.)
  - threshold for perfect hypergraph matchings (Johansson-Kahn-Vu '08)
    $$p_E \asymp n^{-(r-1)} \xrightarrow{\text{KKC}} p_c \lesssim \log n / n^{r-1}$$
  - threshold for bounded degree spanning trees ("tree conjecture"; Montgomery '19)
$p_E(\mathcal{F})$: the expectation threshold

For abstract $\mathcal{F}$, it’s unclear whose expectation we want to compute, so need a careful definition for the "threshold for $E$."
\( p_E(\mathcal{F}) \): the expectation threshold

**Observation**

\( p_c(\mathcal{F}) \geq q \) if \( \exists \: G \subseteq 2^X \) such that

1. "\( G \) covers \( \mathcal{F} \)": \( \forall A \in \mathcal{F} \: \exists B \in G \) such that \( A \supseteq B \) \( (\mathcal{F} \subseteq \langle G \rangle) \)

2. \( \sum_{S \in G} q^{|S|} \leq \frac{1}{2} \) ("\( q \)-cheap")

\text{e.g. in Ex 2, } X = \begin{pmatrix} n \end{pmatrix}_2, \mathcal{F}: \text{contain a copy of } K

- \( G_1 = \{ \text{all (labeled) copies of } K \} \)

  \[ \rightarrow \sum_{S \in G_1} q^{|S|} \leq 1/2 \text{ for } q \lesssim n^{-5/6} \]

  \[ \rightarrow n^{-5/6} \lesssim p_c(\mathcal{F}) \]

- \( G_2 = \{ \text{all (labeled) copies of } H \} \)

  \[ \rightarrow \sum_{S \in G_2} q^{|S|} \leq 1/2 \text{ for } q \lesssim n^{-4/5} \]

  \[ \rightarrow n^{-4/5} \lesssim p_c(\mathcal{F}) \]
**$p_E(\mathcal{F})$: the expectation threshold**

**Observation**

\[ p_c(\mathcal{F}) \geq q \text{ if } \exists \mathcal{G} \subseteq 2^X \text{ such that} \]

1. "\(\mathcal{G}\) covers \(\mathcal{F}\)" : \(\forall A \in \mathcal{F} \ \exists B \in \mathcal{G} \text{ such that } A \supseteq B \) \((\mathcal{F} \subseteq \langle \mathcal{G} \rangle)\)
2. \(\sum_{S \in \mathcal{G}} q^{\lvert S \rvert} \leq \frac{1}{2}\)

\[ p_E(\mathcal{F}) := \max\{q : \exists \mathcal{G}\} \quad \rightarrow \text{a trivial lower bound on } p_c(\mathcal{F}) \]

**The Kahn-Kalai Conjecture ('06)**

There exists a universal \(K > 0\) such that for every finite \(X\) and increasing \(\mathcal{F} \subseteq 2^X\),

\[ (p_E(\mathcal{F}) \leq) \ p_c(\mathcal{F}) \leq K p_E(\mathcal{F}) \log \lvert X \rvert \]
Results and Proof Sketch
Conj of Talagrand: fractional version of Kahn-Kalai Conj

- $p^*_E(\mathcal{F})$: the fractional expectation threshold for $\mathcal{F}$
  - skip def: roughly, replace cover $\mathcal{G}$ by "fractional cover"
- Easy. $p_E(\mathcal{F}) \leq p^*_E(\mathcal{F}) \leq p_c(\mathcal{F})$

Conj (Talagrand ‘10); proved by Frankston-Kahn-Narayanan-P. (‘19).

There exists a universal $K > 0$ such that for every finite $X$ and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq Kp^*_E(\mathcal{F}) \log \ell(\mathcal{F}).$$

* $\ell(\mathcal{F})$: the size of a largest minimal element of $\mathcal{F}$

- Weaker than KKC, but in all known applications, $p_E(\mathcal{F}) \preccurlyeq p^*_E(\mathcal{F})$
- Proof inspired by Alweiss-Lovett-Wu-Zhang
  "Erdős-Rado Sunflower Conjecture"
Recall. In all known applications, \( p_E(\mathcal{F}) \approx p^*_E(\mathcal{F}) \).

**Conjecture (Talagrand ’10)** \( p_E(\mathcal{F}) \approx p^*_E(\mathcal{F}) \)

There exists a universal \( K \) such that for every finite \( X \) and increasing \( \mathcal{F} \subseteq 2^X \),

\[
(p_E(\mathcal{F}) \leq) \ p^*_E(\mathcal{F}) \leq K p_E(\mathcal{F})
\]

- Implies equivalence of KKC and fractional KKC
  – the most likely way to prove KKC?
- Even simple instances of the conjecture are not easy to establish; Talagrand suggested two test cases, proved by (respectively) DeMarco-Kahn (’15) and Frankston-Kahn-P. (’21)

**FKNP (19’)** \( p_c(\mathcal{F}) \leq K p^*_E(\mathcal{F}) \log \ell(\mathcal{F}) \)
**New result**

**Conjecture (Kahn-Kalai '06); proved by P.-Pham ('22)**

There exists a universal $K > 0$ such that for every finite $X$ and increasing $\mathcal{F} \subseteq 2^X$,

$$p_c(\mathcal{F}) \leq Kp_E(\mathcal{F}) \log \ell(\mathcal{F})$$

* $\ell(\mathcal{F})$: the size of a largest minimal element of $\mathcal{F}$

- Proofs inspired by ALWZ (sunflower) and FKNP (fractional Kahn-Kalai) but implementation very different
- Reformulation – think: $\mathcal{H} = \{\text{minimal elements of } \mathcal{F}\}$

**Theorem (P.-Pham '22)**

$\exists L > 0$ such that $\forall \ell$-bdd $\mathcal{H}$, if $p > p_E(\langle \mathcal{H} \rangle)$, then, with $m = Lp \log \ell|X|$, 

$$\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)$$
Proof sketch

\exists L > 0 \text{ such that } \forall \ell\text{-bdd } \mathcal{H}, \text{ if } p > p_E(\langle \mathcal{H} \rangle), \text{ then, with } m = Lp \log \ell|X|,

\mathbb{P}(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1)

- Choose \( W(= X_m) \) little by little: \( W = W_1 \sqcup W_2 \sqcup \ldots \)
- At the end, want \( W \supseteq S \in \mathcal{H} \) whp.
- Run algorithm: no assump \( \rightarrow \) two possible outputs
- (Recall) \( p > p_E(\langle \mathcal{H} \rangle) \) means:

\( \langle \mathcal{H} \rangle \) does not admit a \( p \)-cheap cover.
\[ \exists L > 0 \text{ such that } \forall \ell\text{-bdd } \mathcal{H}, \text{ if } p > p_e(\langle \mathcal{H} \rangle), \text{ then, with } m = Lp \log \ell|X|, \]
\[ P(X_m \in \langle \mathcal{H} \rangle) = 1 - o_\ell(1) \]

- \( W = W_1 \cup W_2 \cup \ldots \)

- At \( i \)th step: choose \( W_i \) of size \( \approx Lp|X| \) at random
  \[ \rightarrow \textbf{Construct} \text{ cover } \mathcal{U}_i = \mathcal{U}_i(W_i) \text{ of some } \mathcal{G}_i = \mathcal{G}_i(W_i) \subseteq \mathcal{H}_{i-1} \]

- When \textbf{terminates}, with \( \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots \) (“partial cover”) either
  \[ (1) \mathcal{U} \text{ covers } \mathcal{H}; \quad \text{or} \quad (2) \ W \in \langle \mathcal{H} \rangle \]

- \textbf{[Main Point]} Typically, \( \mathcal{U} \) is “\( p \)-cheap.”

\[ \text{Ass. } p > p_e(\langle \mathcal{H} \rangle) \quad \rightarrow \quad (1) \text{ is unlikely} \quad \square \]
Open Questions
Gap between $p_E(\mathcal{F})$ and $p_C(\mathcal{F})$

**Theorem (P.-Pham ’22)**

\[(p_E(\mathcal{F}) \leq) \ p_C(\mathcal{F}) \lesssim p_E(\mathcal{F}) \log \ell(\mathcal{F})\]

**Question**

What characterizes the gap between $p_E(\mathcal{F})$ and $p_C(\mathcal{F})$?

- In many cases the $\log \ell(\mathcal{F})$ gap is tight:
  e.g. perfect hypergraph matchings, spanning trees with bounded degree, Hamiltonian cycle, fixed subgraphs…

- There are some cases for which $\log \ell(\mathcal{F})$ is not tight:
  e.g. clique factors, the $k$-th power of a Hamilton cycle, non-spanning large graphs… → good test cases!
Test cases: gaps smaller than $\log \ell(F)$

First successful test case

$F$: contain the square of a Hamilton cycle ($HC^2$)

Conjecture (Kühn-Osthus '12)

$$p_c(F) \asymp n^{-1/2}$$

- $p_e(F)(\asymp p^*_e(F)) \asymp n^{-1/2}$ → no gap!
  - Kühn-Osthus ('12) $p^* \lesssim n^{-1/2+o(1)}$
  - Nenadov-Škorić ('16) $p^* \lesssim n^{-1/2} \log^4 n$
  - Fischer-Škorić-Stege-Trujić ('18) $p^* \lesssim n^{-1/2} \log^3 n$
  - Montgomery $p^* \lesssim n^{-1/2} \log^2 n$
  - Frankston-Kahn-Narayanan-P. $p^* \lesssim n^{-1/2} \log n$

Kahn-Narayanan-P. ('20)

$$p_c(F) \asymp n^{-1/2}$$
Good test cases: gaps smaller than \( \log \ell(F) \)

**[Ex 1]** \( F \): contain a **triangle-factor** (or a \( H \)-factor for fixed \( H \))

Johansson-Kahn-Vu ('08)

\[
p_c(F) \preceq n^{-2/3}(\log n)^{1/3}
\]

**[Ex 2]** Perfect matchings in the ”\( k \)-out model”

Frieze ('86)

\[
\lim_{\begin{array}{c} n \to \infty \\ n \text{ even} \end{array}} \mathbb{P}(G_{k\text{-out}} \text{ has a perfect matching}) = \begin{cases} 
0 & \text{if } k = 1 \\
1 & \text{if } k \geq 2
\end{cases}
\]
Thank you!