On the extension complexity of low-dimensional polytopes

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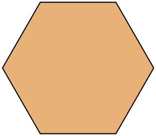
Joint work with Matthew Kwan and Yufei Zhao.

Introduction



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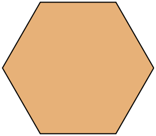

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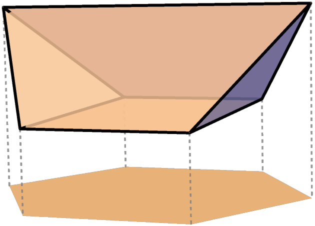

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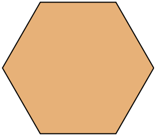

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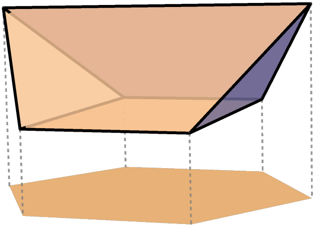
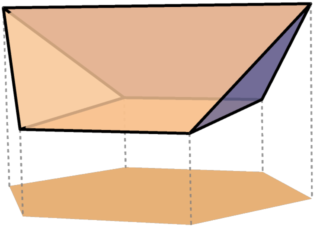


(Note that this construction only works for certain hexagons, for most hexagons it doesn't work).

A regular hexagon can be obtained as the projection of a 3-dimensional polytope with only 5 facets:



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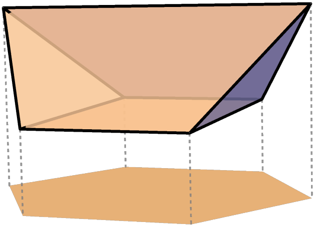

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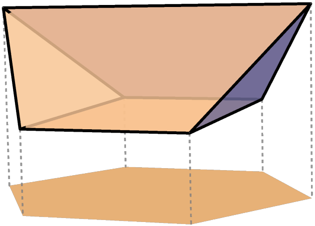

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Using the observation above, one can see that $\mathrm{xc}(\text{regular hexagon}) = 5$.

More generally, a regular (2-dimensional) n-gon has extension complexity $\Theta(\log n)$ (Ben-Tal, Nemirovski, 2001).

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In particular, there has been a lot of work on the extension complexity of specific polytopes relevant in optimization problems, like the correlation polytope, the traveling salesman polytope, and the perfect matching polytope.

In general, it is a difficult problem to determine the extension complexity of a given polytope.

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The answer to the question is n, since an $(n-1)$-dimensional n-vertex simplex has extension complexity n.

But what if we restrict the dimension of the polytope?

Open question

For a fixed dimension d, and large n, what is the maximum possible extension complexity of a d-dimensional polytope with n vertices?

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Theorem (Fiorini, Rothvoß, Tiwary, 2012)

Almost all n-gons have extension complexity at least $\Omega(\sqrt{n})$.

Theorem (Padrol, 2016)

Almost all d-dimensional polytopes with n vertices have extension complexity at least $\Omega(\sqrt{dn})$.

It seems plausible that a random polytopes are good candidates for having high extension complexity.

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This is also a natural question in itself, given the rich literature studying random polytopes.

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Model II

Choose m independent uniformly random points in the unit ball in $\mathbb{R}^d$, and let the polytope P be their convex hull.

In this model, P has a.a.s. $\Theta\left(m^{(d-1)/(d+1)}\right)$ vertices (Reitzner, 2005).

Our results

Fix a dimension $d \geq 2$.

Theorem (Kwan, S., Zhao, 2020+)

Let P be the convex hull of n random points on the unit sphere in $\mathbb{R}^d$. Then, a.a.s. $\mathrm{xc}(P) = \Theta(\sqrt{n})$.

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The lower bound in the second theorem is implied by a result of Padrol (2016), and for both theorems the lower bounds can easily be proved with an argument of Fiorini, Rothvoß and Tiwary (2012).

The interesting part of our results above is the *upper bound*.

For fixed dimension $d \geq 2$, for both of the two models of random polytopes that we considered, we proved that the extension complexity is a.a.s. on the order of the square root of the number of vertices.

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For a fixed dimension d, and large n, what is the maximum possible extension complexity of a d-dimensional polytope P with n vertices? Is the extension complexity always $O(\sqrt{n})$?

For $d = 2$, the best upper bound is $O(n^{2/3})$ (Shitov, 2020). For $d \geq 3$, no nontrivial upper bounds are known.

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It seems plausible that random polytopes exhibit the maximum possible extension complexity, but this is not at all clear.

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We proved that the answer is yes for $d = 2$.

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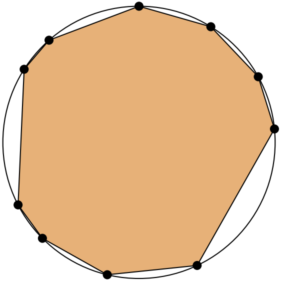

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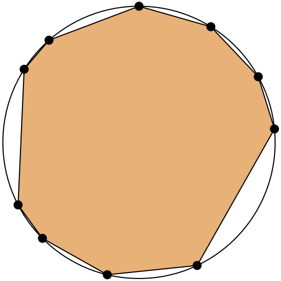

The proof follows a somewhat similar strategy as for our results for random polytopes.

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We proved that there are polytopes of dimension $n^{o(1)}$ such that the extension complexity is close to n:

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For any n, there exists a polytope with at most n vertices, dimension at most $n^{o(1)}$, and extension complexity at least $n^{1-o(1)}$.

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For given n, what is the minimum dimension d such that there exists a d-dimensional polytope with n vertices and extension complexity equal to n?

Open question

What about the extension complexity of random n-vertex polytopes if the dimension d grows slowly with n?

Nonnegative rank

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Definition

The *rank* of an $m \times n$ matrix $M \in \mathbb{R}^{m\times n}$ is the minimum number r such that there is a factorization $M = TU$ with matrices $T \in \mathbb{R}^{m\times r}$ and $U \in \mathbb{R}^{r\times n}$.

Definition

The *nonnegative rank* of a nonnegative $m \times n$ matrix $M \in \mathbb{R}_{\geq 0}^{m\times n}$ is the minimum number r such that there is a factorization $M = TU$ with nonnegative matrices $T \in \mathbb{R}_{\geq 0}^{m\times r}$ and $U \in \mathbb{R}_{\geq 0}^{r\times n}$.

The nonnegative rank of a matrix $M \in \mathbb{R}_{\geq 0}^{m\times n}$ is at least its (oridnary) rank, but may be much larger.

Given a polytope $P \subseteq \mathbb{R}^d$, we can describe P by a list of linear constraints (inequalities) corresponding to the facets of P.

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In order to prove the our desired upper bounds for the extension complexity, we bound the nonnegative rank of the slack matrices of the polytopes.

Proof overview for our random polytope results

Theorem (Kwan, S., Zhao, 2020+)

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Let P be the convex hull of m random points in the unit ball in $\mathbb{R}^d$, and let $n = m^{(d-1)/(d+1)}$. Then, a.a.s. $\operatorname{xc}(P) = \Theta(\sqrt{n})$.

In the first theorem, all vertices of P are on the unit sphere. In the second theorem, all vertices of P are typically very close to the unit sphere.

In both cases, the vertices of P are fairly well-distributed on (or close to) the unit sphere.

The proofs of these two theorems are very similar (but the second theorem requires a bit more work).

Fix a dimension $d \geq 2$. Let P be a d-dimensional polytope with vertices on (or close to) the unit sphere.

Let V be the set of vertices of P, and F the set of facets of P, and M the slack matrix of P (with rows indexed by V and columns indexed by F).

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The following is a key lemma for our argument, and is inspired by a similiar lemma due to Shitov (2014) for $d = 2$.

Lemma (Kwan, S., Zhao, 2020+)

Suppose $F' \subseteq F$ is a small "patch" of facets of P, and $V' \subseteq V$ is a set of vertices of P which are "far away" from the facets in F'.

Consider the $V' \times F'$ submatrix $M[V', F']$ of the slack matrix M. Then the nonnegative rank of this submatrix $M[V', F']$ is $O(1)$.

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Shitov's version of this lemma for dimension $d = 2$ does not require the "far away" assumption (the vertices in V' just need to lie outside the "patch").

For $d > 2$, the proof of the lemma is geometrically much more involved.

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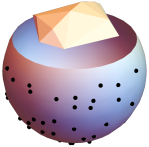

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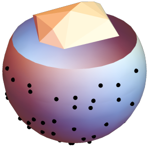
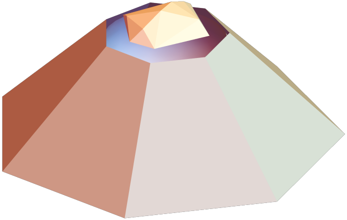
Proof sketch:
Construct a "polyhedral lampshade" polytope Q with $O(1)$ vertices, such that

- Q contains all vertices in V'.
- Q lies entirely on the "positive slack" side of all facets in F'.





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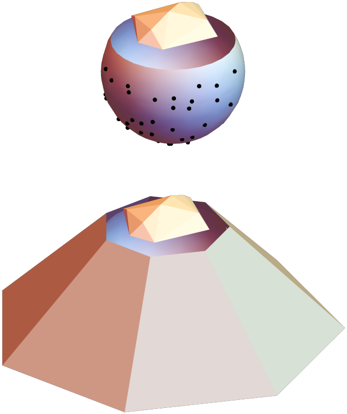

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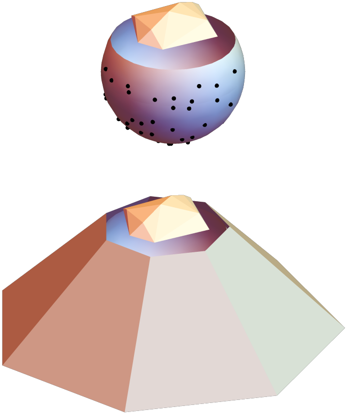

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So the vector of slacks of v with respect to the factes in F' is a convex combination of the "slack vectors" of the vertices of Q.

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However, the challenge is to deal with the slacks of vertices with respect to nearby facets. To handle this, we actually need a stronger version of the lemma (allowing to make certain subtractions from the matrix).

With the stronger (and more technical) version of the lemma, we can then show that the slack matrix of P has nonnegative rank at most $O(\sqrt{n})$.

Thank you very much for your attention!