Ryser's conjecture and more

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The Ryser-Brualdi-Stein conjecture

A **Latin square** of order n is an n by n square with cells filled using n symbols so that every symbol appears once in each row and once in each column. A **transversal of order** k in a Latin square is a set of k cells from distinct rows and columns, containing distinct symbols. A transversal of order n is called **full transversal**.

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- $2n/3 + O(1)$ Koksma, '69
- $3n/4 + O(1)$ Drake, '77
- $n - \sqrt{n}$ Brouwer, De Vries, and Wieringa, '78 and Woolbright,'78
- $n - O(\log^2 n)$ Shor, '82, contained an error
- $n - O(\log^2 n)$ Hatami, Shor, 2008

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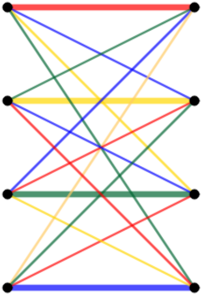
Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

Every $n \times n$ Latin square contains a transversal of order $n - O(\frac{\log n}{\log \log n})$.

Rainbow matchings in $K_{n,n}$

A transversal in a Latin square of order n with R rows, C columns and S symbols corresponds to a perfect rainbow matching in $K_{n,n}$ with bipartition (R, C) and colours S such that $color(r_i c_j) = s_{ij}$.



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Matchings in hypergraphs

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- $V(\mathcal{H}) = R \cup C \cup S$ where R are the rows of L, C the columns of L, and S the symbols of L.
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A **transversal of size** k in $L \leftrightarrow$ **a matching of size** k in $\mathcal{H}$.

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- $4n/15 - O(1)$ Lindner and Phelps, '78
- $n/3 - O(n^{2/3})$ Brouwer, '81
- $n/3 - O(n^{1/2} \log^{3/2} n)$ Alon, Kim, and Spencer 1997

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Every Steiner triple system on n vertices has a matching of size at least $n/3 - O(\log n/\log\log n)$.

Our proof setting

- Large transversals in $n \times n$ Latin squares filled with symbols $\{1, 2, \ldots, n\}$
- **Large rainbow matchings in properly edge-coloured $K_{n,n}$ with n colours**
- Large matchings in linear 3-partite Steiner systems

Our proof setting

- Large transversals in $n \times n$ Latin arrays
- **Large rainbow matchings in coloured quasirandom graphs**
- Large matchings in 3-partite, 3-uniform, linear hypergraphs

Typical (quasirandom) graphs

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(P1) $d(v) = pn(1 \pm n^{-\varepsilon})$,

(P2) for every $u, v \in X$ or $u, v \in Y$ we have $d(u, v) = p^2 n(1 \pm n^{-\varepsilon})$,

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We call a properly edge-coloured bipartite graph G with parts X, Y with $|X| = |Y| = n$ and n colors **coloured** (ε, p, n)-**typical** if

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(P3) $c \in C(G)$ has $|E_G(c)| = (1 \pm n^{-\varepsilon})pn$,

(P4) c, c' have $|V_G(c) \cap V_G(c') \cap X| = (1 \pm n^{-\varepsilon})p^2 n$ and $|V_G(c) \cap V_G(c') \cap Y| = (1 \pm n^{-\varepsilon})p^2 n$.

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We call a properly edge-coloured bipartite graph G with parts X, Y with $|X| = |Y| = n$ and n colors coloured (ε, p, n)-typical **coloured (ε, p, n)-regular** if

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Theorem (Frankl and Rödl, 1985 and Pippenger,unpublished)

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At this point better results are known in the literature where $o(1)$ term can be take to be $n^{-\gamma}$, for some small $\gamma > 0$.

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Note that our first main result is a direct corollary of this statement since $K_{n,n}$ is (ε, p, n)-typical for $p = 1$ and (any) ε.

Corollary (Keevash, Pokrovskiy, Sudakov, Y.)

Every properly n-edge-coloured $K_{n,n}$ has a rainbow matching of size $n - O(\frac{\log n}{\log \log n})$.

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With some more work, via random sampling we can also obtain the second result.

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I will sketch the proof of this result for $G = K_{n,n}$ and the size of the resulting matching being $n - O(\log n)$ rather than $n - O(\log n / \log\log n)$.

Rödl's nibble, back to 1985

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- **Repeat:** Repeatedly take bites and delete the vertices/colours used in each bite from the rest of the graph to obtain M until the remainder has size $< O(n^{1-\varepsilon})$, i.e. $|M| \geq n - O(n^{1-\varepsilon})$.

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Our key new idea: We show that M_0 (and thus, M) has nice **pseudorandom properties w.r.t. colours** which allows us to do further modifications to M until the remainder is $O(\log n)$.

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(S2) Delete vertices and colours of M_0 from $K_{n,n}$. The remaining graph will be colour-typical, therefore we can extend M_0 to a larger rainbow matching M of size $n - n^{1-\varepsilon}$ via Rödl's nibble as a black box. The pseudorandom properties that M_0 had get transferred to M.

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(S3) Do **switching-type** arguments to increase M as long as we have $\log n$ unused colours. We do this iteratively, at each step obtaining a rainbow matching of size $|M_i| + 1$ but such that the edit distance between each M_i and M is still sufficiently small.

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(S4) After at most $O(n^{1-\varepsilon})$ times we get a matching with remainder at most $O(\log n)$.

Expansion

Definition (Expander)

Suppose we are given $K_{n,n}$ with bipartition (X, Y). For any matching M in $K_{n,n}$ and a set of d colours D we say (D, M) is an expander if

- every vertex set $S \subseteq X$ or $S \subseteq Y$ with $|S| \approx n/d$

$$|N_{D,M,D,M}(S)| = (1 - o(1))n.$$

Here $N_{D,M,D,M}(S)$ is defined as the set of vertices which can be reached from some $s \in S$ via a $D - M$-alternating rainbow path of length four. We will use $d = \log n$.

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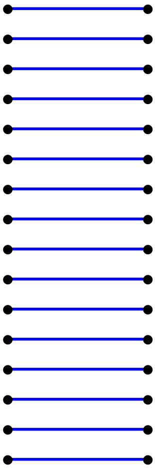
$|S| \approx n/d$



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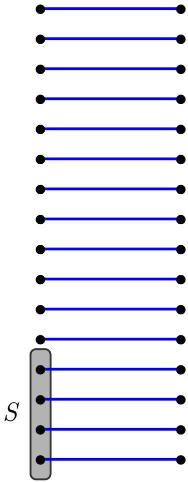

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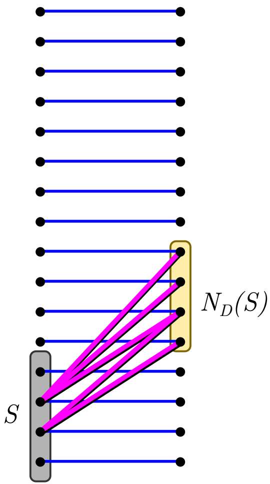

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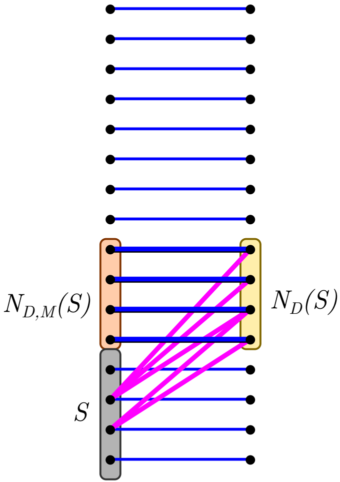

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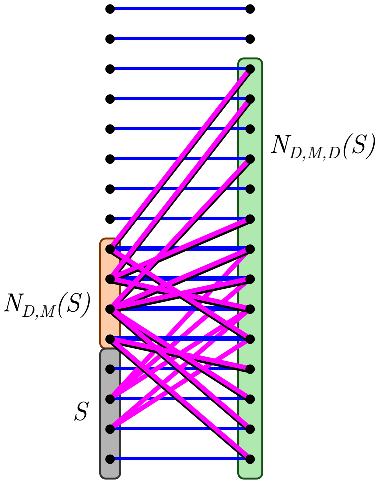

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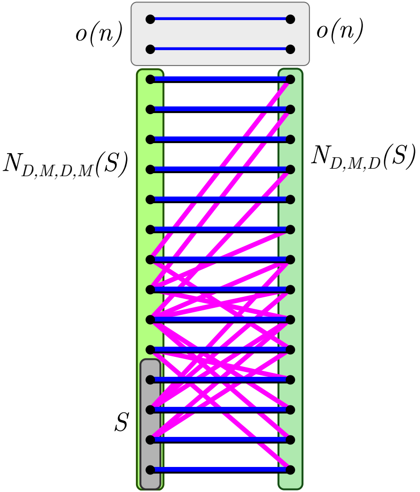

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- every vertex set $S \subseteq X$ or $S \subseteq Y$ with $|S| \approx n/d$ has

$$|N_{D,M,D,M}(S) = (1 - o(1))n.$$

Expansion

Actually our definition of expander is slightly more technical.

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- every vertex set $S \subseteq X$ or $S \subseteq Y$ with $|S| \approx n/d$ has a subset S' with $|S'| \approx n/d^2$ and

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Here $N_{D,M,D,M}(S')$ is defined as the set of vertices which can be reached from some $s \in S'$ via a $D - M$-alternating rainbow path of length four. We will use $d = \log n$.

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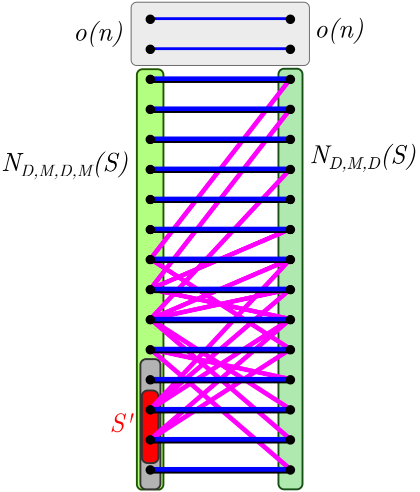
$|S| \approx n/d$, $|S'| \approx n/d^2$



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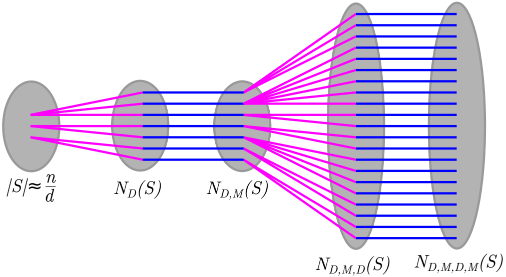
Lemma

For any set of d colours D, (D, M) is an expander.

For proving this lemma we only analyse M_0.

Rainbow paths from expansion

Expansion properties of M allow us to get short rainbow $D - M$-alternating paths between almost all vertices, for any set of colours D. Here is how.



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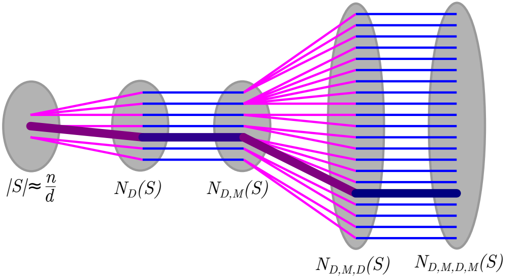
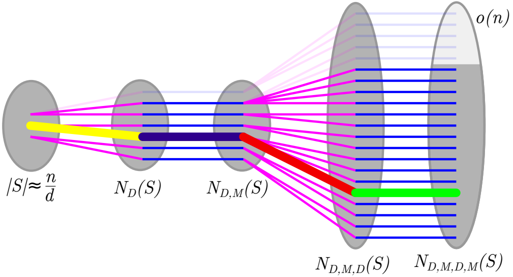


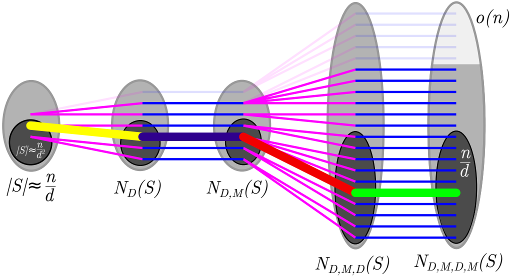
Rainbow paths from expansion

Expansion properties of M allow us to get short rainbow $D - M$-alternating paths between almost all vertices, for any set of colours D. Here is how.



Rainbow paths from expansion

We can iterate this step: find S' of size n/d^2 such that almost all vertices have rainbow $D-M$-alternating paths of length eight going to S'.



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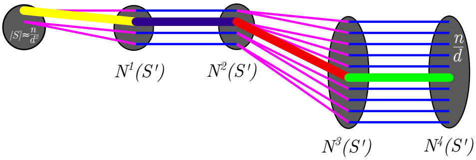

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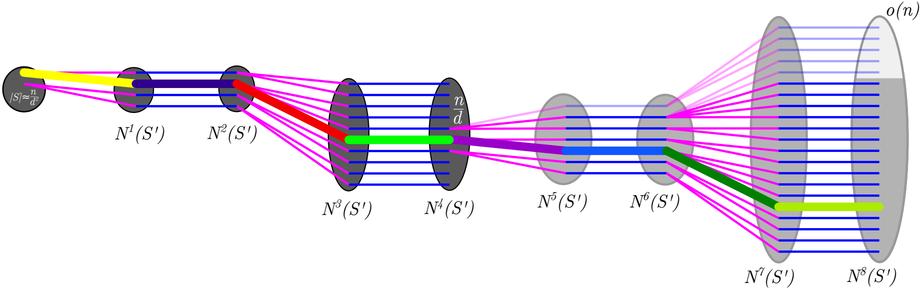
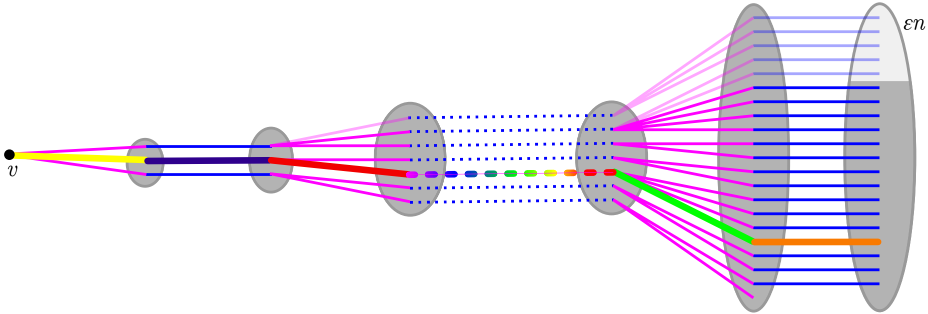


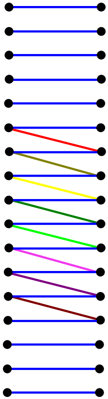
Rainbow paths from expansion

Applying this t times, we reach to S_t, such that $|S_t| = 1$ (i.e. $n/d^t = 1$ which implies $t \approx \log n / \log d = O(\log n / \log \log n)$). This shows that all but $o(n)$ vertices have rainbow $D - M$-alternating paths of length at most $4t$ going to all but $o(n)$ vertices.



Rainbow paths from expansion

This implies for almost all $x \in X$ and $y \in Y$ there is a rainbow $D - M$-alternating path between x and y of length at most $O(\log n / \log \log n)$. This allows us to do modifications to M via augmenting paths.



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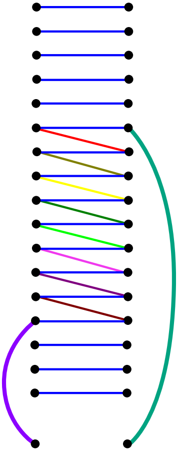

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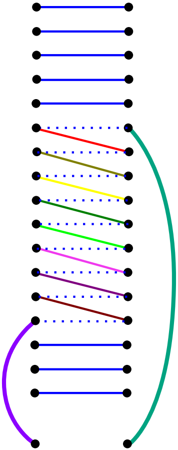
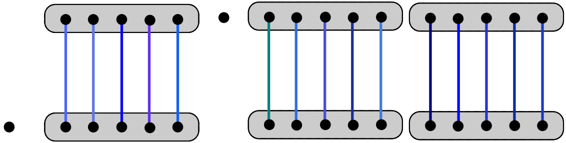


Switchings

Preprocessing step: We split the graph $K_{n,n}$ into three random pieces G_1, G_2, G_3 by selecting each colour/vertex with probability $1/3$. Then find a rainbow matching M as described before with expansion properties and $|M| \geq n - n^{1-\varepsilon}$.

Switchings

As long as there are some $|D| = \log n$ many colours unused on M we can do the following switchings.



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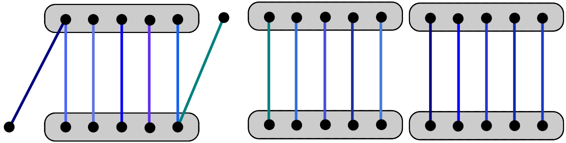

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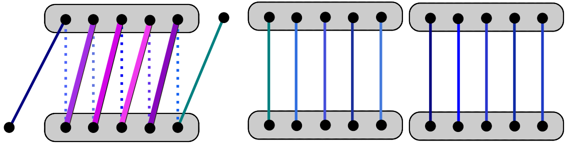

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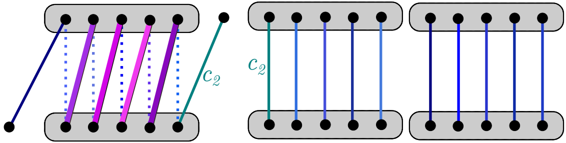

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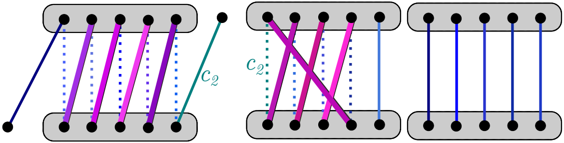

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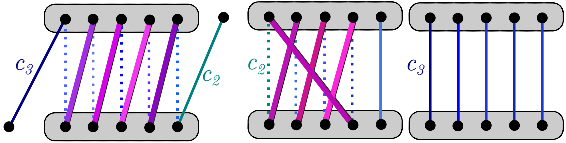

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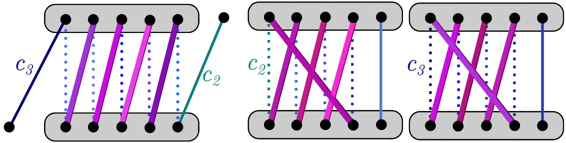

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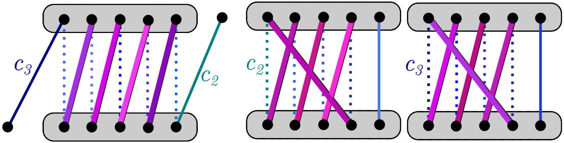

At each step $|M_i \triangle M_{i+1}| = O(\log n / \log\log n)$. Because of this after $O(n^{1-\varepsilon})$ steps, $|M_i \triangle M| \leq O(n^{1-\varepsilon} \log n / \log\log n) \ll |M|$, thus M_i will still have the expansion properties we discussed before thus we can find the alternating paths.

Recap: proof sketch

Find rainbow matchings of size $n - O(\log n)$ in properly n-edge coloured $K_{n,n}$.

(S1) Obtain M_0 rainbow matching via the first bite and show it satisfies certain pseudorandom properties w.r.t. colours, we call it **expansion properties**.

(S2) Delete vertices and colours of M_0 from $K_{n,n}$. The remaining graph will be colour-typical, therefore we can extend M_0 to a larger rainbow matching M of size $n - n^{1-\varepsilon}$ via Rödl's nibble as a black box. The pseudorandom properties that M_0 had get transferred to M.

(S3) Do **switching-type** arguments to increase M as long as we have $\log n$ unused colours. We do this iteratively, at each step obtaining a rainbow matching of size $|M_i| + 1$ but such that the edit distance between each M_i and M is still sufficiently small.

(S4) After at most $O(n^{1-\varepsilon})$ times we get a matching with remainder at most $O(\log n)$.

Our results

Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

If G is a coloured typical bipartite graph then it has a rainbow matching of size $n - O(\log n / \log\log n)$.

Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

Every properly n-edge-coloured $K_{n,n}$ has a rainbow matching of size $n - O(\frac{\log n}{\log\log n})$.

Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

Every Steiner triple system on n vertices has a matching of size at least $n/3 - O(\log n / \log\log n)$.

Further applications of our methods

Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

Let $\mathcal{H}$ be a 3-uniform linear hypergraph on n vertices. Suppose that

(1) *for every vertex v we have $|N_{\partial\mathcal{H}}(v)| = (1 \pm n^{-\varepsilon})pn$*

(2) *for every pair of vertices u, v, $|N_{\partial\mathcal{H}}(v)| = (1 \pm n^{-\varepsilon})pn$ and $|N_{\partial\mathcal{H}}(u) \cap N_{\partial\mathcal{H}}(v)| = (1 \pm n^{-\varepsilon})p^2 n$.*

Then $\mathcal{H}$ has a matching of size $n - O(\log n / \log\log n)$.

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Theorem (Keevash, Pokrovskiy, Sudakov, Y.)

There exists some k such that every $n \times n$ Latin array filled with $kn \log n / \log\log n$ many symbols contains a full transversal.

Previously known for Latin arrays filled with $n^{2-\varepsilon}$ symbols.

Open problems and further line of research

- Reduce, if possible, the error term in Ryser-Brualdi-Stein conjecture from $O(\log n / \log \log n)$ to some absolute constant c.
- Reduce, if possible, the error term in Brouwer's conjecture from $O(\log n / \log \log n)$ to some absolute constant c.
- Do linear 3-uniform regular hypergraphs have matching covering all but $n^{o(1)}$ vertices?

Thank you!