Turán numbers of sunflowers

Matija Bucić

Institute for Advanced Study and Princeton University

joint work with Domagoj Bradač and Benny Sudakov

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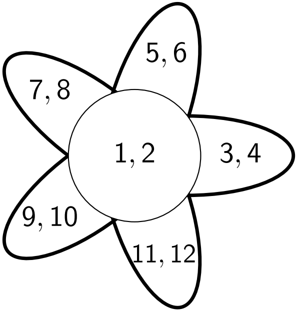

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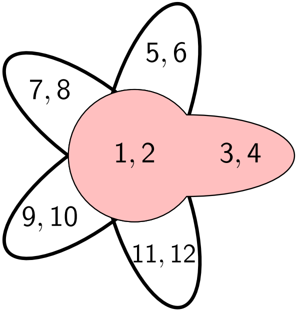

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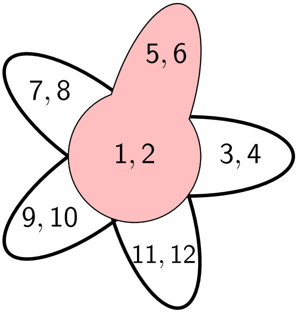

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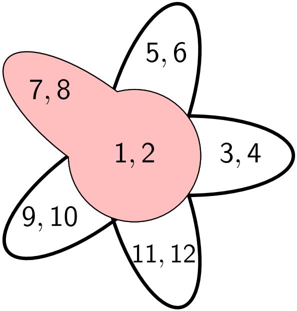

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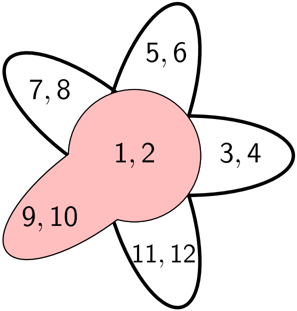

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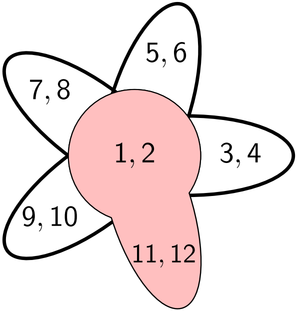

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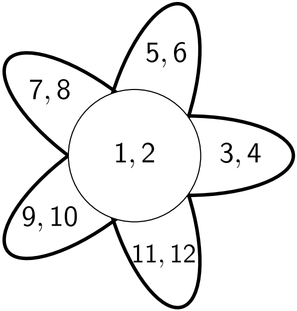

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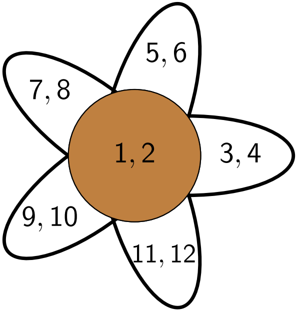

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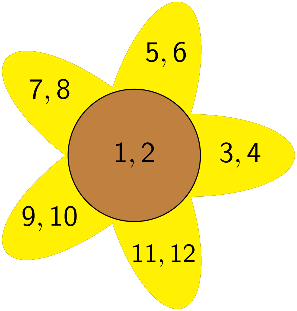

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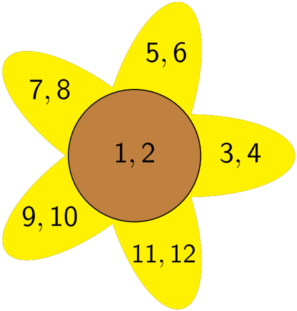

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- *r-uniform* if all sets have size r.

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- Relations to many topics in computer science and probability theory.

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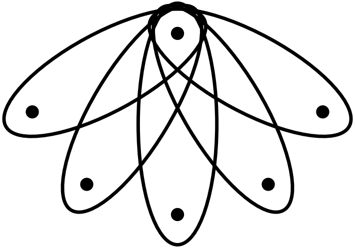

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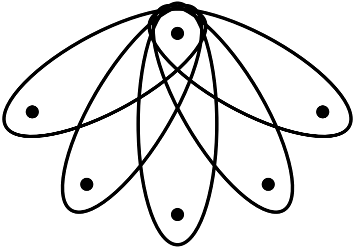

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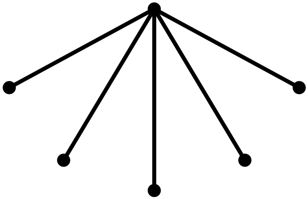

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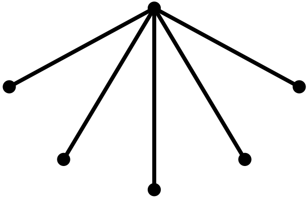

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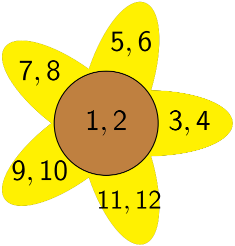

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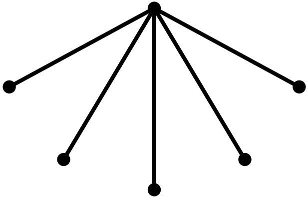

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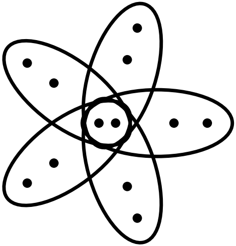

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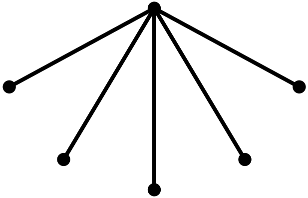

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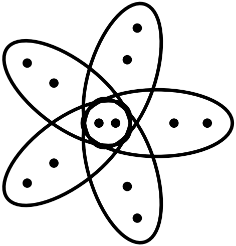

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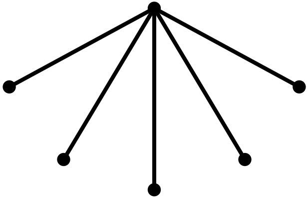

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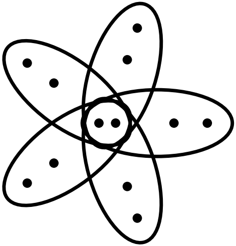

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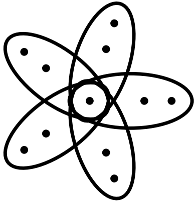

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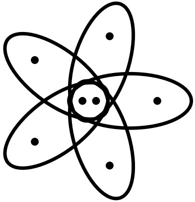

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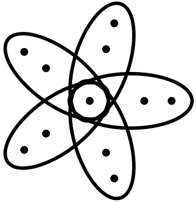

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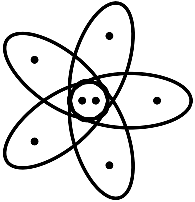

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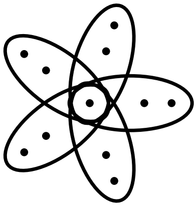

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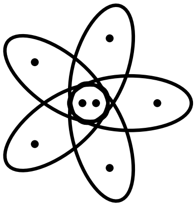

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  - Chung and Frankl determined $\mathrm{ex}(n, \mathcal{S}_1^{(3)}(k))$ precisely.

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1,t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n?
- If $r = 2$ it is trivial to see $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size
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- The $r = 4$ case solved approximately by B., Draganić, Sudakov and Tran.

Main result

Theorem (Bradač, B. and Sudakov)

$$ex(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1}k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

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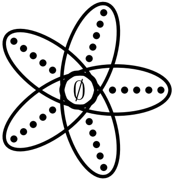

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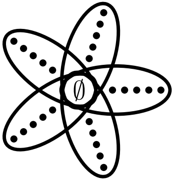

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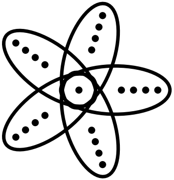

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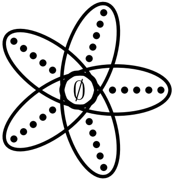

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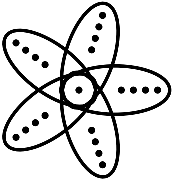

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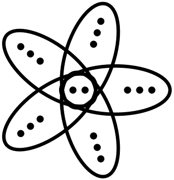

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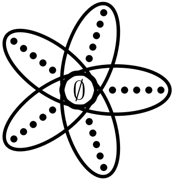

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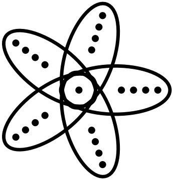

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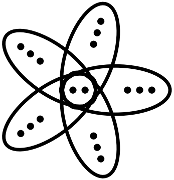

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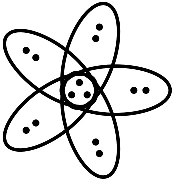

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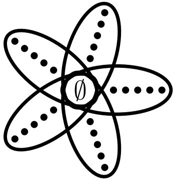

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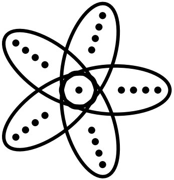

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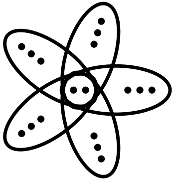

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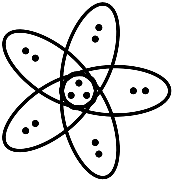

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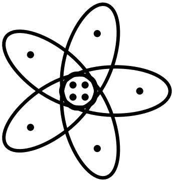

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Upper bounds: overview

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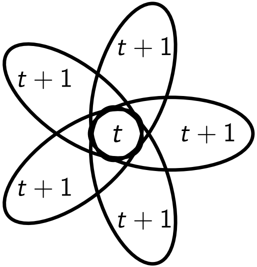

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- **Step 3:** Show there are no $(t+1, t)$-systems on ground set of size $N = 2t + 1$

Reduction to the existence problem for a $(t+1, t)$-system

Lemma

No $(t+1, t)$-system on $2t+1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

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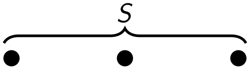

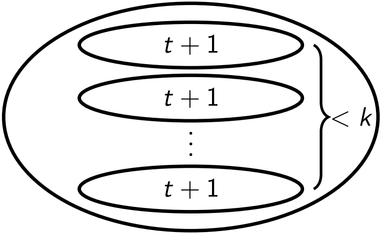

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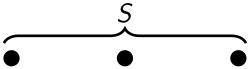
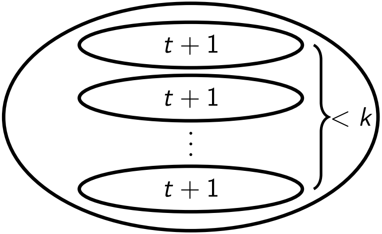
  - Taking the union of a maximal vertex disjoint collection gives τ_S.





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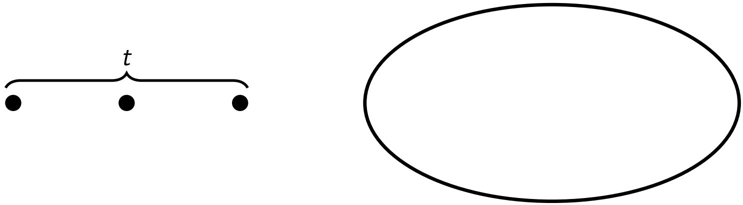

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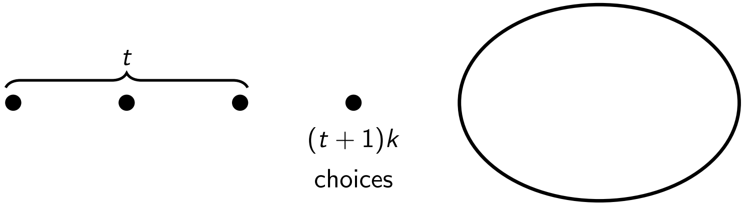

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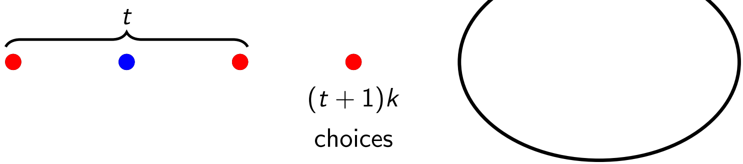

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- Let $H = (V, E)$ be an n-vertex, $2t+1$-uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t+1)k$.
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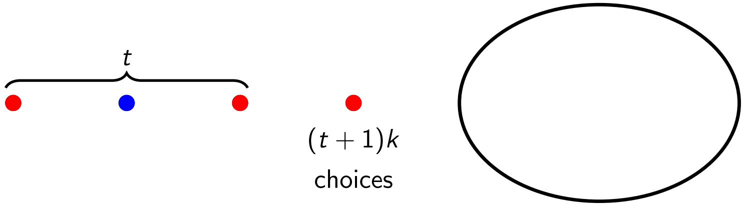

Reduction to the existence problem for a $(t+1,t)$-system

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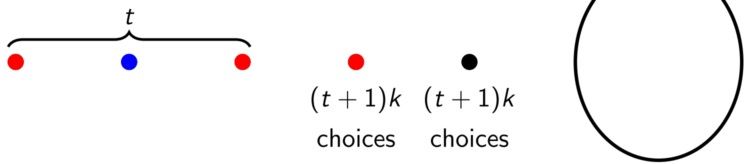

Non-existence of $(t+1, t)$-systems

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- Known for $r \leq 4$, up to constant factor.


рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
faafetai lava
mersi
barka
welalin
tack
spas
kia ora
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
kiitos
dankie
thank you
gracias
хвала
nanni
nandri
dhanyavad
asante
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bedankt
bayarlalaa
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köszönöm
aciu
dankon
akun
tenki
murakoze
enkosi
djiere dieuf
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mochchakkeram
dziękuję
дякую
mamnun
chokrane
chnorakaloutioun
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gràcies
sulpáy
go raibh maith agat
sobodi
dėkuji
obrigado
sagolun
sukriya
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tanemirt
grazie
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dakujem
trugarez
najis tuke
rahmet
kam sah hamnida
rahmat
terima kasih
diolch
dhanyavadagalu
shukriya
mercé
мерси
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감사합니다
xièxie
ευχαριστώ
merci
תודה רבה

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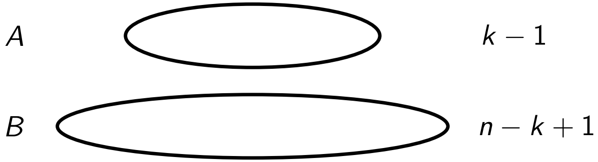

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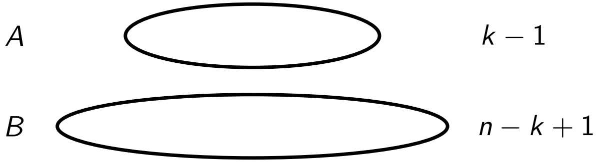

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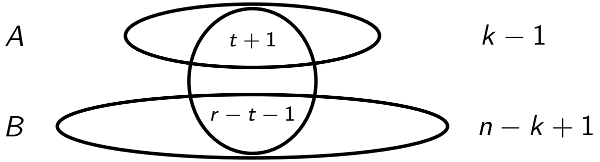

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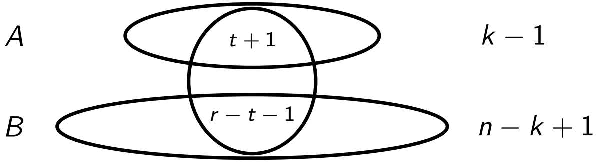

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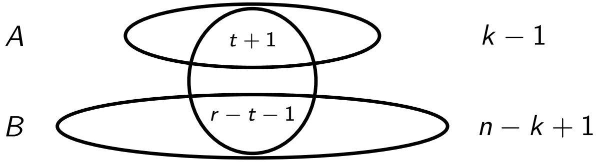

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- Choose as edges of our H any r-vertex subset of an edge of S.
- No $\mathcal{S}_t^{(r)}(k)$ as all its edges must come from the same edge of S.
- The number of edges is at least $\binom{n}{t}/\binom{m}{t} \cdot \binom{m}{r} \geq \Omega_r(n^t k^{r-t}) \implies$

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq \Omega_r(n^t k^{r-t}).$$