Turán numbers of sunflowers

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joint work with Domagoj Bradač and Benny Sudakov
Sunflowers

**Definition**

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets.

The common intersection is the kernel of the sunflower.

$r$-uniform if all sets have size $r$. 

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Oxford Discrete Math and Probability Seminar 2021
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\{1, 2, 7, 8\}
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\{1, 2, 11, 12\}
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Erdős-Rado sunflower conjecture

Question (Erdős-Rado, 1960)

What is the max size of a family of $r$-sets without a $k$ petal sunflower?

Denote the answer by $f_r(k)$.

Erdős-Rado sunflower lemma:

$$(k-1)r \leq f_r(k) \leq (k-1)r \cdot r!.$$ 

Best known upper bound is:

$f_r(k) \leq O(k \log r)$.

Conjecture (Sunflower conjecture, Erdős-Rado, 1960)

$f_r(k) \leq O(k^r)$.

Even $k = 3$ case is open and very interesting.

Relations to many topics in computer science and probability theory.
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Specific sunflowers

- Let $S_t^{(r)}(k)$ be the $r$-uniform sunflower with $k$ petals and kernel of size $t$. 
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![Sunflower diagram]
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![Diagram of sunflowers]

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Sunflower problem: What is the max number of edges in an $r$-graph without any of $S_0^{(r)}(k), S_1^{(r)}(k), \ldots, S_{r-1}(r)$?
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What is the max number of edges in an $n$-vertex $r$-graph without $S_t^{(r)}(k)$?
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What is the max number of edges in an n-vertex r-graph without $S^{(r)}_t(k)$?

- The answer is called the *Turán number* of $S^{(r)}_t(k)$, denoted $ex(n, S^{(r)}_t(k))$. 
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Turán problem for sunflowers

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- Captures several classical problems:
  - Case \( t = 0 \) corresponds to the Erdős matching conjecture.
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- Many results and bounds in various regimes.
- Frankl and Füredi 1985: For fixed \( r \) and \( k \) we have
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  \text{ex}(n, S_t^{(r)}(k)) \approx_{r,k} n^\max\{r-t+1,t\}.
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Chung, Erdős, Graham 1980’s: What if we let $k$ grow with $n$?
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• If $r = 3$ there are two types of sunflowers depending on kernel size
Large sunflowers

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$$S_1^{(3)}(k):$$

$$S_2^{(3)}(k):$$

$$\text{ex}(n, S_2^{(3)}(k)) \approx n^2 k$$
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  - $S_1^{(3)}(k)$:
    \[ \text{ex}(n, S_1^{(3)}(k)) \approx nk^2 \]

  - $S_2^{(3)}(k)$:
    \[ \text{ex}(n, S_2^{(3)}(k)) \approx n^2k \]
Frankl and Füredi 1985: For fixed $r$ and $k$ we have
\[ \text{ex}(n, S_t^{(r)}(k)) \approx r, k \cdot n^{\max\{r-t+1, t\}}. \]

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  - Chung determined $\text{ex}(n, S_1^{(3)}(k))$ up to lower order terms.
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- The $r = 4$ case solved approximately by B., Draganić, Sudakov and Tran.
Main result

**Theorem (Bradač, B. and Sudakov)**

\[ ex(n, S_t^{(r)}(k)) \approx_r \begin{cases} 
  n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\
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\[ \text{ex}(n, S_0^{(5)}(k)) \approx n^4 k \]
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\[ \text{ex}(n, S_{0}^{(5)}(k)) \approx n^{4}k \]
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\[\emptyset\]

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\begin{align*}
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\text{ex}(n, S_3^{(5)}(k)) &\approx n^3 k^2 & \text{ex}(n, S_4^{(5)}(k)) &\approx n^4 k
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Upper bounds: overview

**Theorem (Bradač, B. and Sudakov)**

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\]

- **Step 1:** Use induction to reduce to the balanced case:

  \[
  \text{ex}(n, S_t^{(2t+1)}(k)) \leq O(n^tk^{t+1}).
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Upper bounds: overview

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A balanced sunflower:
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- **Step 2:** Reduce the balanced case to an existence problem for \((t+1, t)\)-systems
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**Definition**

\( A \subseteq \mathcal{P}([N]) \) is a \((t + 1, t)\)-system if:
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Definition

\[ A \subseteq \mathcal{P}([N]) \text{ is a } (t + 1, t)\text{-system if:} \]

- \( A \) is intersection closed, i.e. \( \forall A, B \in A \) we also have \( A \cap B \in A \),
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  - any subset of \([N]\) of size at most \( t \) is contained in some set in \( A \) and
  - \( \forall A \in A \) we have \( |A| \not\equiv N \pmod{t + 1} \).
Step 1: Use induction to reduce to the balanced case:

\[ \text{ex}(n, S_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}). \]

Step 2: Reduce the balanced case to an existence problem for \((t + 1, t)\)-systems

### Definition

\(A \subseteq \mathcal{P}([N])\) is a \((t + 1, t)\)-system if:

- \(A\) is intersection closed, i.e. \(\forall A, B \in A\) we also have \(A \cap B \in A\),
- any subset of \([N]\) of size at most \(t\) is contained in some set in \(A\) and
- \(\forall A \in A\) we have \(|A| \equiv N \pmod{t + 1}\).

- Nägele, Sudakov, Zenklusen: no \((t + 1, t)\)-system exists if \(t + 1\) is a prime power
Upper bounds: overview

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**Definition**
\[ \mathcal{A} \subseteq \mathcal{P}([N]) \text{ is a } (t + 1, t)\text{-system if:} \]
- \(\mathcal{A}\) is intersection closed, i.e. \(\forall A, B \in \mathcal{A} \text{ we also have } A \cap B \in \mathcal{A},\)
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- **Step 3:** Show there are no \((t + 1, t)\)-systems on ground set of size \(N = 2t + 1\)
Lemma

No $(t+1, t)$-system on $2t+1$ points $\implies \text{ex}(n, S^{(2t+1)}(k)) \leq O(n^t k^{t+1})$
Let $H = (V, E)$ be an $n$-vertex, $2t + 1$-uniform, $S_t^{(2t+1)}(k)$-free hypergraph.
Lemma

No \((t + 1, t)\)-system on \(2t + 1\) points \(\implies\) \(\text{ex}(n, S_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})\)

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Reduction to the existence problem for a \((t + 1, t)\)-system

**Lemma**

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Reduction to the existence problem for a \((t + 1, t)\)-system

**Lemma**

*No \((t + 1, t)\)-system on \(2t + 1\) points* \(\implies\) \(\text{ex}(n, S_{t}^{(2t+1)}(k)) \leq O(n^t k^{t+1})\)

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  - Among \(t + 1\)-sets extending \(S\) into an edge there are no \(k\) vertex disjoint ones
  - Taking the union of a maximal vertex disjoint collection gives \(\tau_{S}\).

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Reduction to the existence problem for a \((t + 1, t)\)-system

**Lemma**

No \((t + 1, t)\)-system on \(2t + 1\) points \(\implies\) \(\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})\)

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Reduction to the existence problem for a \((t + 1, t)\)-system

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Reduction to the existence problem for a \((t + 1, t)\)-system

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\(t\) choices
Reduction to the existence problem for a \((t + 1, t)\)-system

**Lemma**

No \((t + 1, t)\)-system on \(2t + 1\) points \(\implies\) \(\text{ex}(n, S_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})\)

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\[ t \]

\[(t + 1)k \]
choices
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Non-existence of \((t + 1, t)\)-systems

- Let \(t + 1 = p^\alpha\), for \(p\) prime and assume that \(A \subseteq \mathcal{P}([N])\) satisfies:
  1. \(A\) is intersection closed
  2. all \(t\)-subsets of \([N]\) are covered
  3. \(\forall A \in A\) we have \(|A| \not\equiv N \pmod{t + 1}\)

By adding dummy vertices to every \(A \in A\) we may assume \(N \equiv -1 \pmod{t + 1}\).

Let \(A = \{A_1, \ldots, A_m\}\).

Double counting the \# of \(t\)-sets covered by some \(A_i\):

- \(N_t = \# \text{ of covered } t\)-subsets

Lucas' theorem implies:

- \(a_t \equiv 0 \pmod{p}\) \iff \(a \not\equiv -1 \pmod{p^\alpha}\)
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\[
\binom{N}{t} = \# \text{ of covered } t\text{-subsets}
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Non-existence of \((t + 1, t)\)-systems

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\]

\[
= \binom{|A_1|}{t} + \binom{|A_2|}{t} + \ldots + \binom{|A_m|}{t} - \ldots
\]

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- Let \(\mathcal{A} = \{A_1, \ldots, A_m\}\). Double counting the \(\#\) of \(t\)-sets covered by some \(A_i\):

\[
\binom{N}{t} = \# \text{ of covered } t\text{-subsets} \n= \binom{|A_1|}{t} + \binom{|A_2|}{t} + \ldots + \binom{|A_m|}{t} - \binom{|A_1 \cap A_2|}{t} - \binom{|A_1 \cap A_3|}{t} - \ldots - \binom{|A_{m-1} \cap A_m|}{t} + \ldots
\]
Non-existence of \((t+1, t)\)-systems

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(-1)^{m-1} \binom{|A_1 \cap \ldots \cap A_m|}{t}
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\[
\binom{N}{t} = \# \text{ of covered } t \text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{|\bigcap_{i \in I} A_i|}{t}
\]
Non-existence of \((t + 1, t)\)-systems

Let \(t + 1 = p^\alpha\), for \(p\) prime and assume that \(\mathcal{A} \subseteq \mathcal{P}(\{N\})\) satisfies:

- \(\mathcal{A}\) is intersection closed
- all \(t\)-subsets of \([N]\) are covered
- \(\forall A \in \mathcal{A}\) we have \(|A| \not\equiv N \pmod{t + 1}\)

By adding dummy vertices to every \(A \in \mathcal{A}\) we may assume \(N \equiv -1 \pmod{t + 1}\)

Let \(\mathcal{A} = \{A_1, \ldots, A_m\}\). Double counting the \# of \(t\)-sets covered by some \(A_i\):

\[
\binom{N}{t} = \# \text{ of covered } t\text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{\bigcap_{i \in I} A_i}{t}
\]

Lucas’ theorem implies: \(\binom{a}{t} \equiv 0 \pmod{p}\) \(\iff a \not\equiv -1 \pmod{p^\alpha}\)
Non-existence of \((t + 1, t)\)-systems

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- Lucas’ theorem implies: \(\binom{a}{t} \equiv 0 \pmod{p} \iff a \not\equiv -1 \pmod{p^\alpha}\)
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Lucas’ theorem implies: \(\binom{a}{t} \equiv 0 \pmod{p}\) \(\iff\) \(a \not\equiv -1 \pmod{p^\alpha}\)
We determined the dependency of \( \text{ex}(n, S_t^{(r)}(k)) \) on \( n \) and \( k \).
Further directions

- We determined the dependency of $\text{ex}(n, S_t^{(r)}(k))$ on $n$ and $k$.

**Problem 1**

*What is the dependency on $r$?*
Further directions

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Problem 1

*What is the dependency on \( r \)?*

Problem 2

*What if we forbid a collection of \( r \)-uniform sunflowers?*
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What is the dependency on $r$?

Problem 2
What if we forbid a collection of $r$-uniform sunflowers?

Problem 3 (Chung-Erdős unavoidability problem, 1983)
Among $r$-uniform hypergraphs with $e$ edges which is hardest to avoid?
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What if we forbid a collection of $r$-uniform sunflowers?

Problem 3 (Chung-Erdős unavoidability problem, 1983)

Among $r$-uniform hypergraphs with $e$ edges which is hardest to avoid?

- Known for $r \leq 4$, up to constant factor.
Theorem (Bradač, B. and Sudakov)

\[
ex(n, S^{(r)}_t(k)) \approx_r \begin{cases} 
  n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\
  n^t k^{r-t} & \text{if } t > \frac{r-1}{2}.
\end{cases}
\]
First lower bound

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- Partition the vertex set into $A$ and $B$ s.t. $|A| = k - 1$ and $|B| = n - k + 1$
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![Diagram](visual_representation)
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- No $S_t^{(r)}(k)$ since every petal needs to have a vertex in $A$.
- The number of edges is at least \( \binom{k-1}{t+1}\binom{n-k+1}{r-t-1} = \Omega_r(n^{r-t-1}k^{t+1}) \)

\begin{center}
\begin{tikzpicture}
  \node[draw, ellipse, minimum width=2cm, minimum height=2cm] (A) at (0,0) {$t + 1$};
  \node[draw, ellipse, minimum width=2cm, minimum height=2cm] (B) at (0,-3) {$r - t - 1$};
  \node[draw, ellipse, minimum width=2cm, minimum height=2cm] (C) at (3,0) {$k - 1$};
  \node[draw, ellipse, minimum width=2cm, minimum height=2cm] (D) at (3,-3) {$n - k + 1$};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (A);
\end{tikzpicture}
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\[
ex(n, S_t^{(r)}(k)) \geq \Omega_r(n^{r-t-1}k^{t+1}).
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Let \( m := |S_t^{(r)}(k)| - 1 \) and \( S \) be an \( m \)-uniform \( n \)-vertex hypergraph s.t.
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Let \( m := |S_t^{(r)}(k)| - 1 \) and \( S \) be an \( m \)-uniform \( n \)-vertex hypergraph s.t.

- any subset of \( t \) vertices is contained in precisely one edge of \( S \)
Second lower bound

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- No \( S_t^{(r)}(k) \) as all its edges must come from the same edge of \( S \).
- The number of edges is at least \( \binom{n}{t} / \binom{m}{t} \cdot \binom{m}{r} \geq \Omega_r(n^t k^{r-t}) \implies \text{ex}(n, S_t^{(r)}(k)) \geq \Omega_r(n^t k^{r-t}). \)