Paths in random temporal graphs

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Includes joint results with Nicolas Broutin and Gábor Lugosi.

1) THE ERDŐS - RÉNYI RANDOM GRAPH $G(n,p)$

- vertices $[n] = \{1, 2, \dots, n\}$
- edges: $ij \in E(G(n,p))$ with probability p; mutually independent

- A holds with high probability if $\mathbb{P}[A] \longrightarrow 1$ as $n \longrightarrow \infty$.

- $p = \frac{1}{n}$ is the 'phase transition'
  - for $pn < 1-\varepsilon$, component sizes are $O(\log n)$ whp.
  - for $pn > 1+\varepsilon$, $\exists$ path of size $c\varepsilon^2 n$ whp.

- $p = \frac{\log n}{n}$ is the 'connectivity threshold'
  $\{\text{connected graphs}\} \subseteq \{\text{min-degree-1 graphs}\}$

$$\mathbb{P}[G \text{ is connected}] \longrightarrow \begin{cases} 0, & pn < (1-\varepsilon)\log n \\ 1, & pn > (1+\varepsilon)\log n. \end{cases}$$

2) (RANDOM) TEMPORAL GRAPHS

- A temporal graph $G = (V, E, \omega)$ is a graph (V, E) with time stamps $(\omega_e)_{e \in E}$
- A temporal $(v_0 - v_k)$ path is a path $v_0 v_1 v_2 \dots v_{k-1} v_k$ with
  $\omega(v_{i-1} v_i) \leq \omega(v_i v_{i+1})$
  - we denote $v_0 \rightsquigarrow v_k$
  - ⊛ $\omega \longmapsto$ a permutation π on E

A random temporal graph is $G_p[W] = (V, E, W)$, where
W_e are iid, $W_e \sim Exp(1)$ and
$E = \{e \in K_n : W_e \leq -\log(1-p)\}$
⊛ $G_p[W] \longmapsto G(n,p)$ with random $\pi : E \to E$

Why? $\mathbb{P}[e \in E] = p$
$\mathbb{P}[\omega(e) < \omega(\ell)] = \frac{1}{2}$

CONNECTIVITY?
TYPICAL PATH LENGTHS?
DIAMETER?

3) PREVIOUS RESULTS

Theorem (Casteigts, Raskin, Renken, Zamaraev) ['phase transition(s)']

In $G_p[W]$, the threshold for

- $\{u \rightsquigarrow v$ for typical $(u,v)\}$ is $\log n / n$
- $\{u \rightsquigarrow v$ for all v & typical $u\}$ is $2 \log n / n$
- $\{u \rightsquigarrow v$ for all $(u,v)\}$ is $3 \log n / n$.

- 70's : 'Random Exchanges of Information' – Boyd & Steele; Haigh; Moon.

Theorem (Angel, Ferber, Sudakov, Tassion) ['longest paths']

For $1 \gg pn \gg \log n$, whp, the longest temporal path in $G_p[W]$ has length $\sim epn$.

Def. $L_{u,v}$ = max length of a u–v path

$\ell_{u,v}$ = min length of a u–v path

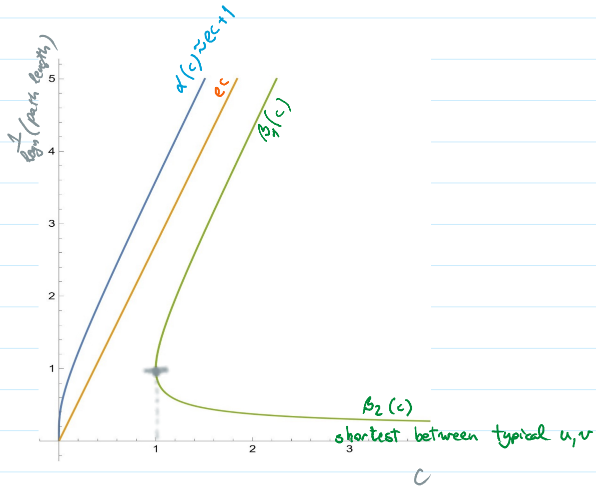
Theorem 1 (Broutin, K, Lugosi) For $p = \frac{c \log n}{n}$, whp in $G_p[W]$

(i) $\max_{u,v} L_{u,v} \sim \alpha(c) \log n$, where $\alpha \log \frac{\alpha}{ec} = 1$

(ii) $\max_{v} L_{1,v} \sim ec \log n$

(iii) $L_{1,2} \sim \beta_1(c) \log n$, where $\beta_1 \log \frac{\beta_1}{ec} = -1$, $c \geq 1$.

$(\& \ \beta_1 \geq 1)$



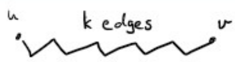
Theorem 2 (Braun, K, Lugosi) For $p = \frac{c \log n}{n}$, whp in $G_p[W]$

- $\max_{u,v} l_{u,v} \sim \beta_2(c-2) \log n$
- $\max_u l_{u,v} \sim (c-1) \log n$ for typical v
- $l_{u,v} \sim (c) \log n$ for typical u, v

METHODS & IDEAS

1) FIRST MOMENT $(np = c \log n)$

$S_k(u,v) = \#$ u-v paths of length $k = \theta \log n$

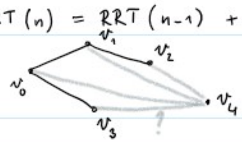


- $\mathbb{E}\left[\sum_{u,v} S_k(u,v)\right] \sim n^{k+1} \cdot \frac{p^k}{k!} \sim n \left(\frac{npe}{k}\right)^k = \exp\left(\log n + \log\left(\frac{ce}{\theta}\right) \cdot \theta \log n\right)$
- $\mathbb{E}\left[\sum_v S_k(1,v)\right] \sim \left(\frac{npe}{k}\right)^k$
- $\mathbb{E}\left[S_k(1,2)\right] \sim \frac{1}{n}\left(\frac{npe}{k}\right)^k$

Digression: for $np >> \log n$,
$L_{uv} = \max_{ij} L_{ij} \quad \forall ij.$

2^-) RANDOM RECURSIVE TREES

RRT (n) = RRT $(n-1)$ + random edge to new vertex v_n

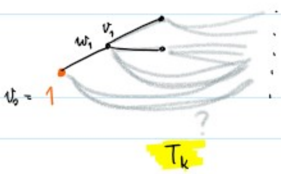


FACTS: Height of RRT (n) is $\sim e \log n$ whp

Most vertices are at distance $\sim \log n$ from v

2) GREEDY TREE IN $G_p[W]$, $p = \frac{\log n}{n}$

$W_e \sim$ Exp (1) $\qquad$ $W_e | W_e > t$ distributed as $t +$ Exp(1)



w_i = min time stamp of a useful edge from $\{v_0, v_1, \dots, v_n\}$

$\mathbb{E}_{i-1}[w_i - w_{i-1}] \sim \frac{1}{in}$ $\qquad (i << n)$

a) At what p do we discover $n^{1-\varepsilon}$ vertices?

$\mathbb{E}[w_k] \sim \frac{1}{n} + \frac{1}{2n} + \dots + \frac{1}{k} \sim \frac{\log k}{n}$

$\mathbb{E}[w_{n^{1-\varepsilon}}] \sim \frac{(1-\varepsilon)\log n}{n}$

$\mathbb{E}[w_{n^{1-\varepsilon}}] \sim \frac{(1-\varepsilon)\log n}{n}$.

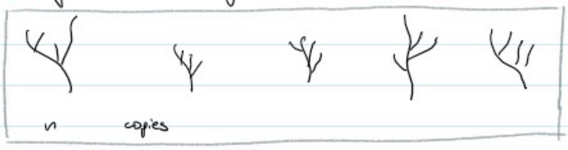
Rmk Can expand from $n^{1-\varepsilon}$ to $n(1-o(1))$ in "few" steps

$\Rightarrow$ "proves" phase transition @ $\frac{\log n}{n}$.

b) T_k distributed as RRT(k)!

$\hookrightarrow$ height of T_k & typical distances at 'time' $p = \frac{\log n}{n}$

(iii) $\max_{u,v} L_{u,v} = \alpha \log n$ = max height of n iid copies of RRT(n)



3) SECOND MOMENT + TWIST (for (i); $p = \frac{\log n}{n}$)

S_k = # temporal paths of length k ($\sim \alpha \log n$)

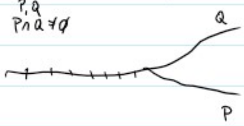
$S_k = \sum_{P \in \mathcal{P}_k} \mathbb{1}_{A_P}$ where $A_P = \{P \text{ is a temporal path}\}$

paths in K_n

$\mathbb{E} S_k = n \left(\frac{npe}{k}\right)^k$

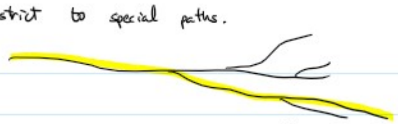
WANT $\operatorname{Var} S_k \leq \sum_{\substack{P,Q \\ P \cap Q \neq \emptyset}} \mathbb{P}[A_P \wedge A_Q] \ll (\mathbb{E} S_k)^2$

No!



'BAD INTERSECTION'

IDEA: restrict to special paths.



$B_P = \{P \text{ is a special temporal path}\} \subseteq A_P$

$P = e_1 \dots e_k$ is a special path of length $\alpha \log n$ if

(i) $W_{e_i} \approx \frac{i}{\alpha n}$ (rate $\frac{1}{\alpha n}$)

(ii) the 'branches of P' increase at rate $\geq \frac{1}{en}$.

Special paths

- forbid bad intersections:

$P, Q \in \mathcal{P}_k$ intersect in 1 segment $\Rightarrow \neg (B_P \wedge B_Q)$

- are still expected to appear:

$\mathbb{P}[B_P \mid A_P] \geq \delta \mathbb{P}[A_P]$.

$\Rightarrow$ Can use Chebyshev's Inequality to say $\sum_P B_P > 0$ whp.

$\sum_{\substack{P,Q \\ \text{intersecting}}} \mathbb{P}[B_P \wedge B_Q] \ll (\mathbb{E} S_k)^2$

OPEN PROBLEMS

- other models of RTG's
- Is there a 'giant component' at $p = \frac{\log n + C}{n}$?

(Becker, Casteigts, Crescenzi, Kodric, Renken, Raskin, Zamaraev)