Combinatorics from the roots of polynomials

Julian Sahasrabudhe
University of Cambridge

November 3, 2020

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Based on joint work with Marcus Michelen

The probability generating function

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$X \in \{0, 1, \ldots, n\}$.

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What is the relationship between the roots of f_X and the distribution of X?

A first example

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Proof.

"Really just the usual central limit theorem, in disguise".

□

Question

What is the "correct" condition on the roots of f to guarantee normal behavior of X?

Godsil's theorem

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For $d \in \mathbb{N}$, let G d-regular graph on n vertices.

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See Kahn (2000) "Normal Law for matchings" for a modern reference.

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If none of the X_i "dominate".

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$$\sup_{t \in \mathbb{R}} |\mathbb{P}(X^* \leqslant t) - \mathbb{P}(Z \leqslant t)| = O\left(\frac{\log n}{\delta\sigma}\right),$$

where $Z \sim N(0, 1)$.

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Corollary (Michelen, Sahasrabudhe)

Pemantle's conjecture is true when

$$\sigma_n \gg \log n$$

and this is best possible.

Another perspective

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Probability generating function of X:

$$f_X(z_1, \ldots, z_d) = \sum_{i_1, \ldots, i_d} \mathbb{P}(X = (i_1, \ldots, i_d)) z_1^{i_1} \cdots z_d^{i_d}.$$

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f_X is *real-stable* if it has no roots in

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Theorem (Borcea, Brändén, Liggett, '07)

The correct definition is

"$X_1, \ldots, X_d$ are negatively dependent random variables if the (multi-variate) probability generating function of $X = (X_1, \ldots, X_d)$ is real stable.

What is the limit shape of random variables with real-stable probability generating functions?

$X = (X_1, \ldots, X_d) \in \mathbb{R}^d,$

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A a $d \times d$ positive semi-definite matrix, let $N(0, A)$ be the centered Gaussian with covarinace matrix A.

Question

What is the limit shape of these distributions?

Conjecture (Ghosh, Liggett, Pemantle, 2017)

For $d \in \mathbb{N}$*, let* $X_n \in \{0, \ldots, n\}^d$ *be a sequence of random variables.*

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$$(X_n - \mu_n)\sigma_n^{-1} \to N(0, A),$$

where $\mathrm{Cov}(X_n) \to A$.

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Theorem (Michelen, Sahasrabudhe)

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Then

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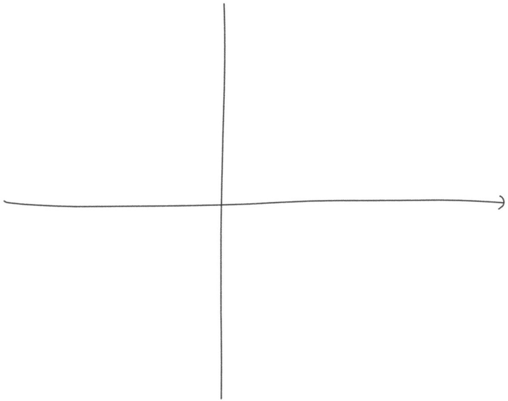

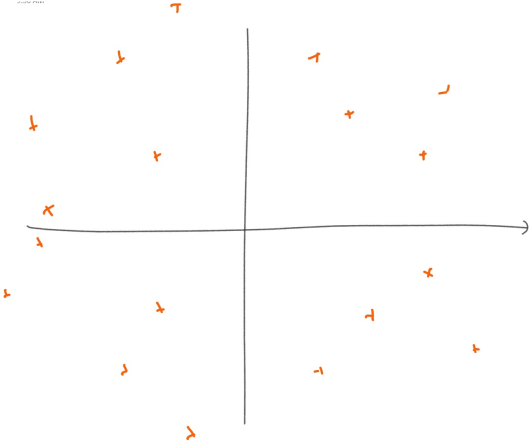

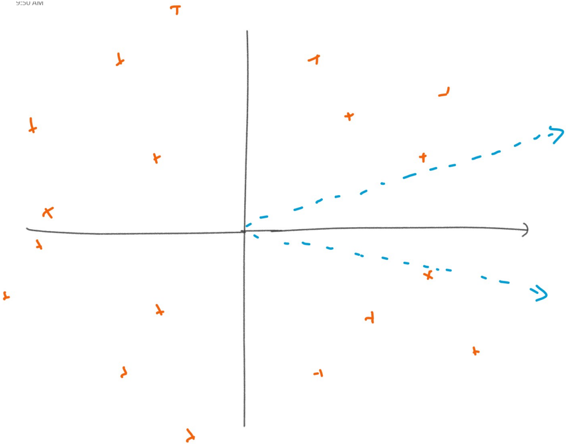

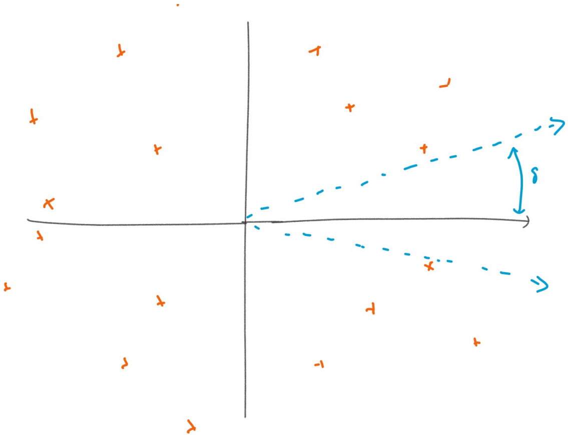
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Proof overview

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Goal

Show $|a_k|/\sigma^k \ll 1$, *uniformly for all* $k \in \mathbb{N}$.

Say that a function $u : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is *weakly positive* if

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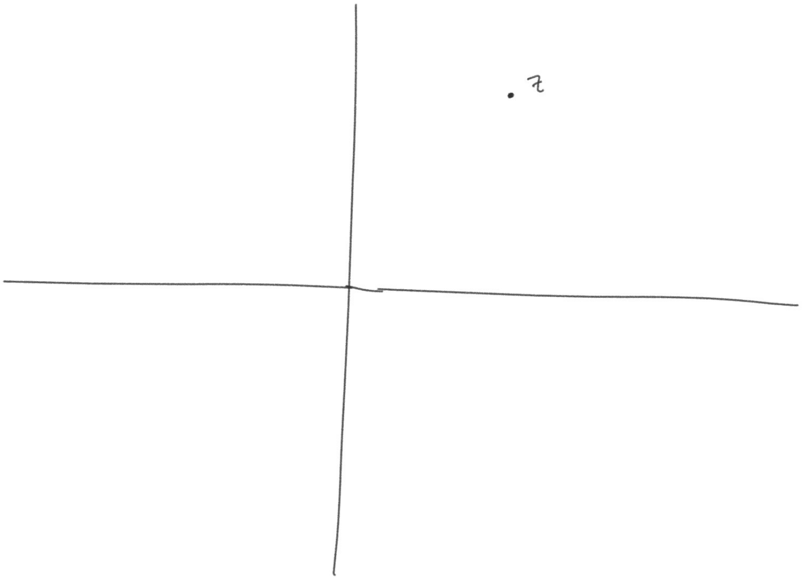
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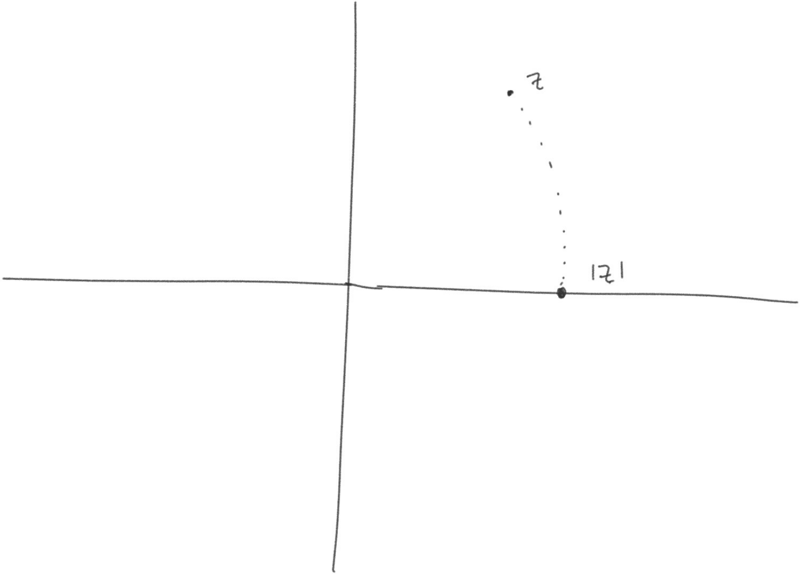
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u is *radially decreasing* if for all $0 < \theta_1 < \theta_2 \leqslant \varepsilon$ and all $r > 0$ we have

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Lemma

Our function u is radially decreasing in a neighbourhood of $1 \in \mathbb{C}$.

Lemma

For all $L \geqslant 2$

$$\frac{\sum_{j \geqslant L} |a_j| \varepsilon^j}{\sum_{j \geqslant 2} |a_j| \varepsilon^j} \leqslant C \cdot 2^{-L}, \tag{1}$$

where $\varepsilon \approx \delta$.

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Claim

$$\sum_{j \geqslant 2} |a_j| \varepsilon^j \geqslant \varphi_\varepsilon(1).$$

Proof of Claim.

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To finish the proof of this claim we need to prove

$$\max_{\theta\in[0,2\pi]} |U_0(2\varepsilon,\theta)|^2 \leqslant C(\varphi_\varepsilon(1))^2.$$

Lemma

For all $L \geqslant 2$

$$\frac{\sum_{j \geqslant L} |a_j| \varepsilon^j}{\sum_{j \geqslant 2} |a_j| \varepsilon^j} \leqslant C \cdot 2^{-L}, \tag{2}$$

where $\varepsilon \approx \delta$.

Lemma

For all $j \in \mathbb{N}$*, there exists a real number* $\varepsilon_0 \geqslant \alpha(\varepsilon, j)$ *for which*

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Theorem

Let $X \in \{0, \ldots, n\}$ be a random variable, let $\zeta_1, \ldots, \zeta_n$ be the roots of f_X and put $\delta = \min_i |\arg(\zeta_i)|$. Then

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