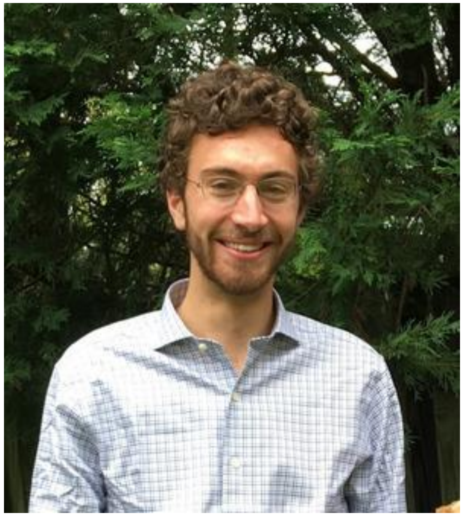
Random Friends Walking on Random Graphs

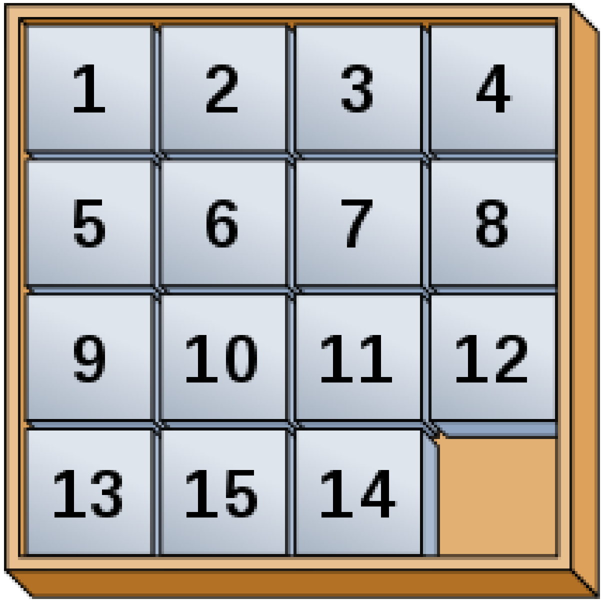
Noga Alon, Princeton and TAU

Joint work with Colin Defant and Noah Kravitz

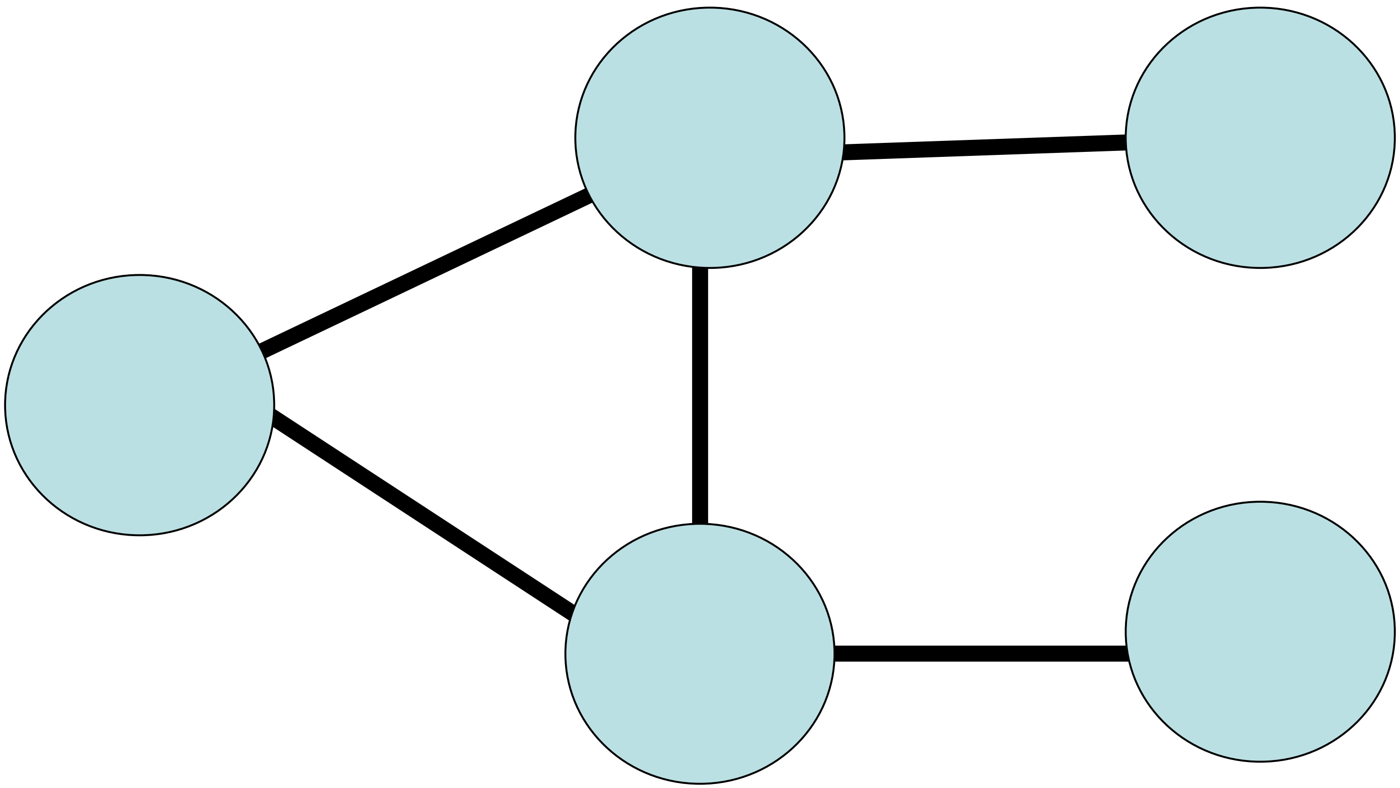


Oxford, Jan. 2021

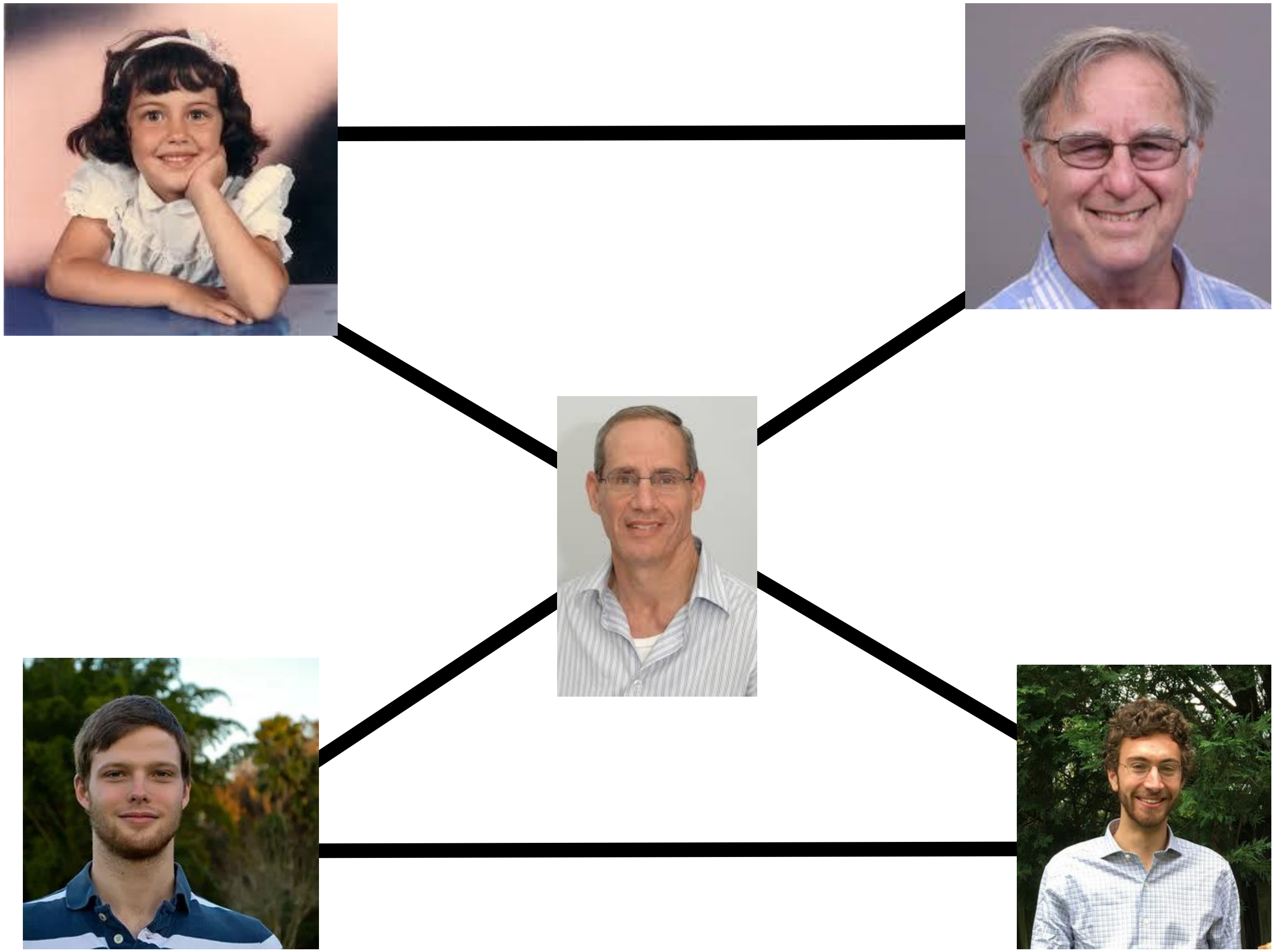
Friends and Strangers Graphs

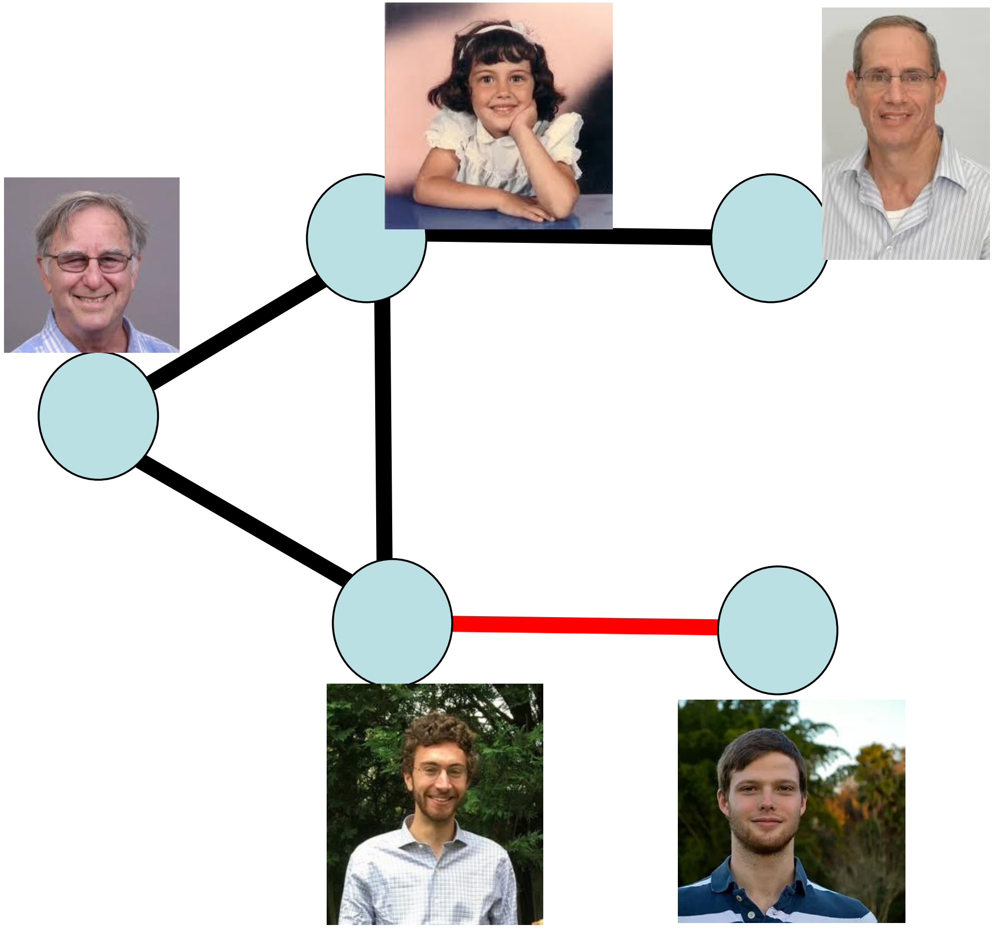


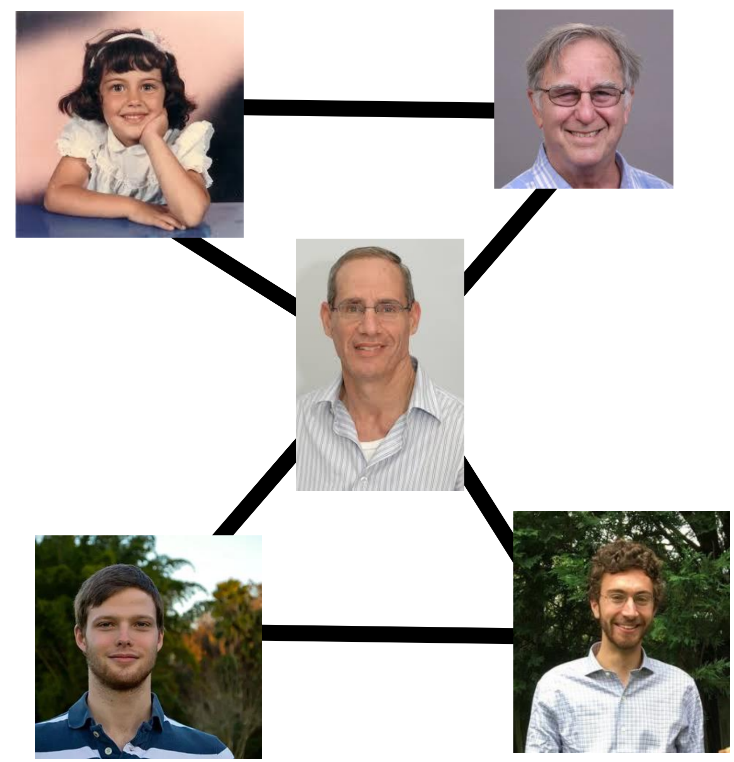
Example

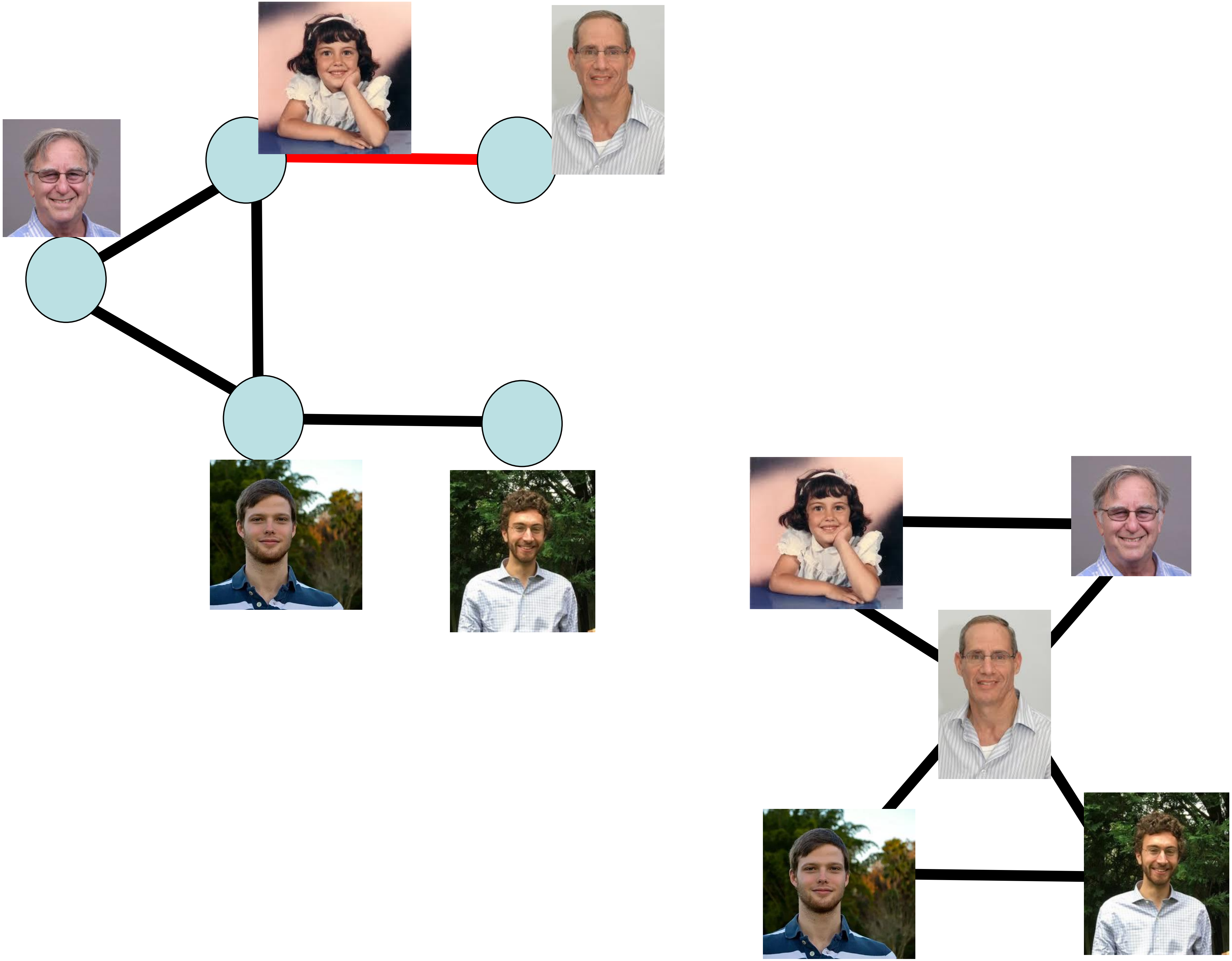


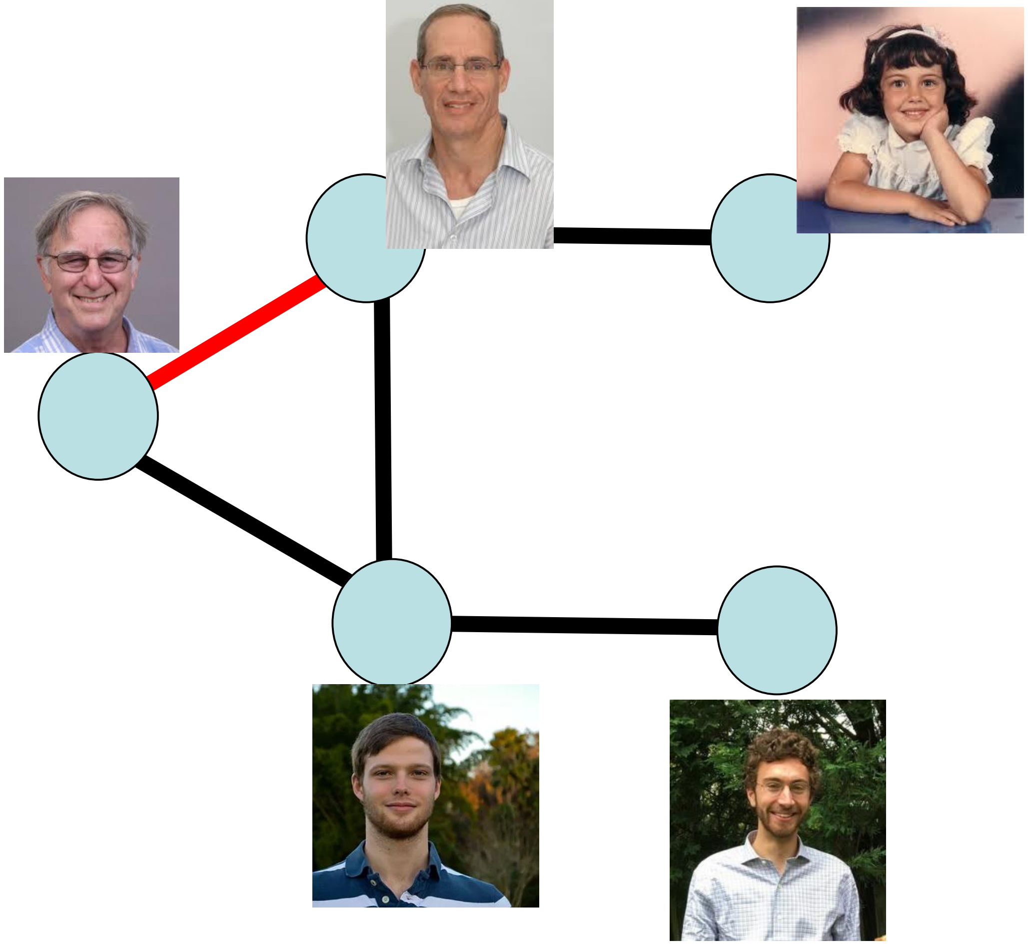
Friendship Graph, example

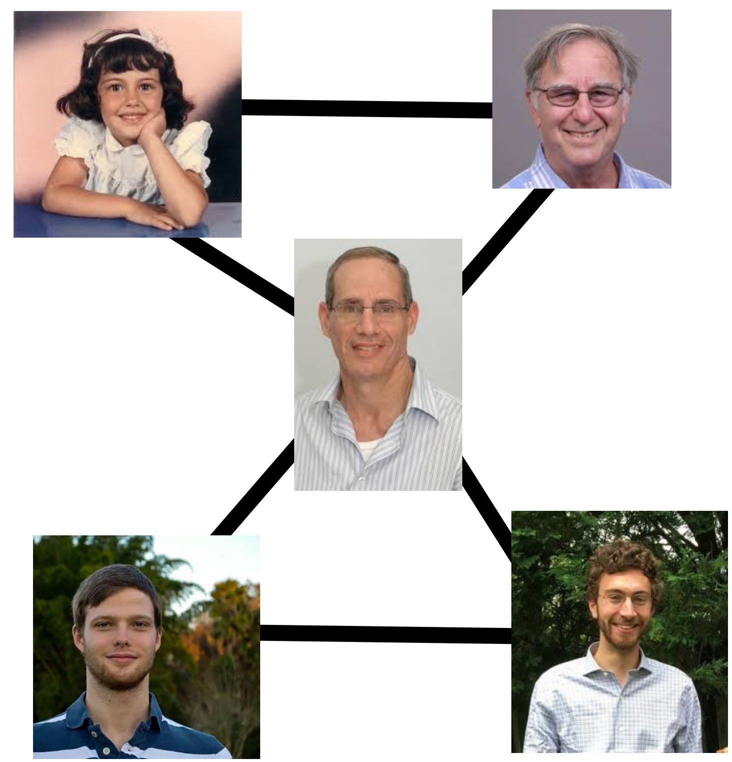


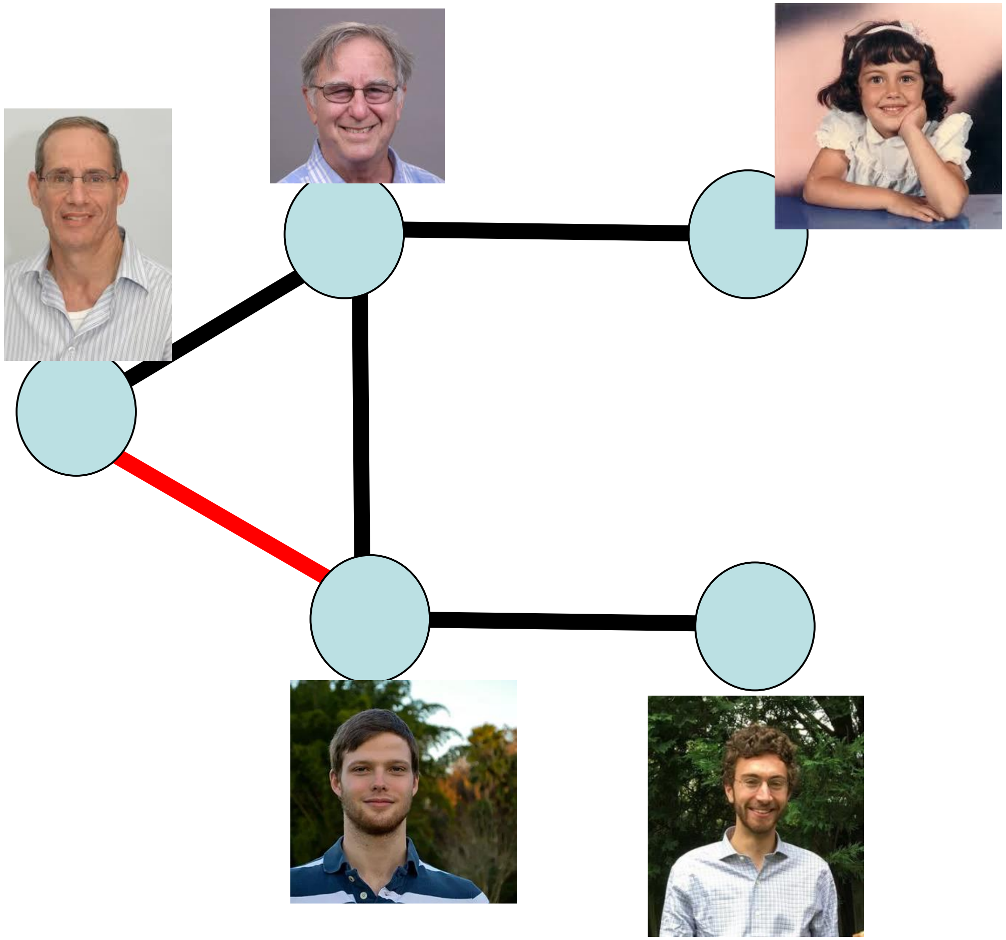


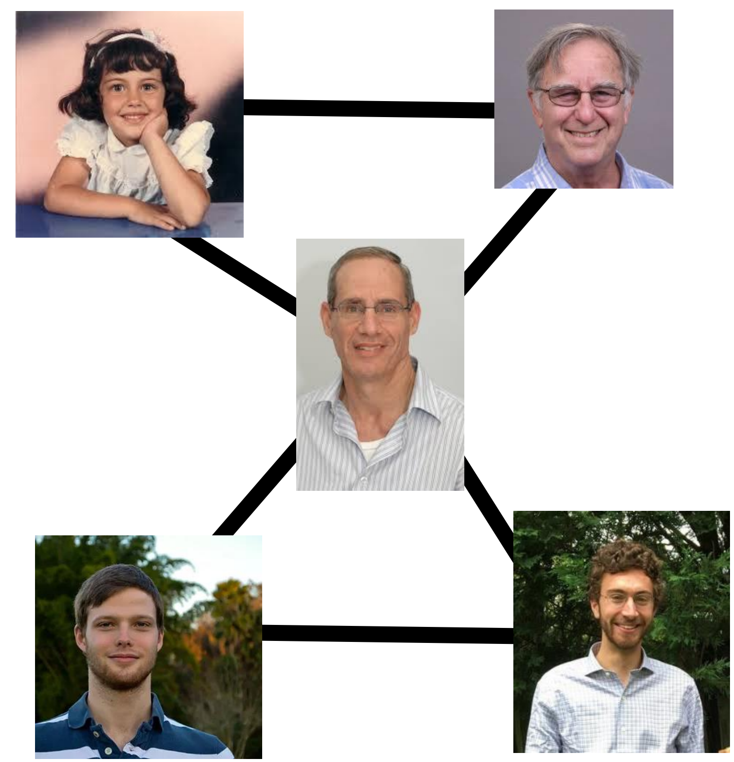


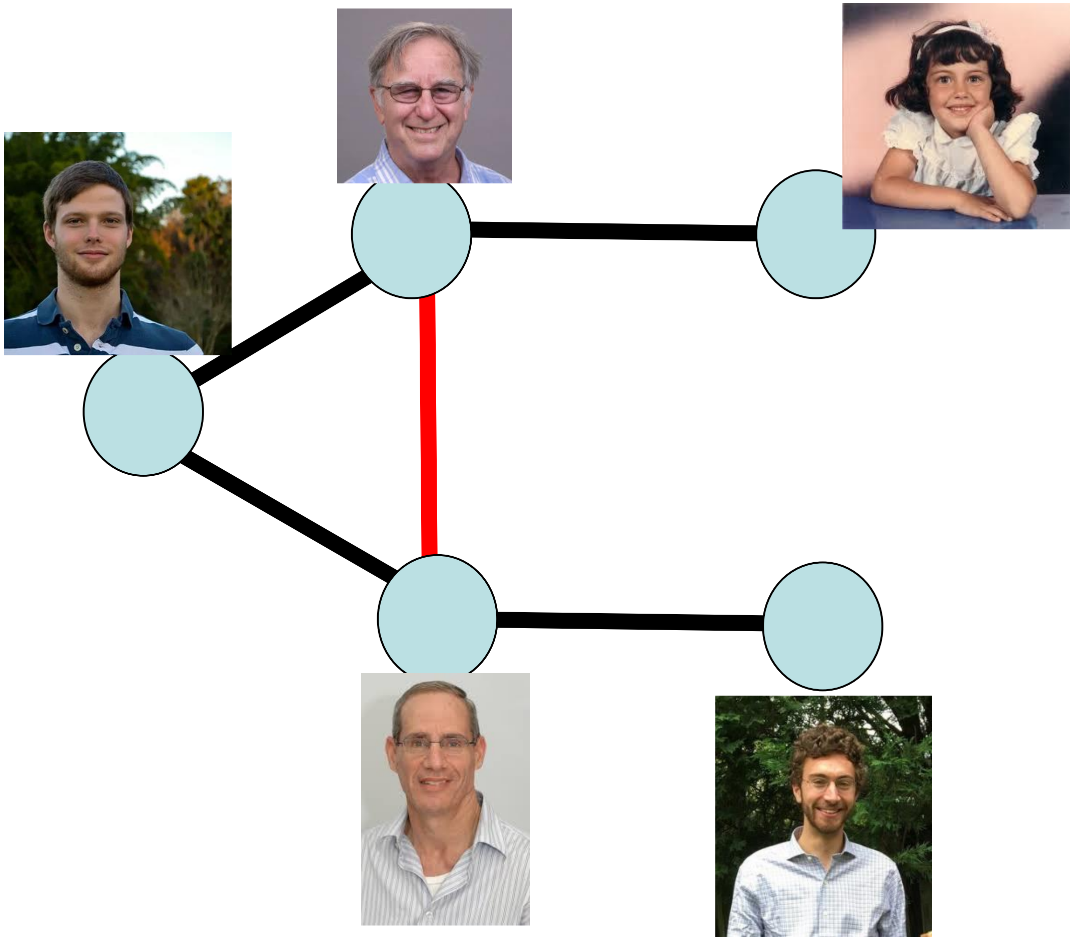


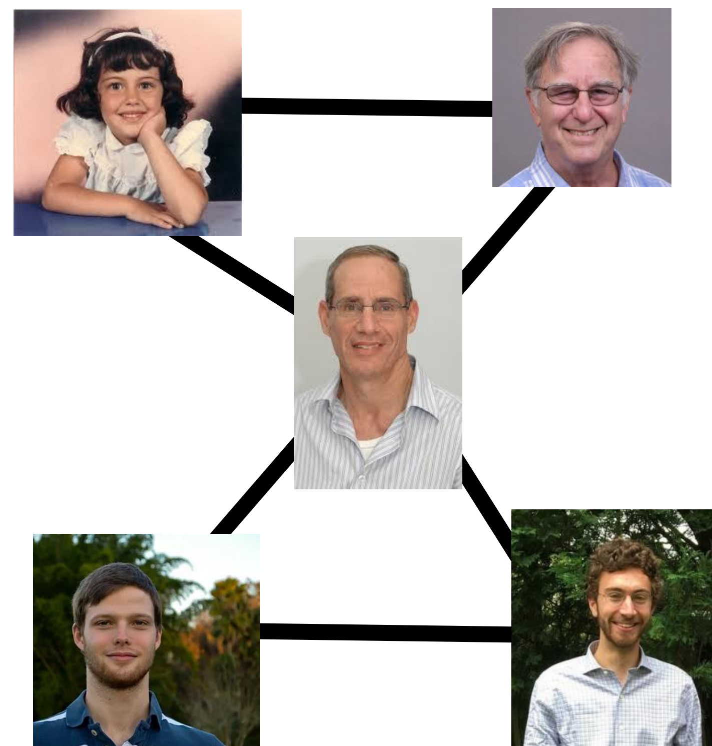


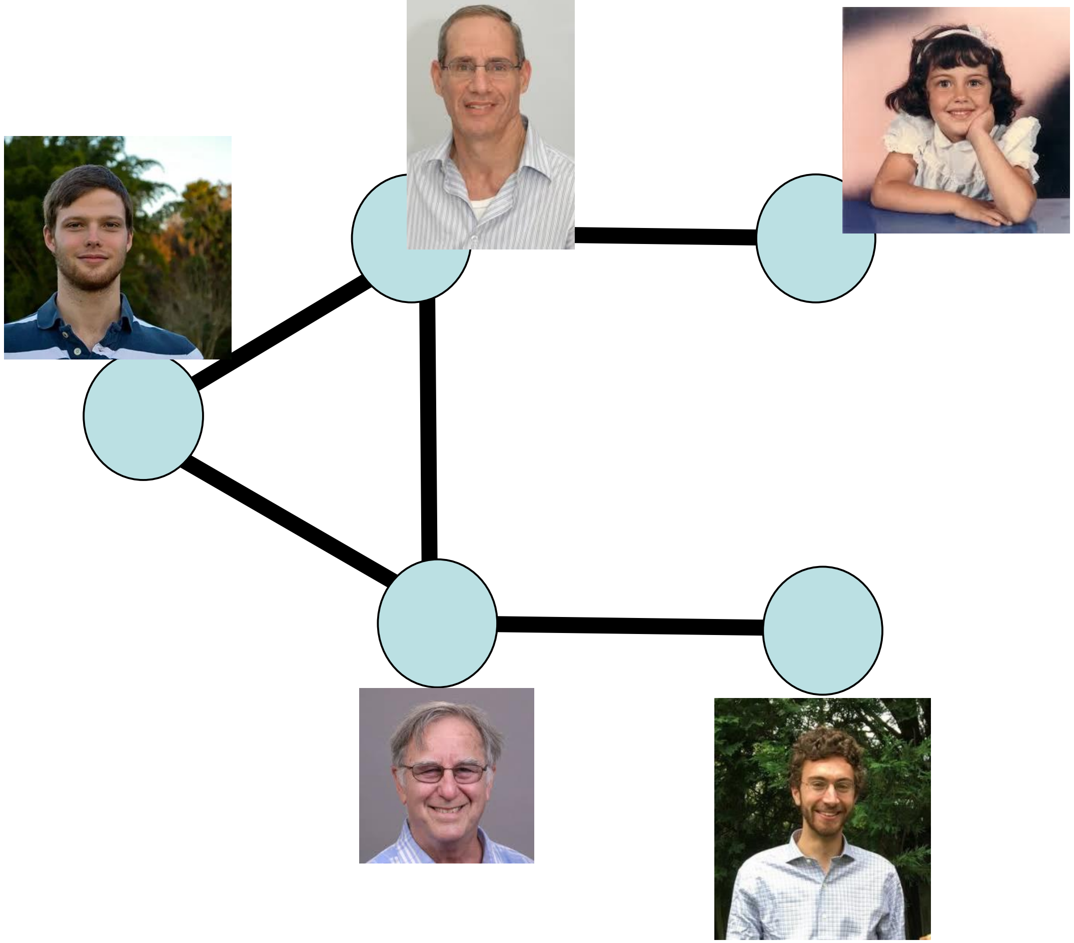


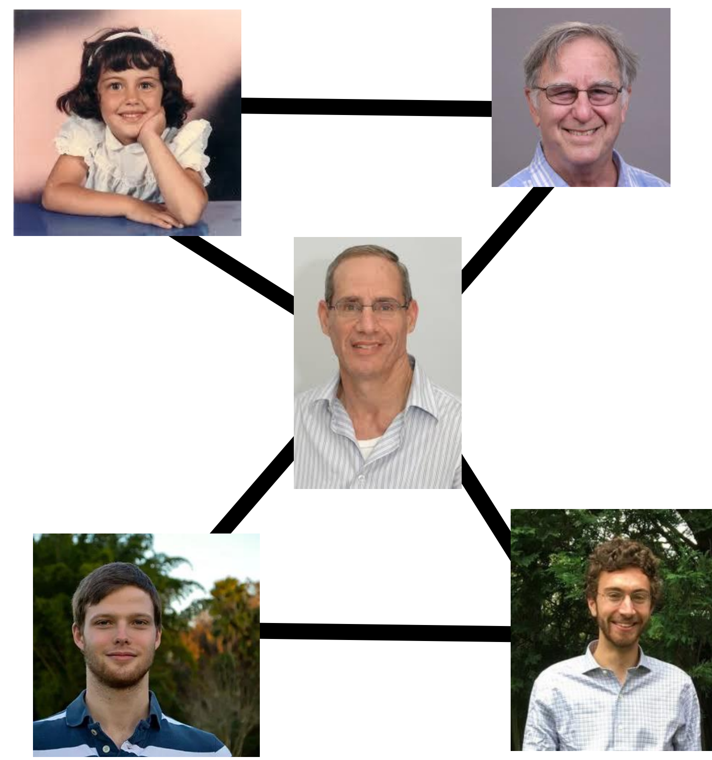








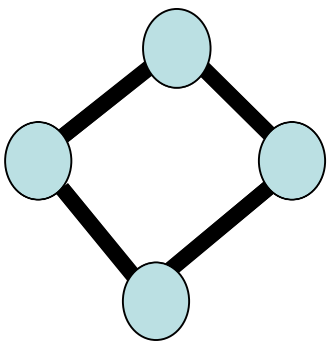




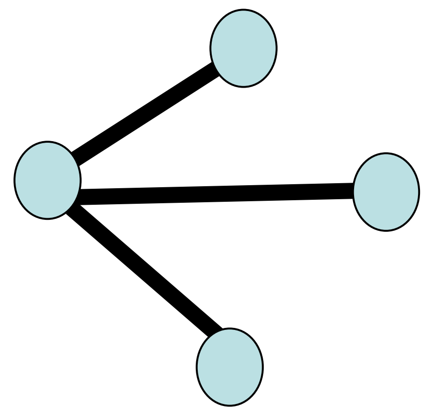
Definition: **Let X, Y be two n-vertex graphs. The Friends and Strangers Graph FS(X,Y) is the graph whose vertices are the bijections from V(X) to V(Y) where two bijections are adjacent if one can be obtained from the other by a friendly swap.**

Example:

X=



Y=

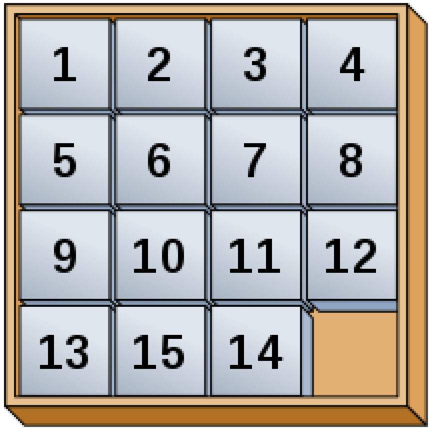


FS(X,Y)=2C_{12}

Previous Work

For any graph Y, FS(K_n ,Y) is the Cayley graph of S_n generated by the transpositions corresponding to the edges of Y

Analyzing the 15-puzzle game is equivalent to analyzing FS(4 by 4 grid, $K_{1,15}$)



Wilson (74) studied the connected components of FS(X,$K_{1,n-1}$) for arbitrary X

Stanley (12) studied the connected components of FS(P_n , P_n).

Defant and Kravitz (20) studied the connected components of FS(X,P_n), FS(X,C_n) for general X

Basic properties

Let X and Y be two n-vertex graphs

Isolated vertices of FS(X,Y) correspond to edge-disjoint packings of X,Y in K_n

FS(X,Y) is isomorphic to FS(Y,X)

If X is disconnected so is FS(X,Y)

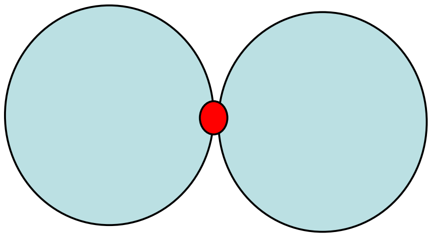
If X,Y are bipartite and n≥3 then FS(X,Y) is disconnected

The star graph $K_{1,n-1}$

When is $FS(X,K_{1,n-1})$ disconnected ?

X is disconnected

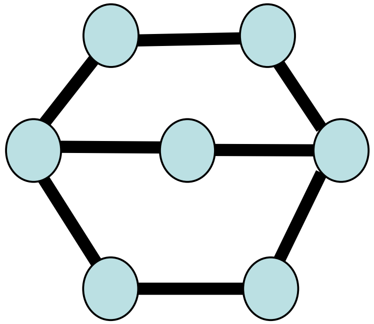
X has a cut-vertex



X is bipartite (n≥ 3)

X is a cycle (n ≥ 4)
[$FS(C_n, K_{1,n-1})$ has (n-2)! components]

X=Z_0 =



$FS(Z_0, K_{1,6})$ has 6 components

Theorem (Wilson (74)): Let X be an n vertex graph, $n \geq 3$. Suppose X is biconnected, neither Z_0 nor a cycle of length at least 4.

If X is non-bipartite, then $FS(X, K_{1,n-1})$ is connected.

If X is bipartite then $FS(X, K_{1,n-1})$ has exactly two connected components

Proof combines the ear decomposition of a 2-connected graph with some group theoretic arguments.

Connectivity: Typical and Extremal Questions

Minimum degree

Question 1a: what is the smallest d_n so that FS(X,Y) is connected for every two n-vertex graphs X,Y each having minimum degree at least d_n ?

Question 1b: what is the smallest $d_{n,n}$ so that FS(X,Y) has exactly two connected components for every two subgraphs X,Y of $K_{n,n}$ each having minimum degree at least $d_{n,n}$?

Random Graphs

Question 2a: **Let X,Y be independent binomial random graphs in G(n,p). For which p=p(n) is FS(X,Y) connected with high probability ?**

Question 2b: **Let X,Y be two independent bipartite random graphs in G(n,n,p). For which p=p(n) does FS(X,Y) have exactly 2 connected components with high probability ?**

Results

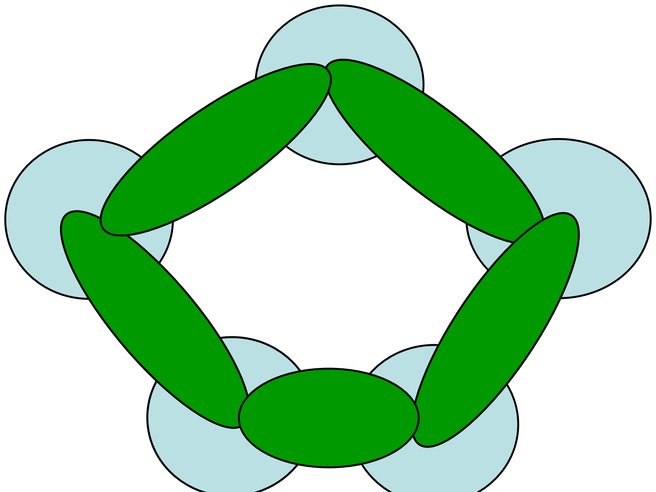
Minimum degree

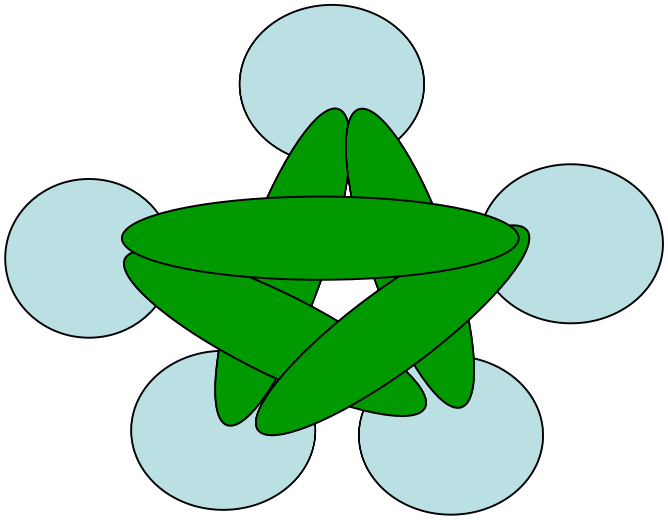
Question 1a: what is the smallest d_n so that FS(X,Y) is connected for every two n-vertex graphs X,Y each having minimum degree at least d_n ?

Theorem (A,Defant,Kravitz):

$$\frac{3n}{5} - 2 \le d_n \le \frac{9n}{14} + 1$$

The lower bound



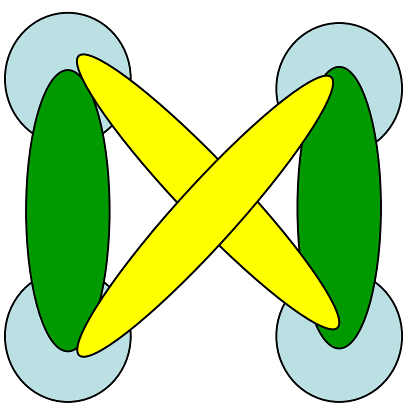


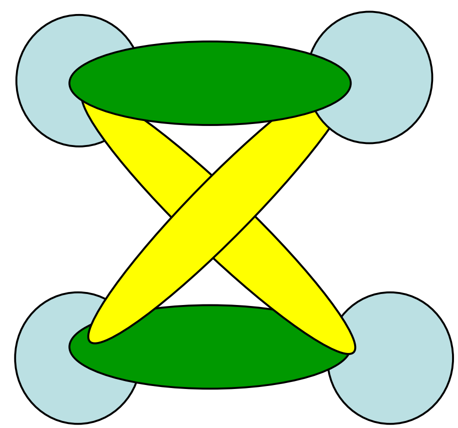
Question 1b: what is the smallest $d_{n,n}$ so that FS(X,Y) has exactly two connected components for every two subgraphs X,Y of $K_{n,n}$ each having minimum degree at least $d_{n,n}$?

Theorem (A, Defant, Kravitz):

$$\left\lceil \frac{3n+1}{4} \right\rceil \leq d_{n,n} \leq \left\lceil \frac{3n+2}{4} \right\rceil$$

The lower bound





Random Graphs

Question 2a: **Let X,Y be independent binomial random graphs in G(n,p). For which p=p(n) is FS(X,Y) connected with high probability ?**

Theorem (A, Defant, Kravitz): **The threshold p=p(n) for connectivity of FS(X,Y) is**

$$p(n) = \frac{1}{n^{\frac{1}{2}+o(1)}}$$

Question 2b: **Let X,Y be two independent bipartite random graphs in G(n,n,p). For which p=p(n) does FS(X,Y) have exactly 2 connected components with high probability ?**

Theorem (A,Defant,Kravitz): **the threshold p(n) for having two components satisfies**

$$\Omega\left(\frac{1}{n^{1/2}}\right) \leq p(n) \leq \tilde{O}\left(\frac{1}{n^{\frac{3}{10}}}\right)$$

A bit about the proofs

Theorem: **the threshold for connectivity of FS(X,Y) for X,Y in G(n,p(n)) is $n^{-1/2+o(1)}$**

Fact: **for** $p(n) \le \frac{2^{-\frac{1}{2}} - \epsilon}{\sqrt{n}}$ **then with high probability FS(X,Y) is disconnected (has isolated vertices).**

Proof: Sauer and Spencer (78) **showed that if**

$$2\,\Delta(X)\Delta(Y) < n$$

then X and Y have an edge disjoint packing in K_n

The main part of the proof is:

Theorem: if $p(n) \geq \frac{exp[(2 \log n)^{2/3}]}{n^{1/2}}$

then with high probability FS(X,Y) is connected.

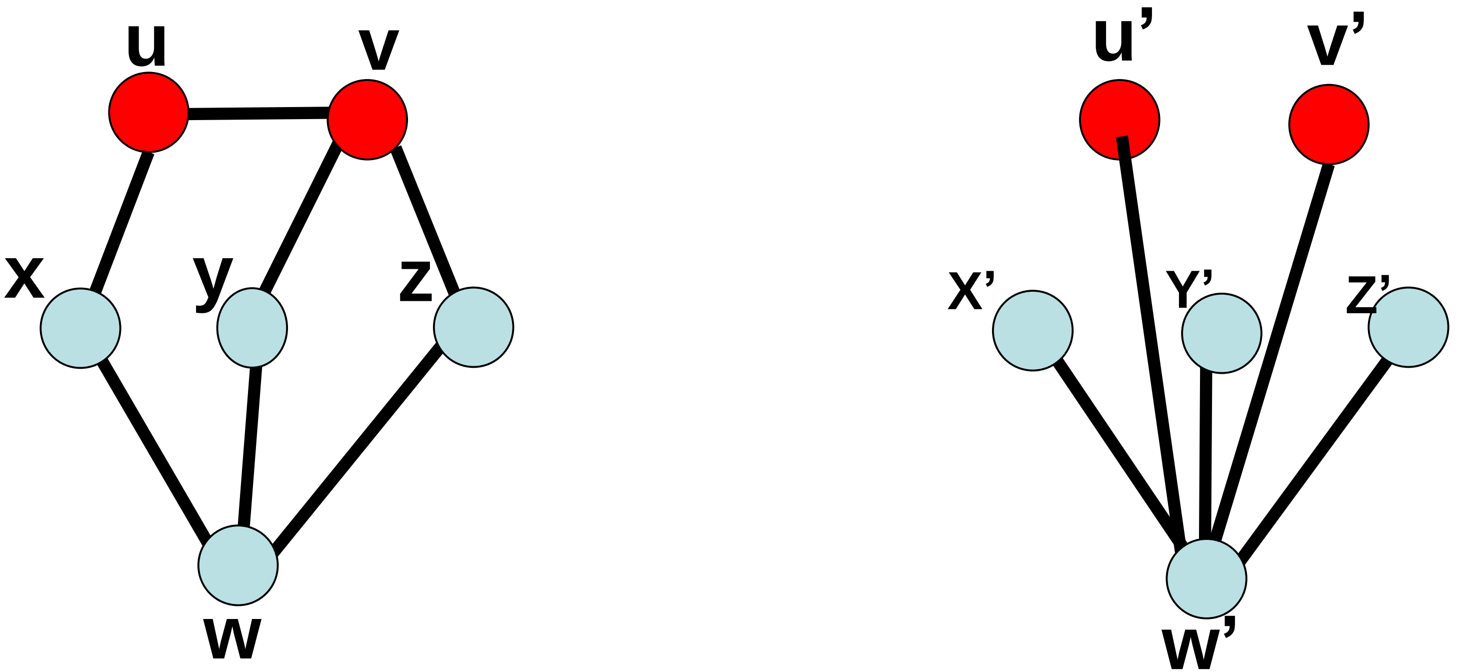
This is proved by showing that with high probability, for every bijection f from V(X) to V(Y) every pair of adjacent elements u,v in Y are exchangeable.

First attempt to establish that: hope that for every such u,v and f there is a common neighbor w of u and v in Y where $w' = f^{-1}(w)$ is a common neighbor of $u' = f^{-1}(u)$ and $v' = f^{-1}(v)$.

But this fails with high probability for all

$$p < {}^{1}/_{\sqrt{2}}$$

Second attempt: show that for every such u,v and f the following two graphs appear in X,Y

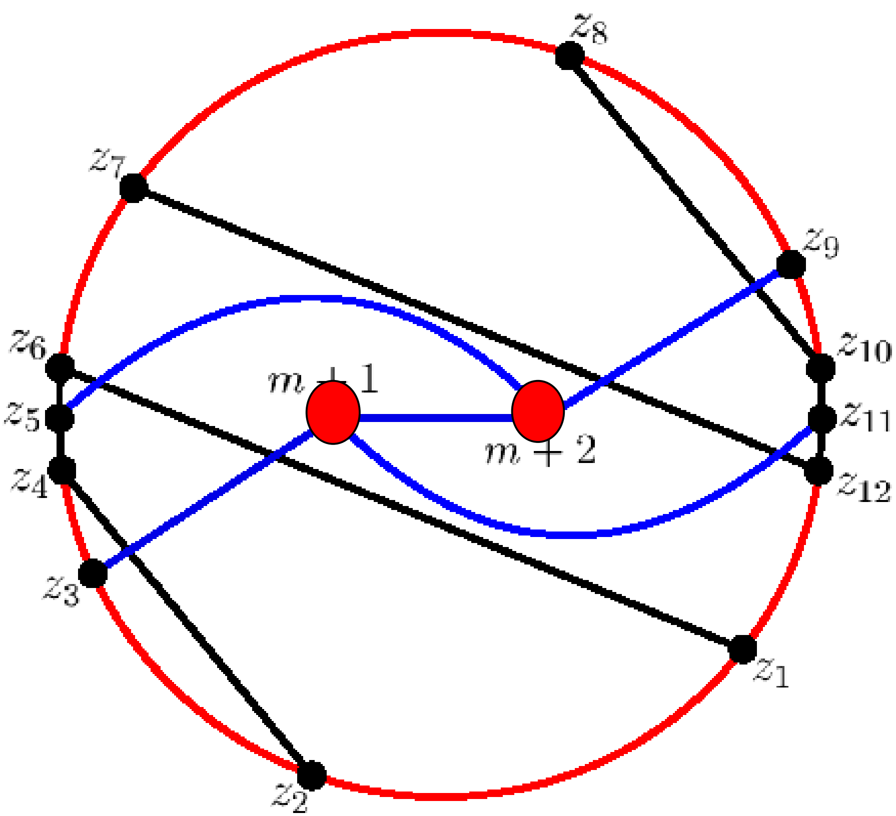


This will suffice by **Wilson's Theorem**

For $p(n) \geq \widetilde{\Omega}\left({}^{1}\!/_{n^{0.25}}\right)$ this holds with high probability, by **Janson Inequalities**

Yet it fails for $p(n)=1/n^{1/2-o(1)}$

Need more complicated graphs



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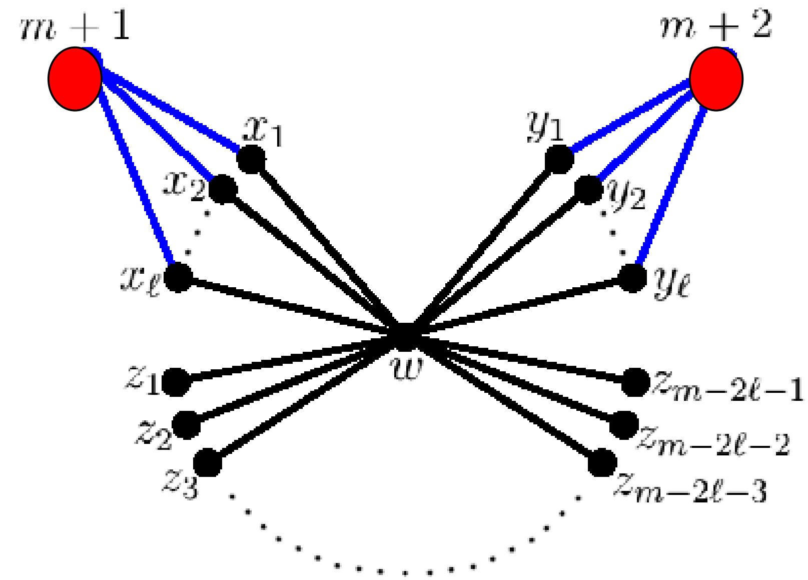
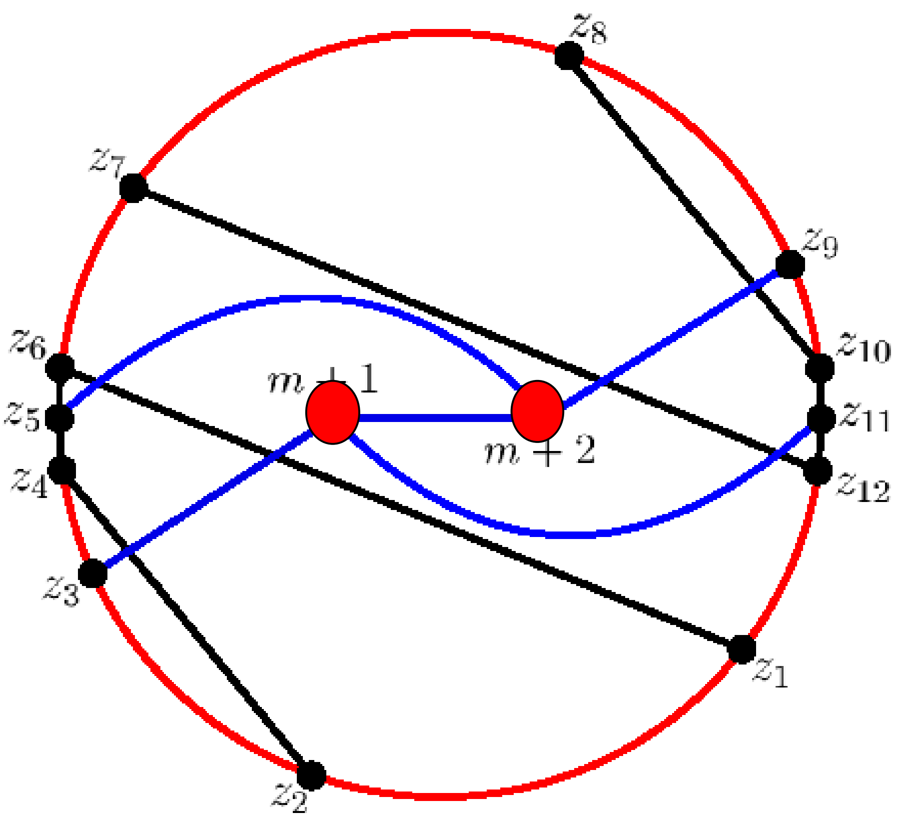


H^*

Here $m = \left\lfloor \log n^{2/3} \right\rfloor$ **grows with n.**

Need more complicated graphs



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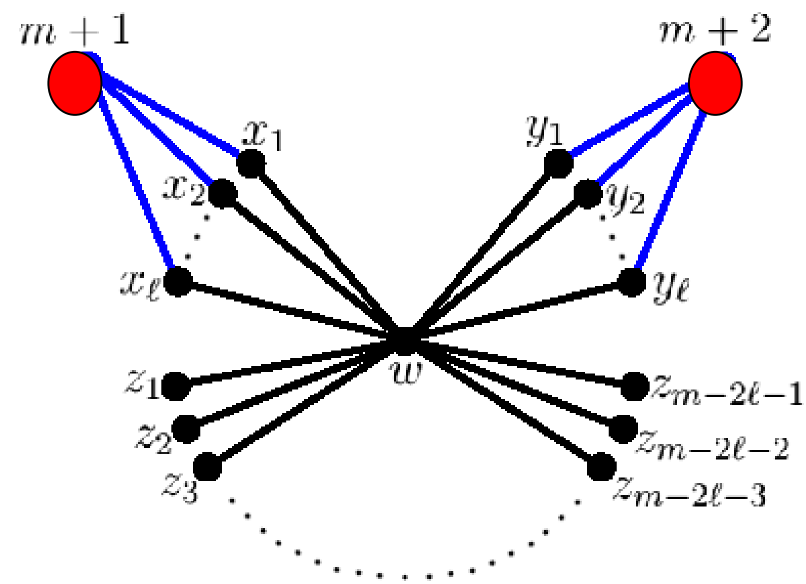
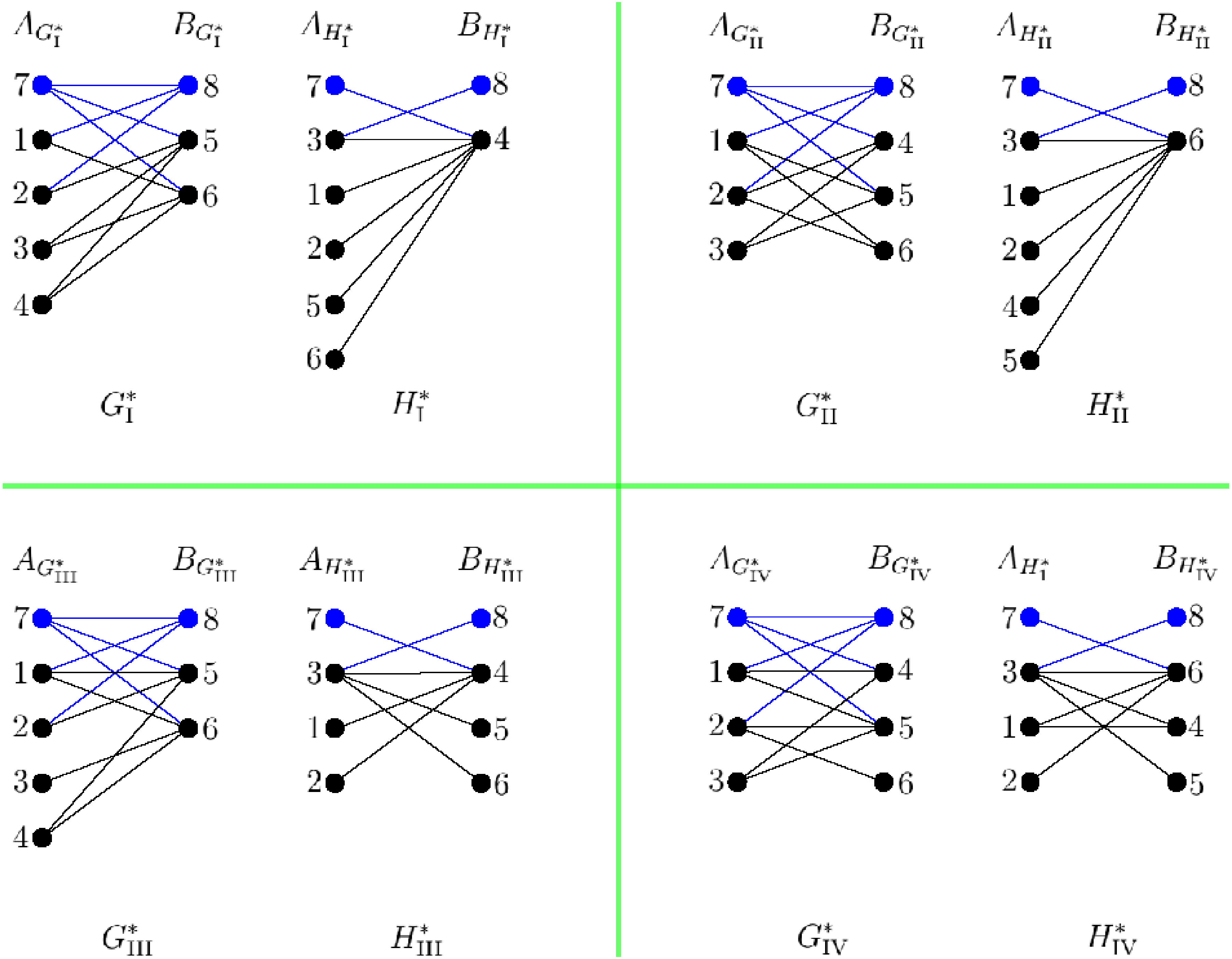


H^*

The proof these imply exchangeability applies Wilson's Theorem several times

The (non-tight) result for the bipartite case is obtained by a similar reasoning using specific constant size pairs of graphs that supply exchangability by sequences of friendly swaps found by computer search.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Open Problems

Is $p(n)=n^{-1/2+o(1)}$ the threshold for ensuring two connected components in the bipartite case too?

Is the smallest value d_n ensuring connectivity of FS(X,Y) for every pair of n-vertex graphs X and Y with minimum degree at least d_n 3n/5+O(1) ?

Is there a hitting time result ? Namely, starting with two edgeless graphs (X_0,Y_0) on n vertices each, let (X_i ,Y_i) be a random sequence of pairs of graphs, where each X_{i+1} is obtained from X_i by adding to it a uniform random yet unchosen edge, and each Y_{i+1} is defined analogously.

Let t_{iso} denote the smallest i so that $FS(X_i ,Y_i)$ has no isolated vertices.

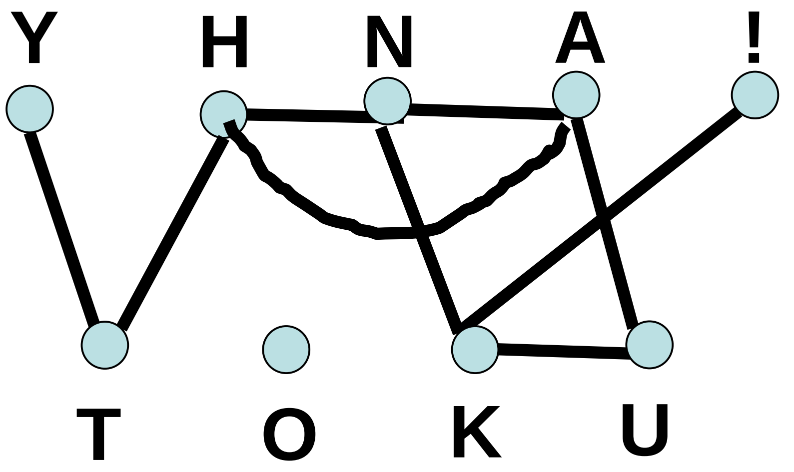
Let t_{con} denote the smallset i so that $FS(X_i ,Y_i)$ is connected.

Is $t_{conn} = t_{iso}$ with high probability ? If not, is $t_{con} =(1+o(1))\, t_{iso}$ with high probability ?

Is the diameter of any connected component of FS(X,Y) at most $n^{O(1)}$?

What about the mixing properties of the random walk on FS(X,Y) ?

[The case FS(X,K_n) is Aldous spectral gap conjecture settled by Caputo, Liggett and Richthammer (10)]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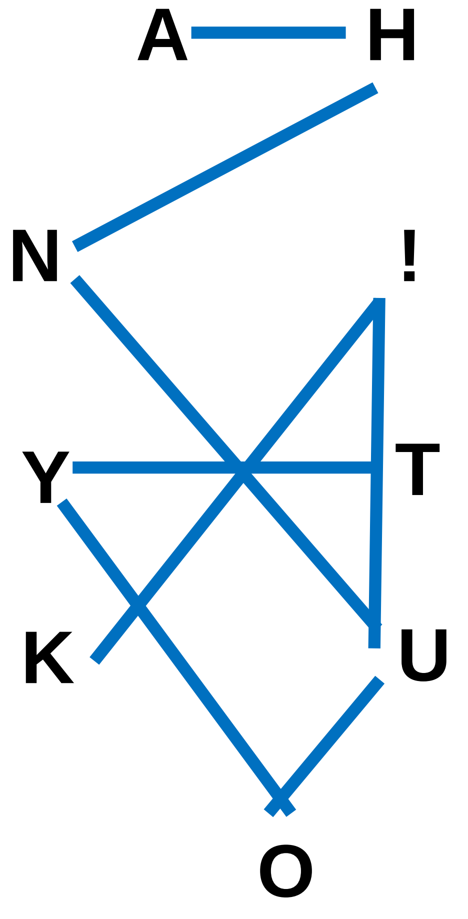
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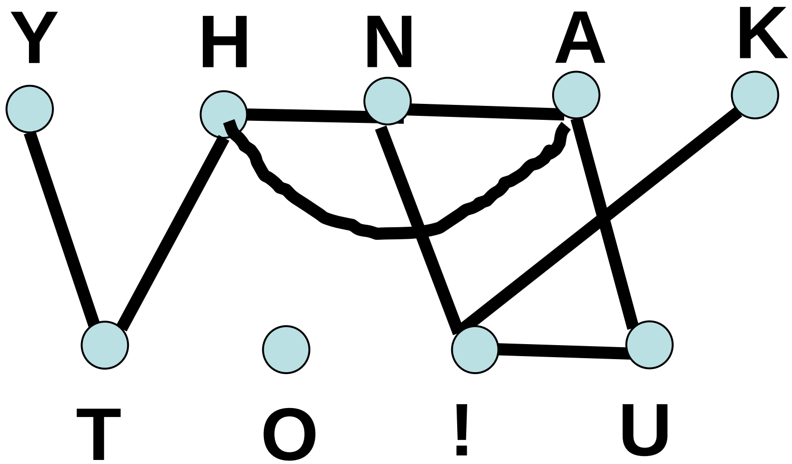
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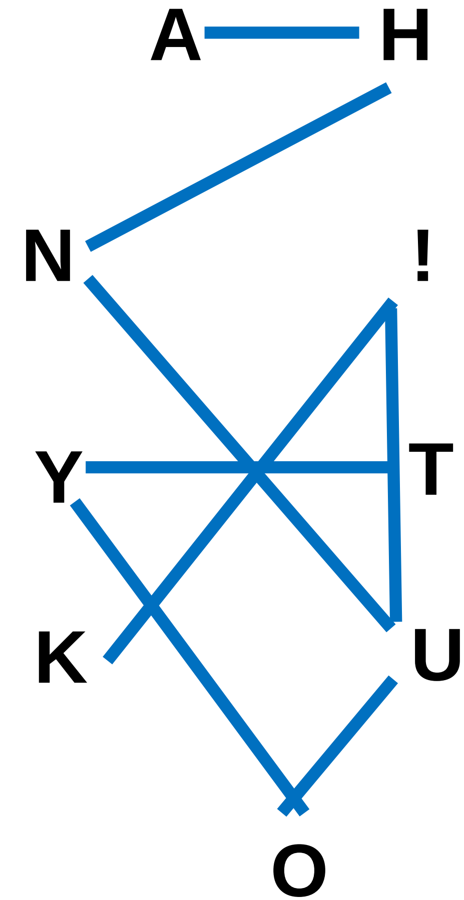
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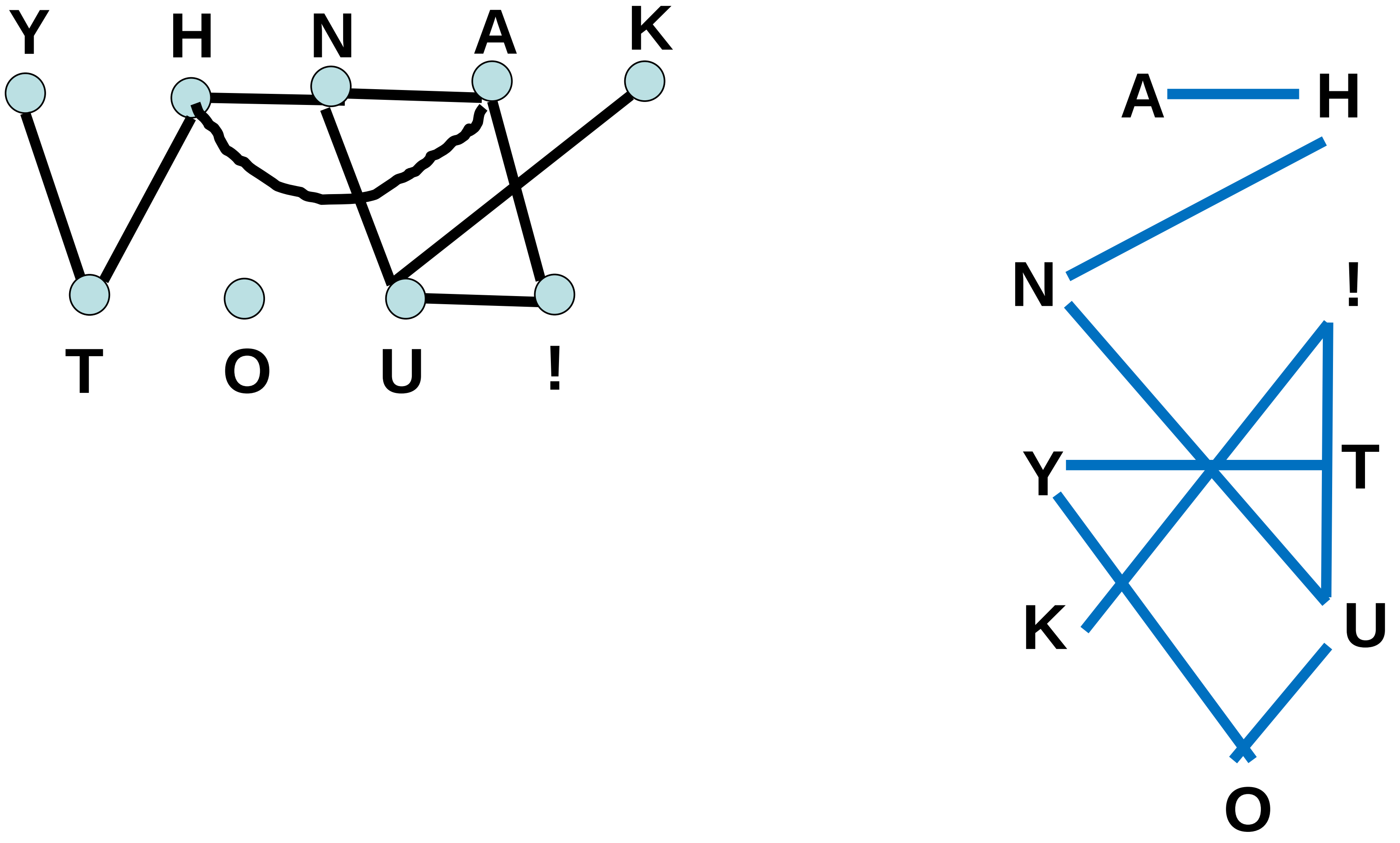
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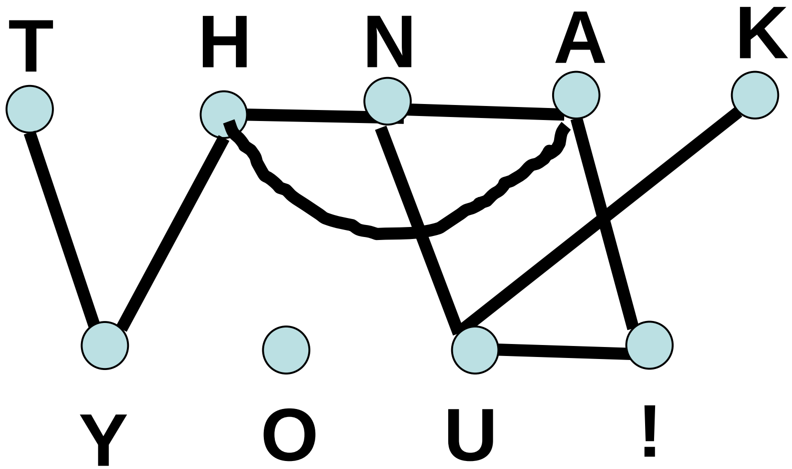
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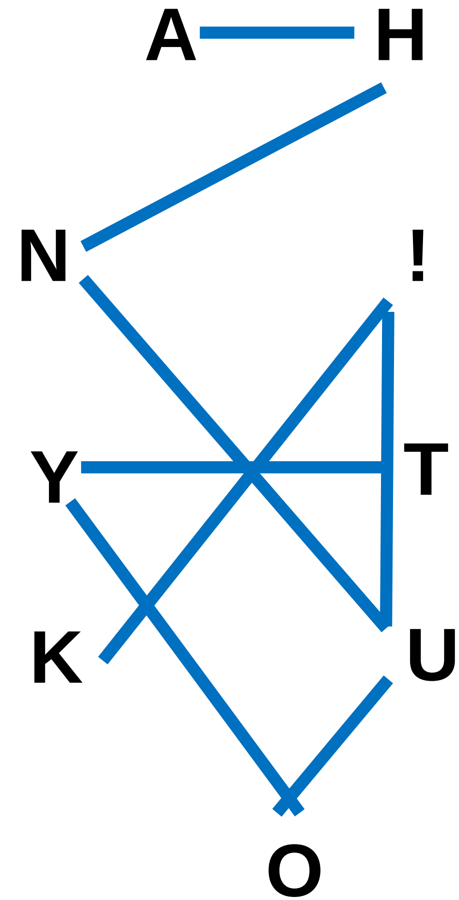
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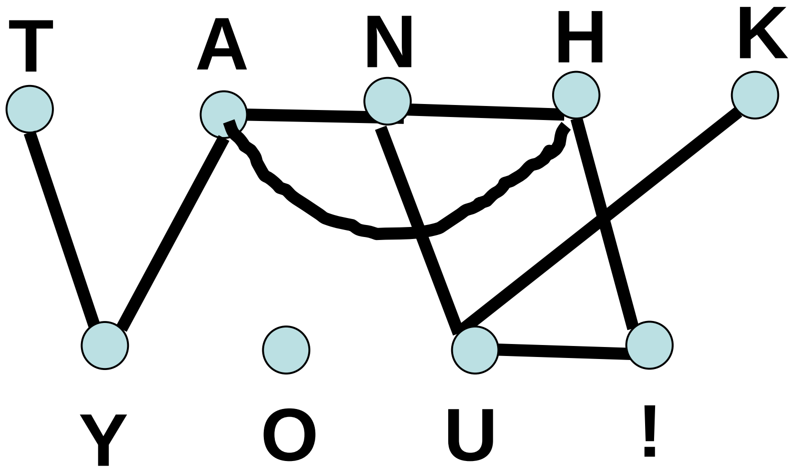
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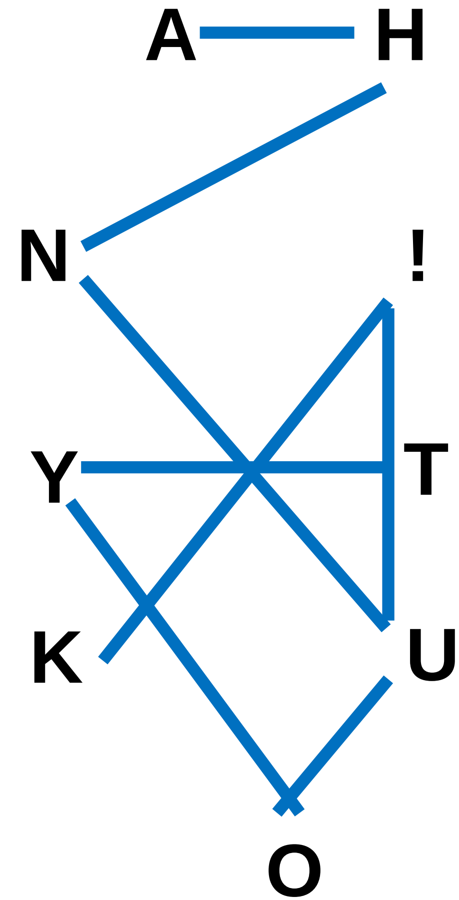
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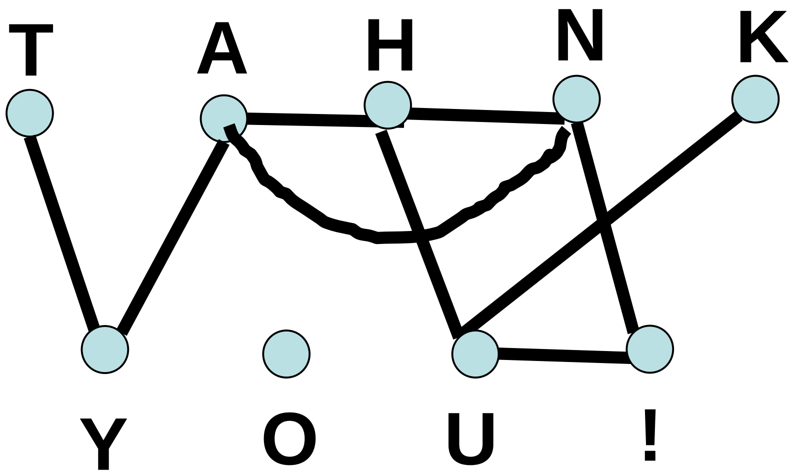
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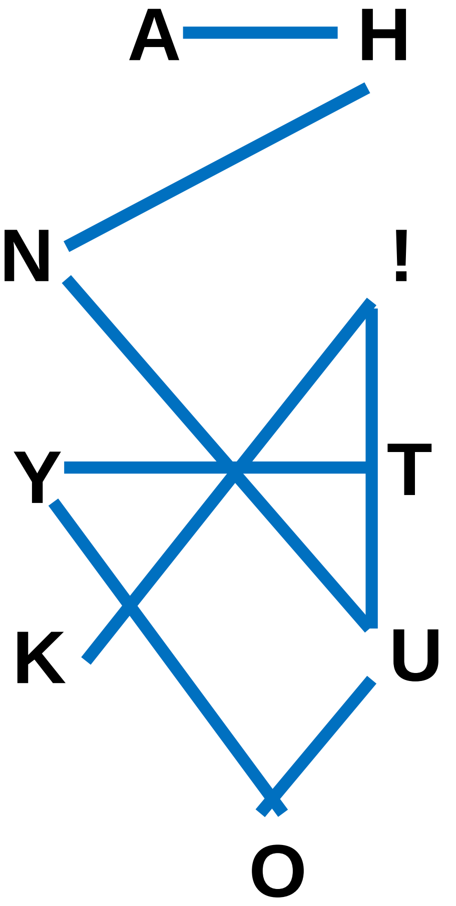
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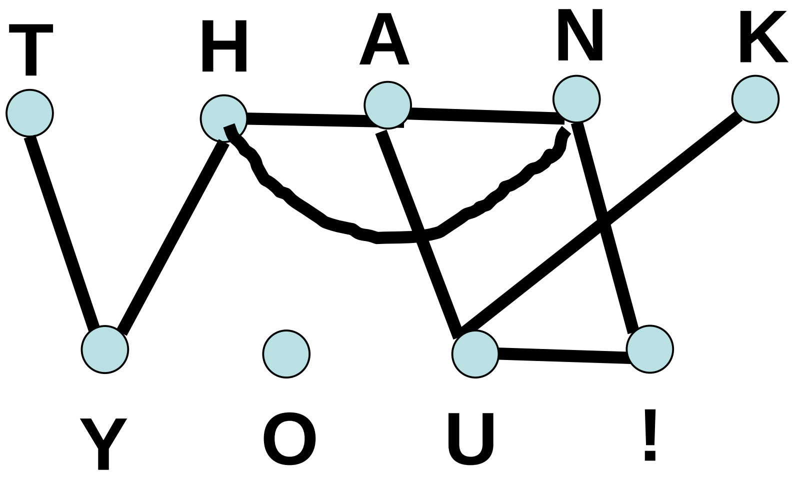
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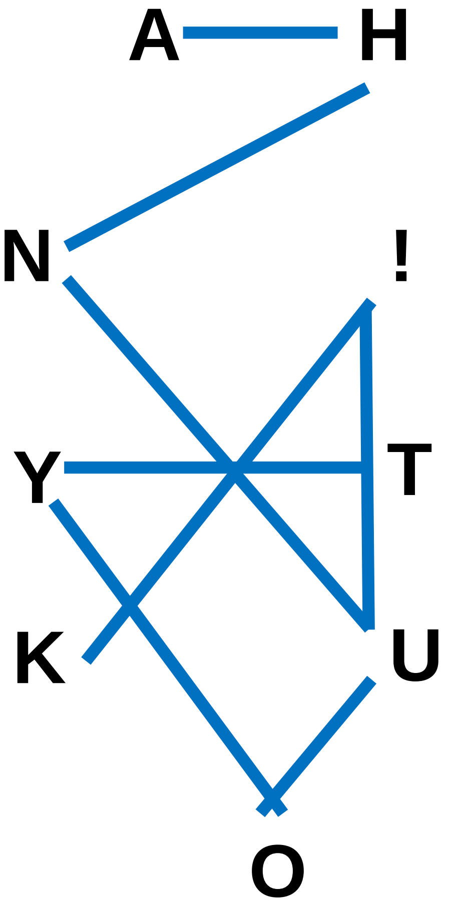
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