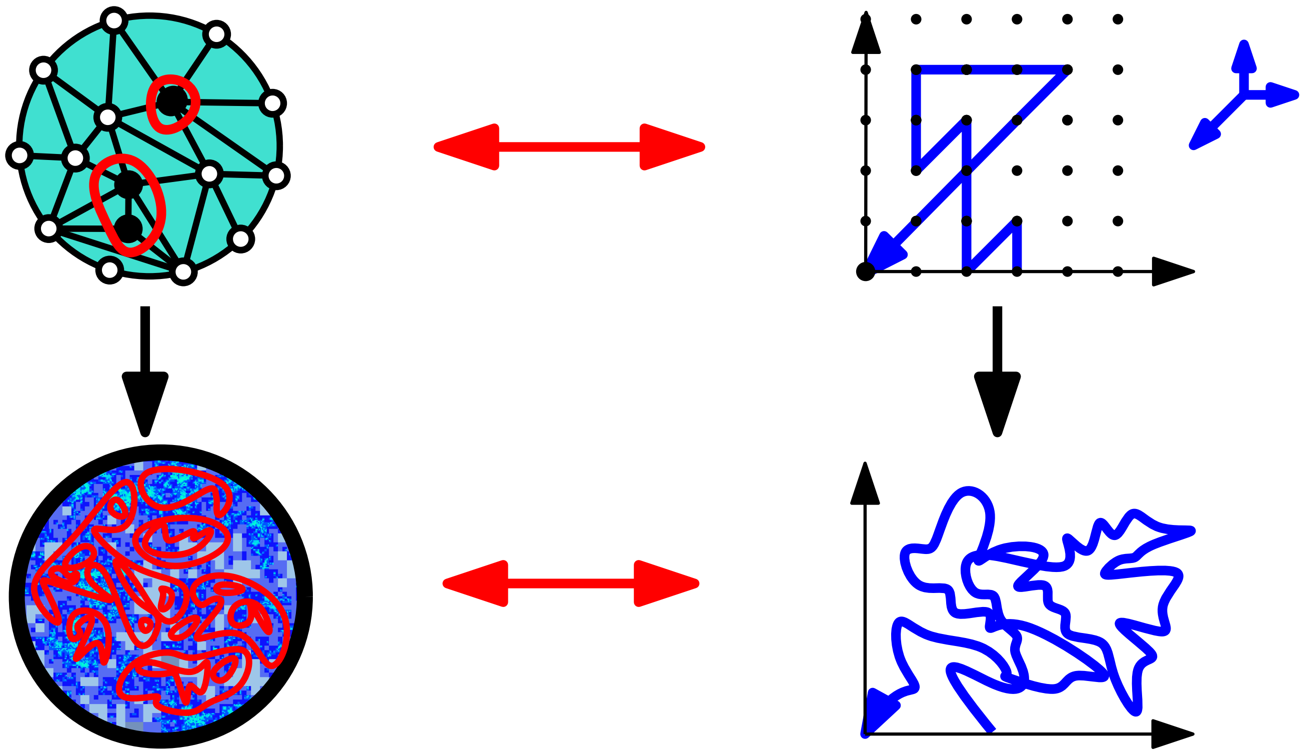
Percolation on triangulations:
A bijective path to Liouville quantum gravity

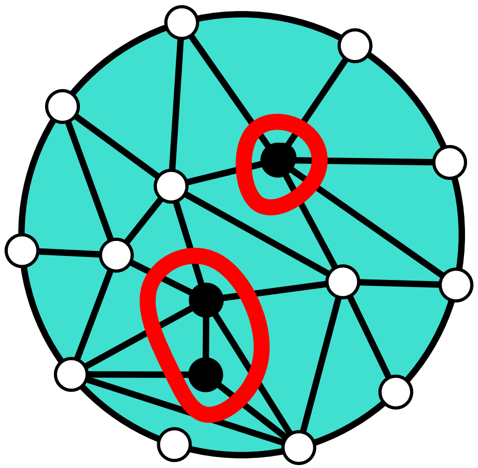
Olivier Bernardi - Brandeis University

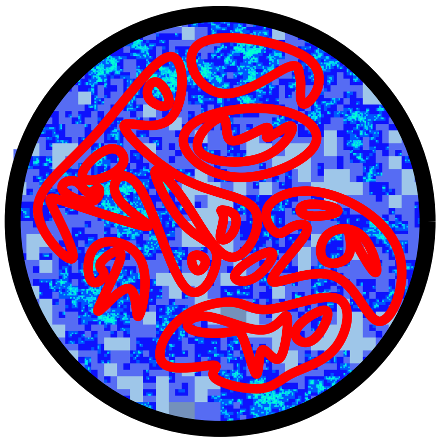
Joint work with **Nina Holden** (ETH) & **Xin Sun** (Columbia)



Oxford Discrete Mathematics & Probability Online Seminar, April 2020

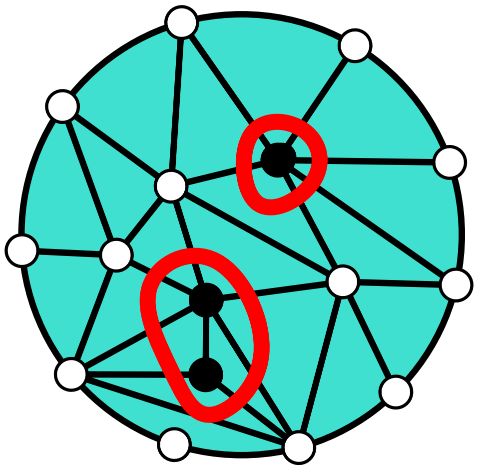
Percolation on triangulations



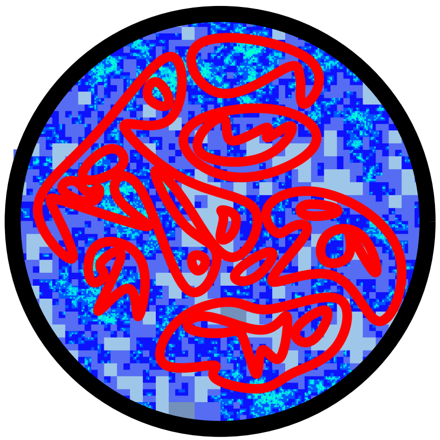


CLE on Liouville Quantum Gravity

Percolation on triangulations



“Random curves on a random surface”



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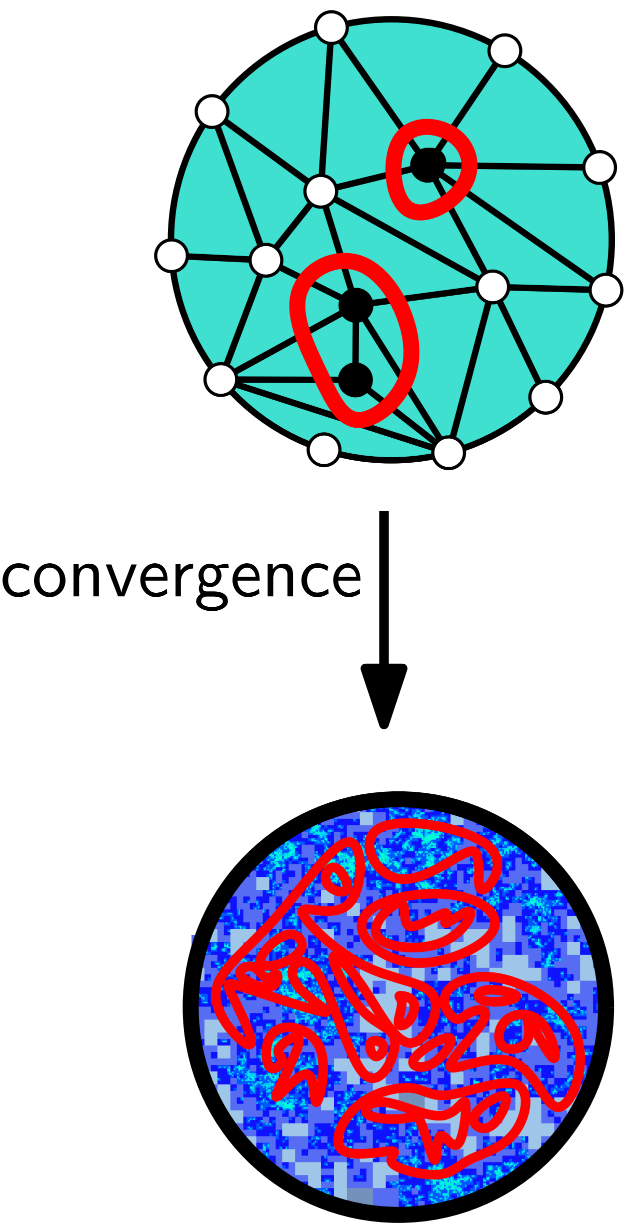

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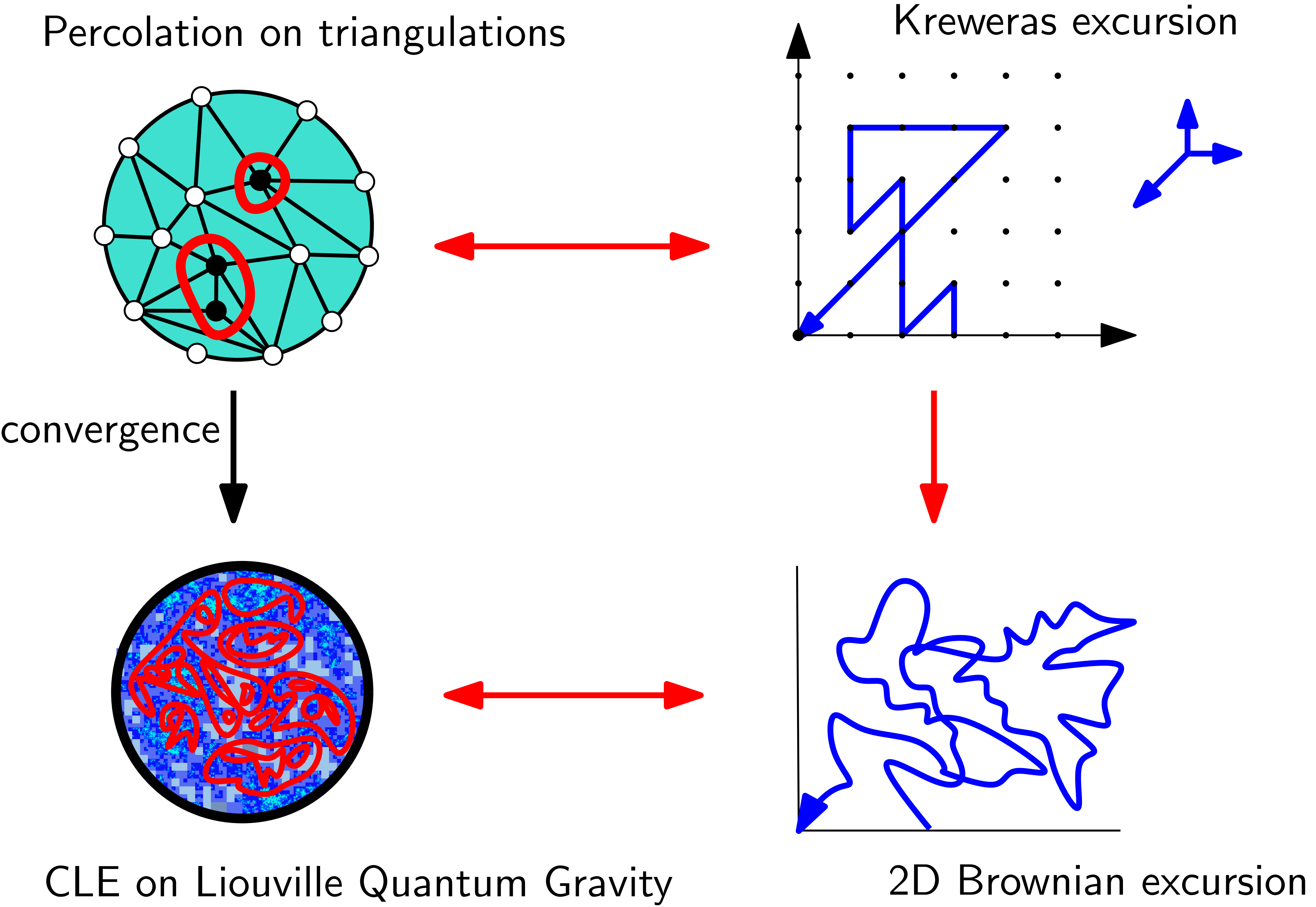
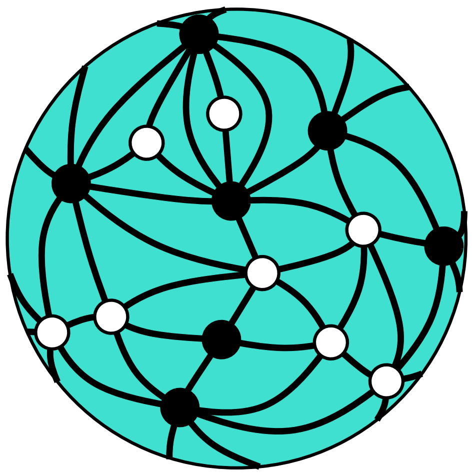
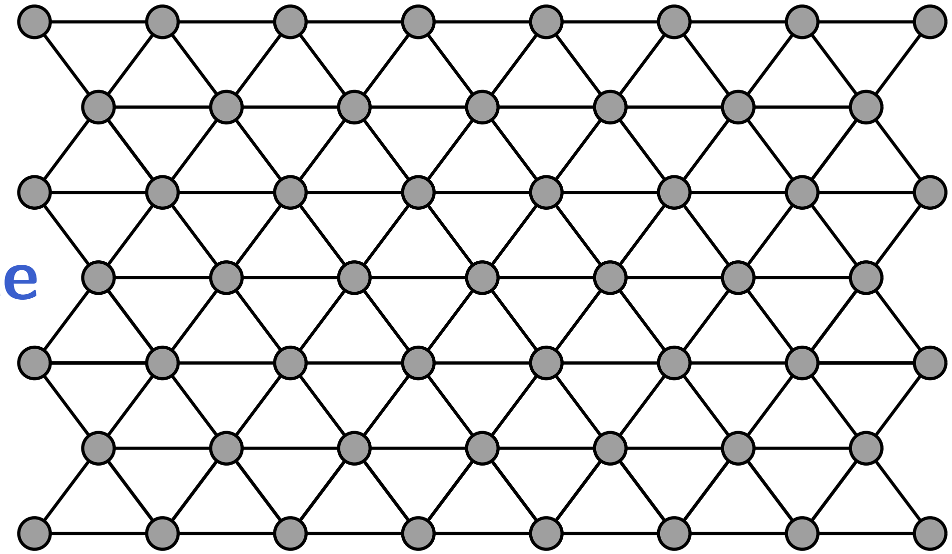

Percolation on triangulations
Kreweras excursion
convergence
CLE on Liouville Quantum Gravity
2D Brownian excursion

Percolation on triangulations



Percolation on a regular lattice

Triangular lattice



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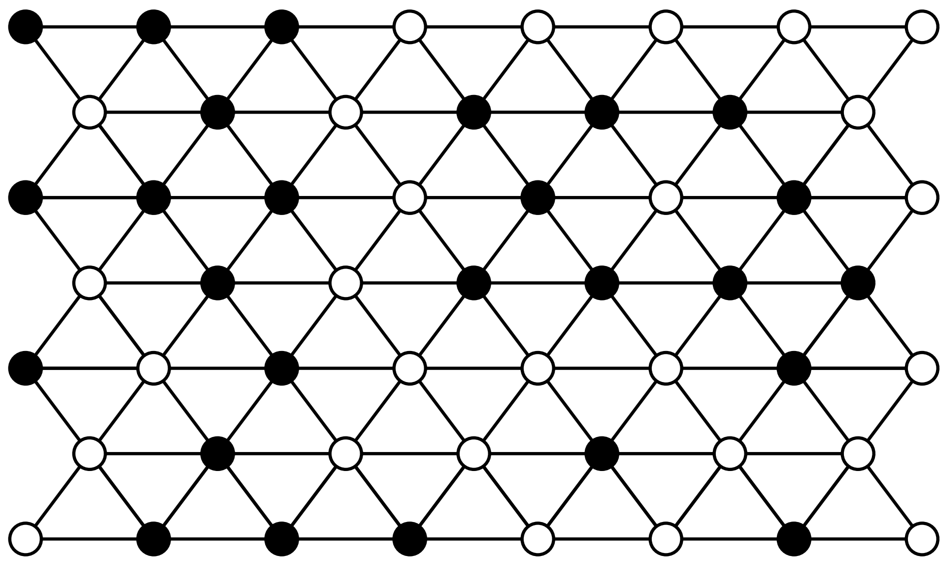
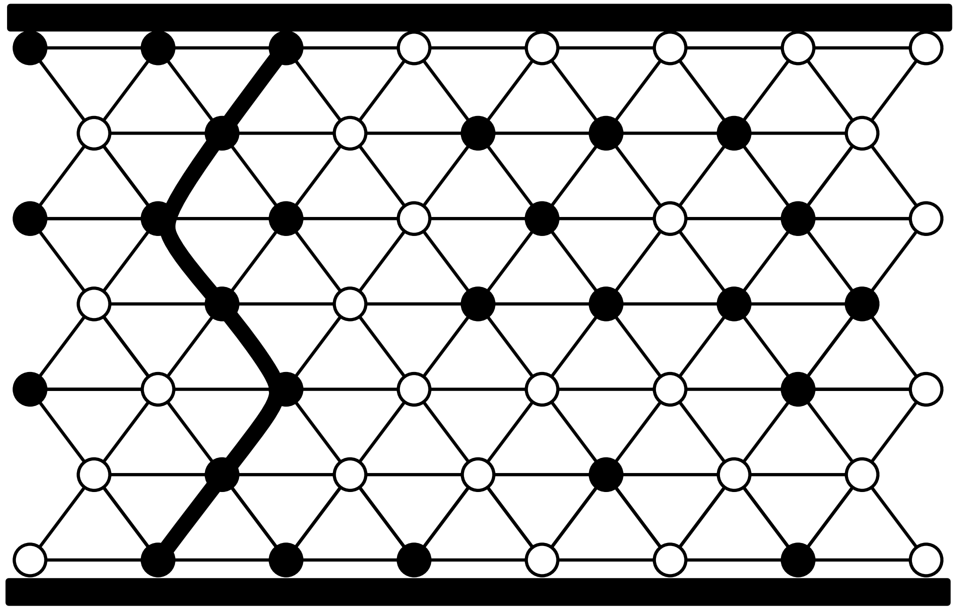


Site percolation: Color vertices *black* or *white* with probability $1/2$.

Percolation on a regular lattice



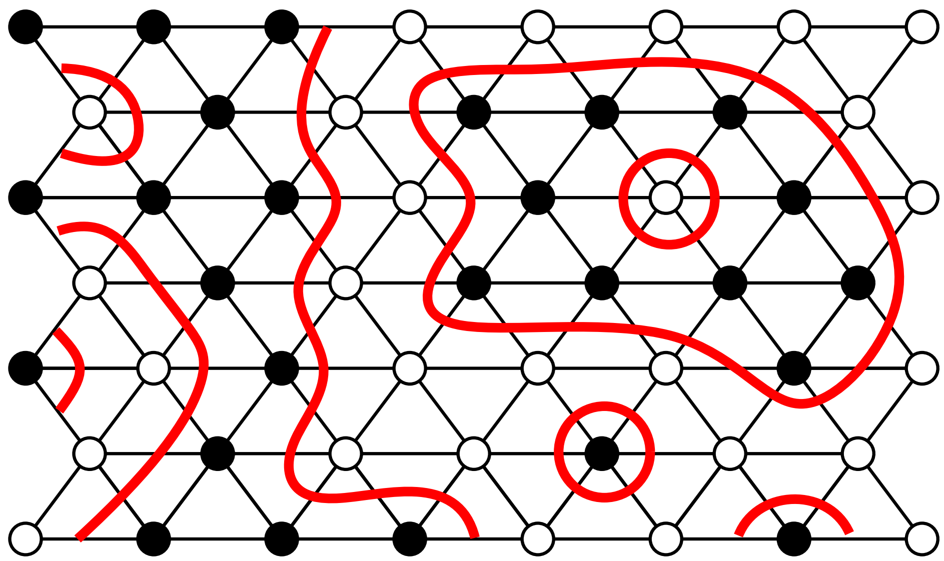
$A\,n \times B\,n$ box

Site percolation: Color vertices $black$ or $white$ with probability $1/2$.

Questions:

- **Crossing probabilities**?

Percolation on a regular lattice



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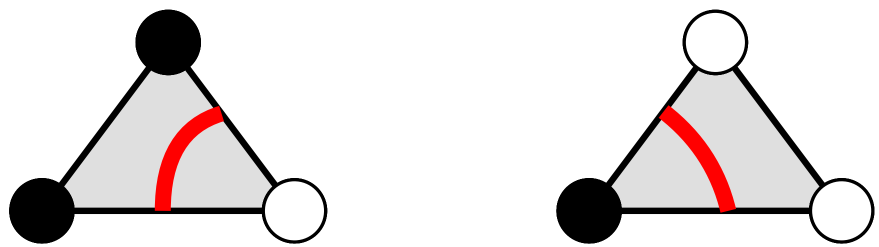

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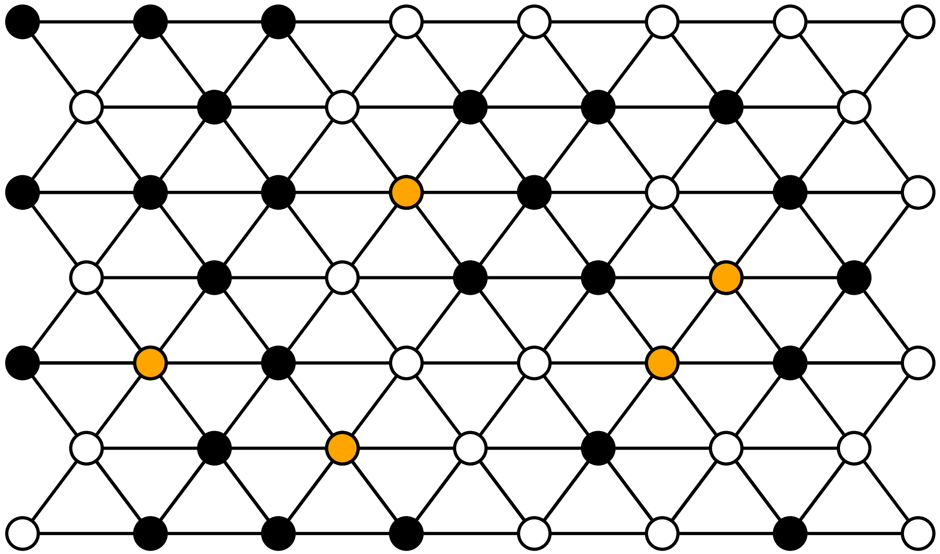

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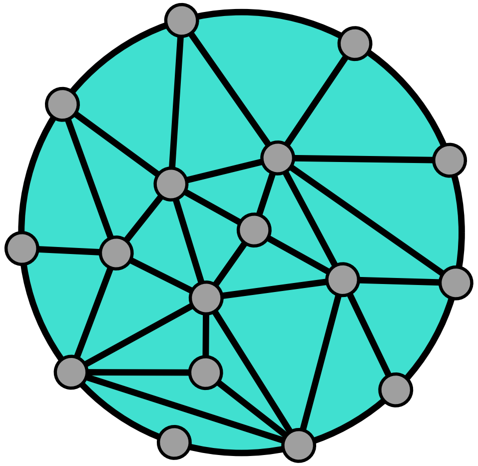
Triangulations (of the disk)

Def. A **triangulation** of the disk is a decomposition into triangles.



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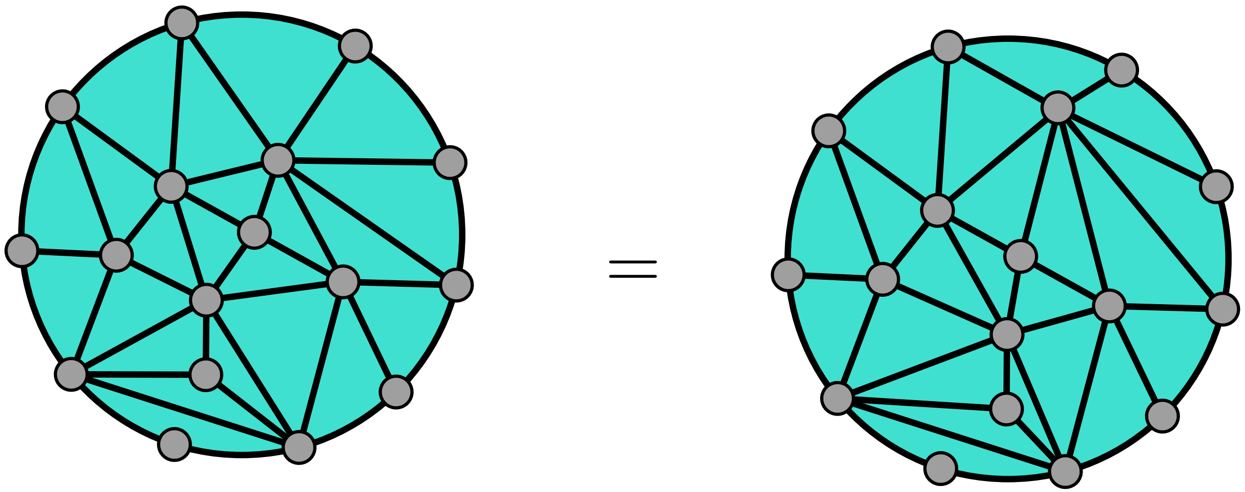

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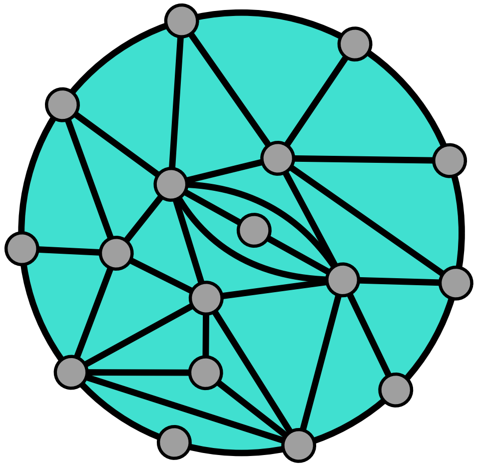

(multiple edges allowed, loops forbidden)

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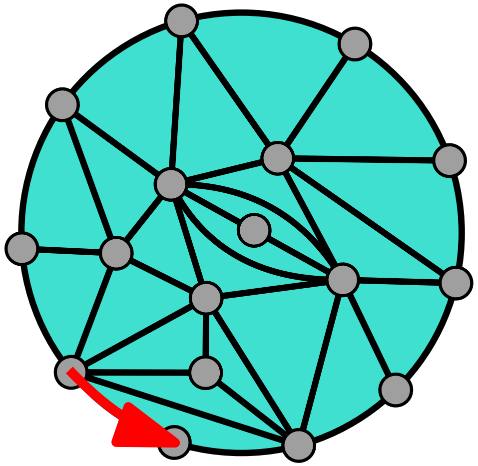
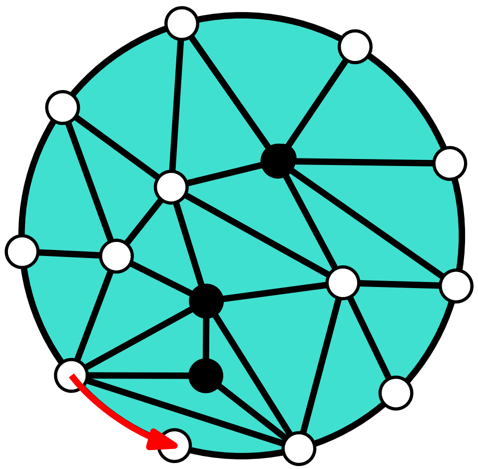


Def. A triangulation is **rooted** by marking an edge on the boundary.

Percolation on triangulations

We can consider percolation on **random triangulations** of the disk.
(k exterior vertices, n interior vertices; uniform probability)



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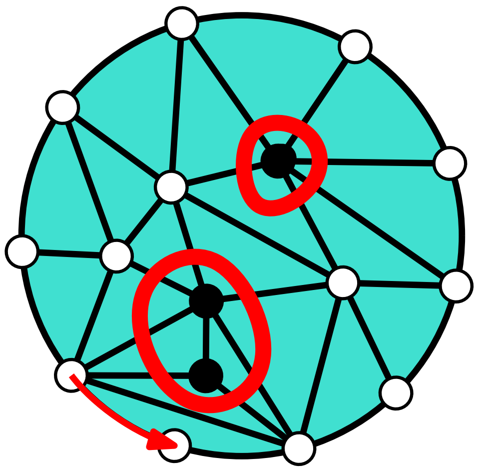

Same questions:

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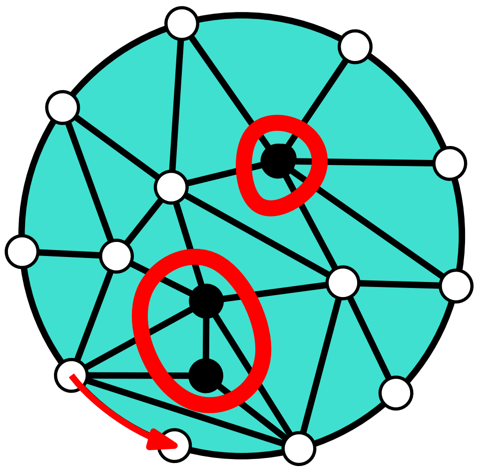

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Goal 1: Answer these questions. (as $n \to \infty$, $k \sim \sqrt{n}$)

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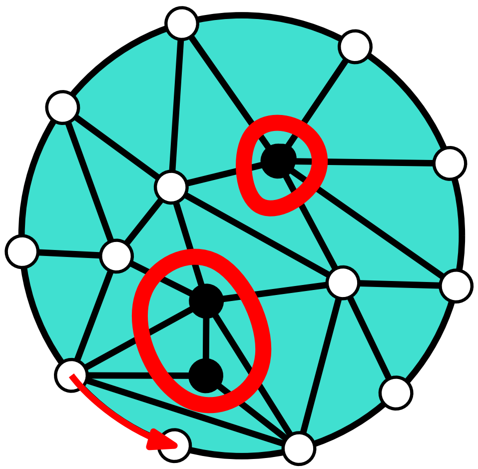

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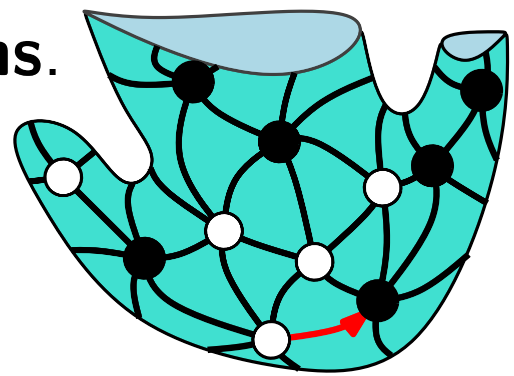
We can alternatively consider **infinite triangulations**.

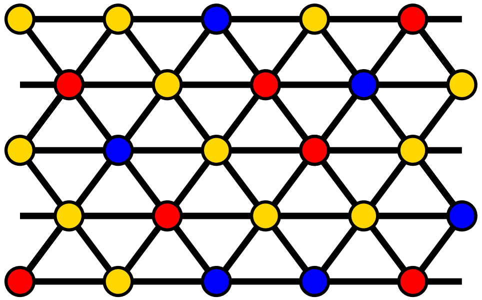
Uniform Infinite Planar Triangulation

[Angel,Schramm 04]



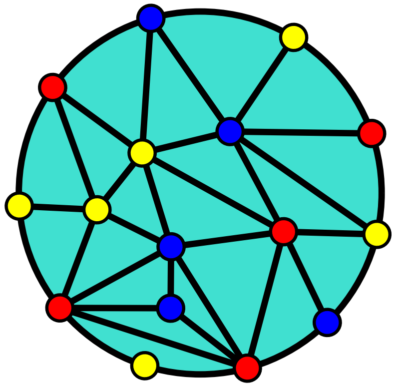
Regular lattices Vs random lattices

Is it interesting to study statistical mechanics on random lattices?



regular lattice

Vs



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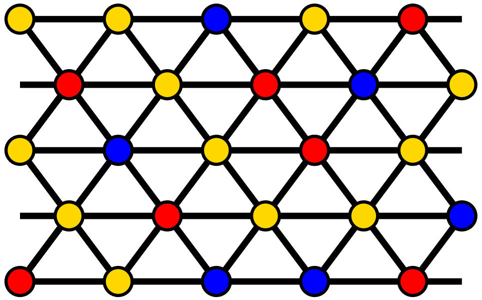

regular lattice

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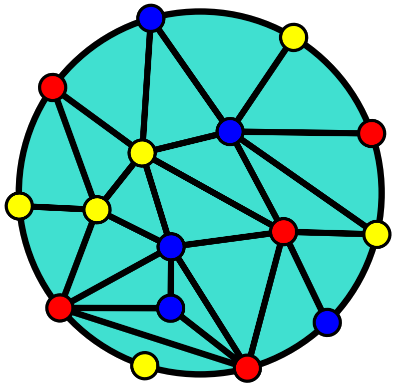

random lattice

Yes! New tools: **random matrices**, **generating functions**, **bijections**.

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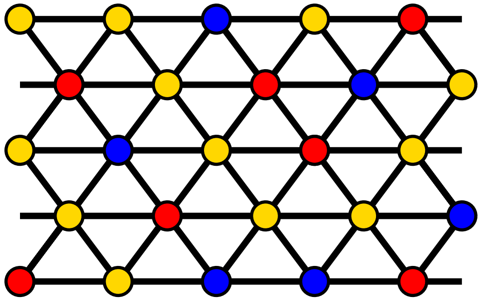

regular lattice

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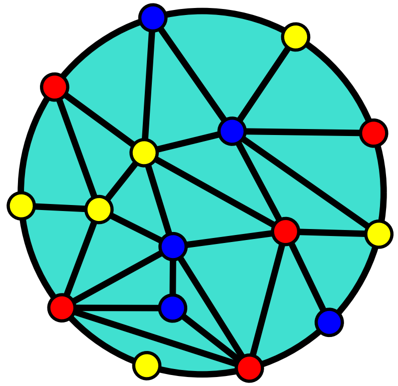

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Yes! The "critical exponents" on **regular Vs random** lattices are related by the **KPZ formula** [Knizhnik, Polyakov, Zamolodchikov].

Regular lattices Vs random lattices

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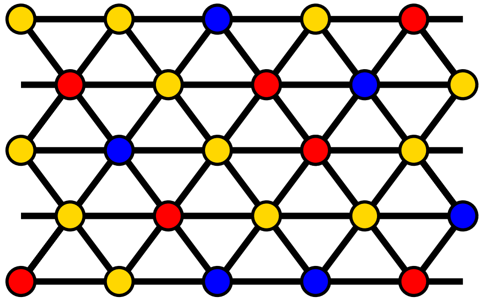
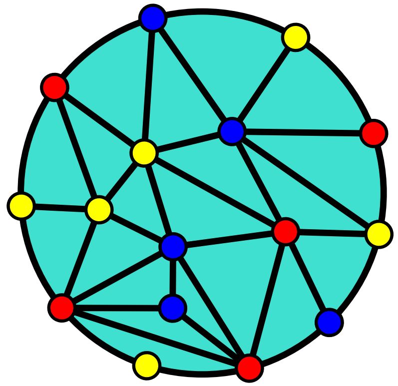


regular lattice

Vs



random lattice

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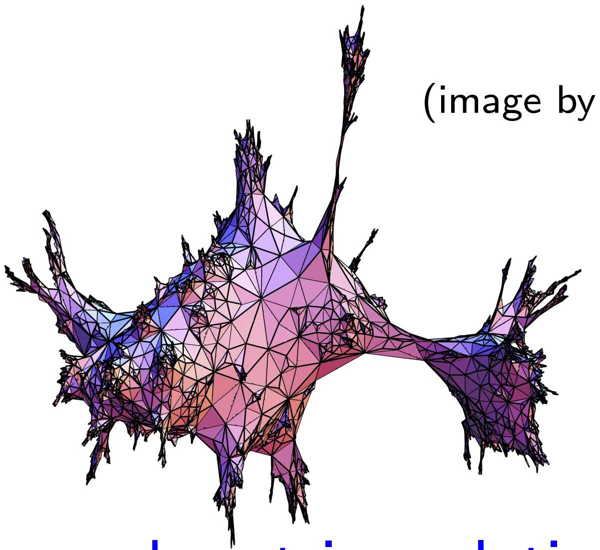
Yes! Critically weighted random lattices $\rightsquigarrow$ **random surfaces**.

Triangulations as a random surface

Uniformly random triangulation with n triangles of side length $n^{-1/4}$.



(image by N. Curien)

random triangulation

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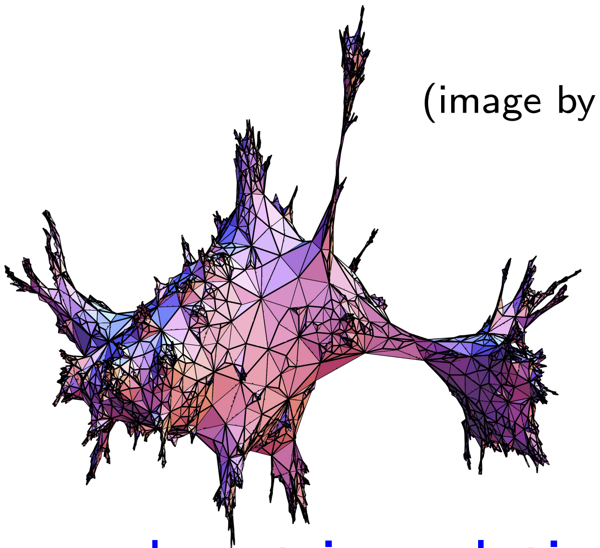

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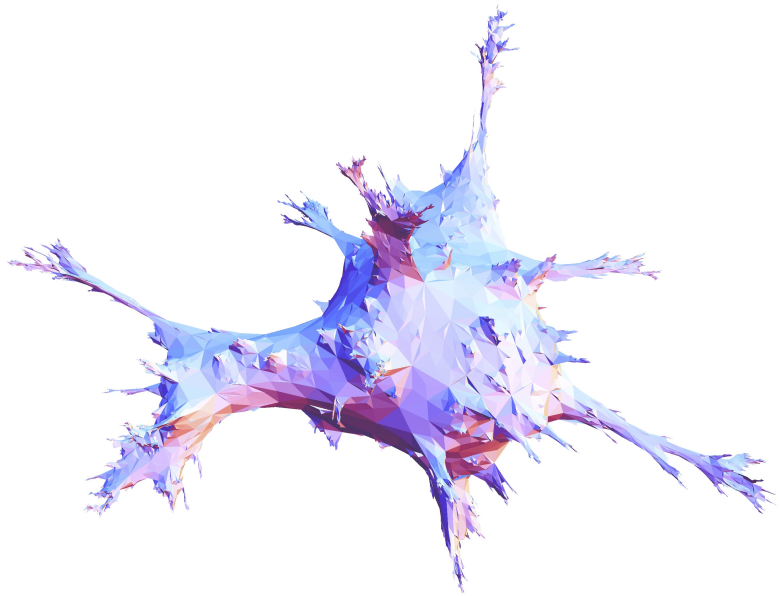


Brownian map

Theorem **[LeGall 2013, Miermont 2013]***,**
Convergence in law as a metric space (Gromov-Hausdorff topology).

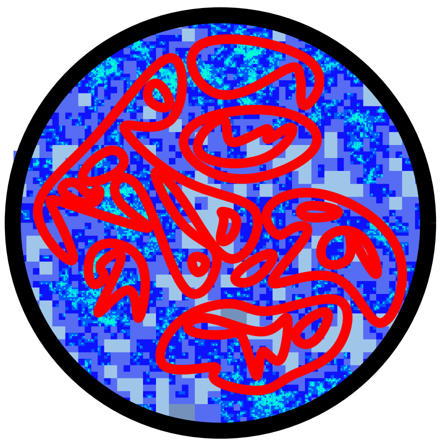
Limit is a random compact metric space homeomorphic to 2D sphere, of Hausdorff dimension 4.

(* for a different family of planar maps) (** based on prior bijective results)

Triangulations as a random surface

Goal 2: Say something new about this random surface.

Conformal Loop Ensemble (CLE)
on
Liouville Quantum Gravity (LQG)



What is . . . Liouville Quantum Gravity?

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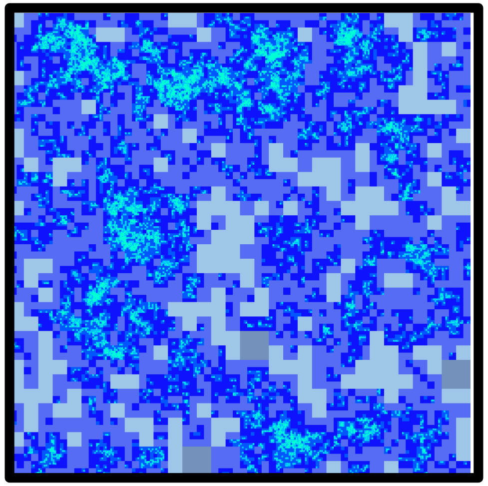
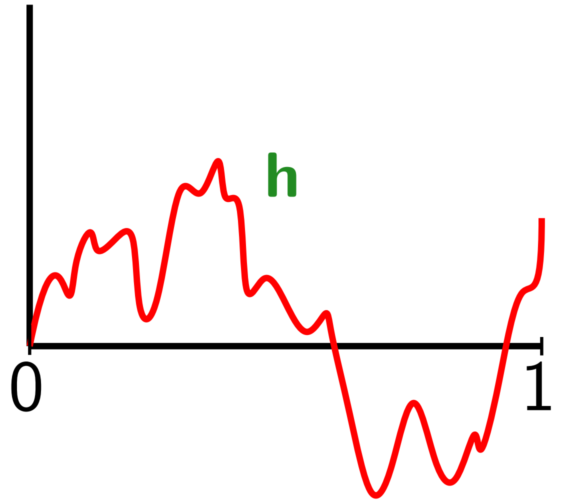


(image by J. Miller)

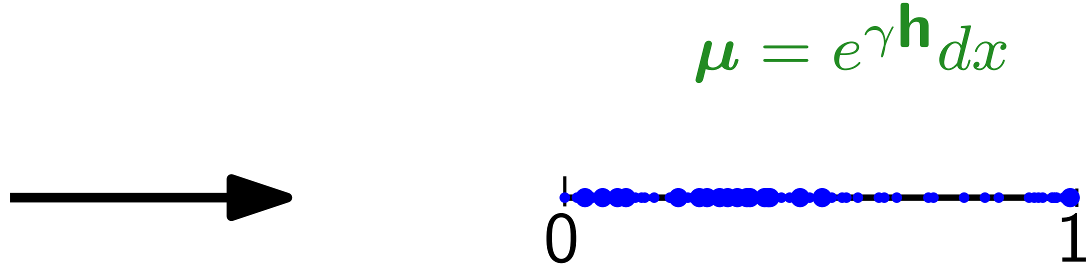
LQG is a random area measure μ on a $\mathbb{C}$-domain which is related to the Gaussian free field.

What is . . . Liouville Quantum Gravity?

1D LQG



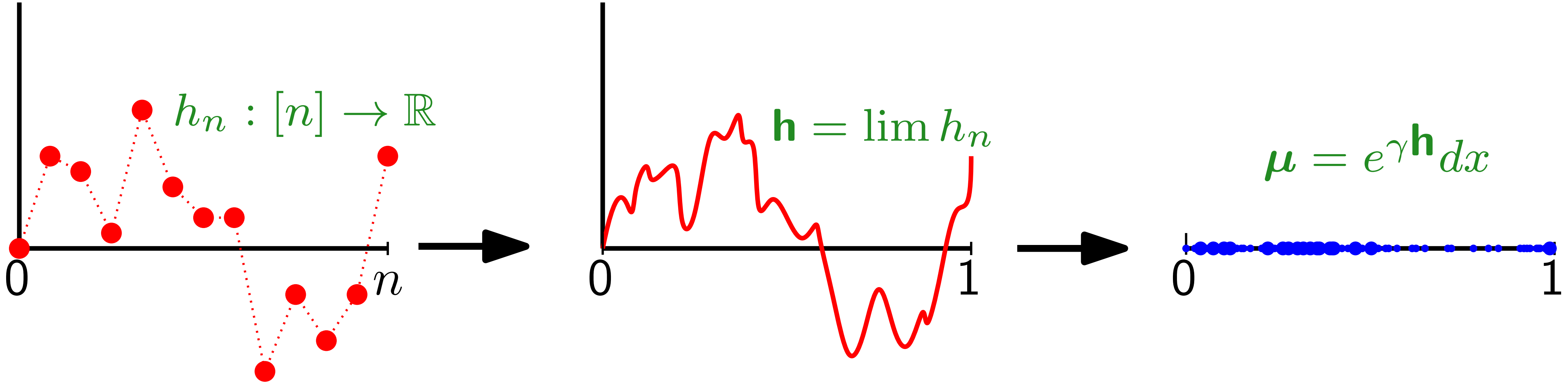
Brownian motion



1D **LQG**

What is . . . Liouville Quantum Gravity?

1D LQG



Random function chosen with probability proportional to

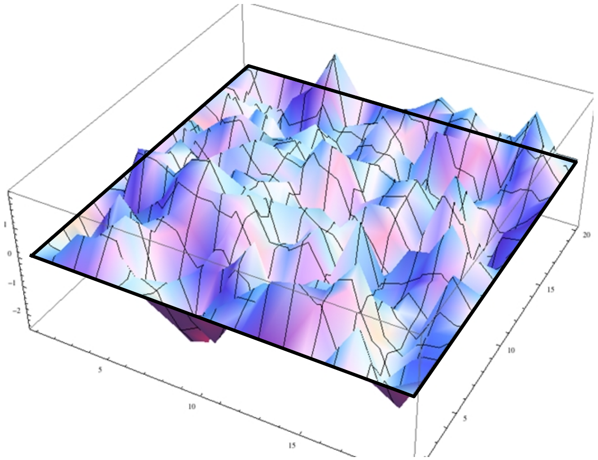
$$e^{-\sum_{i=1}^{n} \frac{(h(i)-h(i-1))^2}{2}}$$

Brownian motion

1D **LQG**

What is . . . Liouville Quantum Gravity?

$h_n : [n]^2 \to \mathbb{R}$



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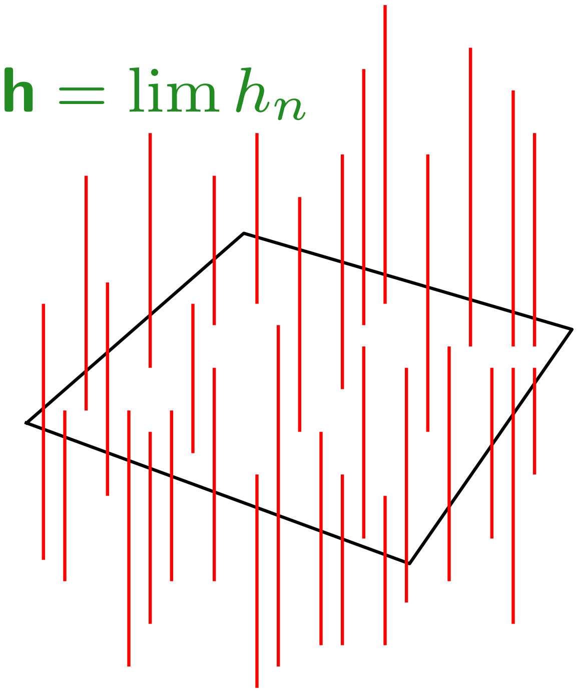
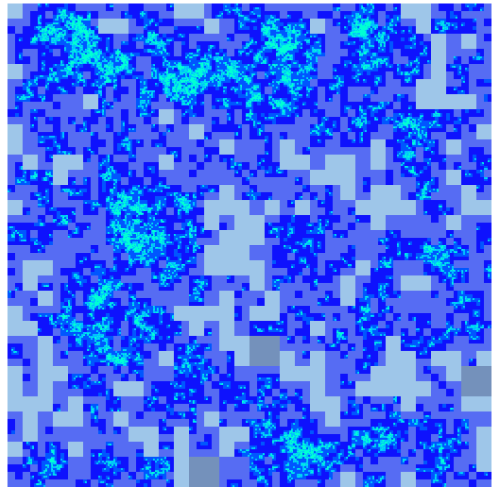
$\mathbf{h} = \lim h_n$



Gaussian Free Field

(a distribution)

$\boldsymbol{\mu} = e^{\gamma \mathbf{h}} dxdy$



LQG

(area measure)

What is . . . Liouville Quantum Gravity?

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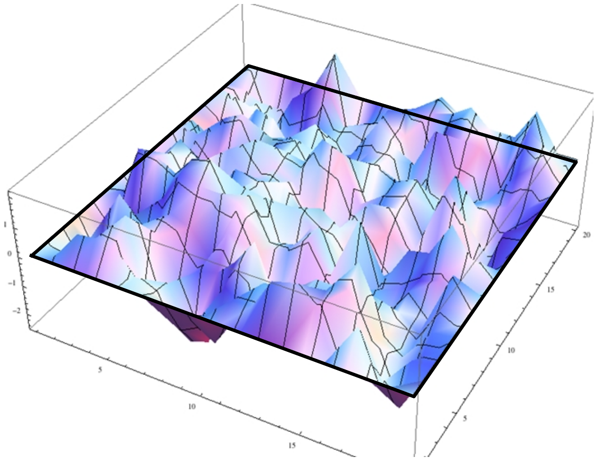

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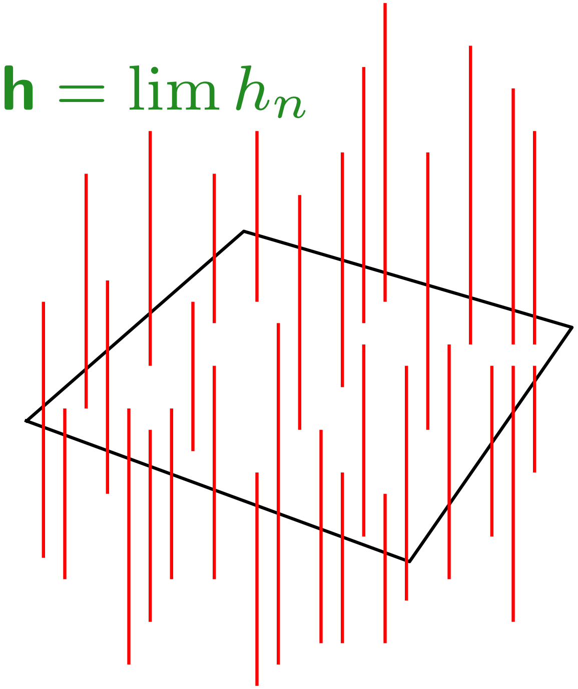

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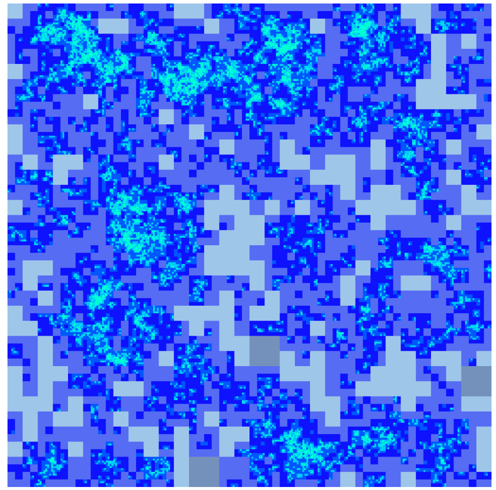

$\gamma \in [0, 2]$ controls how wild LQG measure is.

Today: $\gamma = \sqrt{8/3}$. "pure gravity"

What is... a SLE (Schramm–Loewner evolution)?

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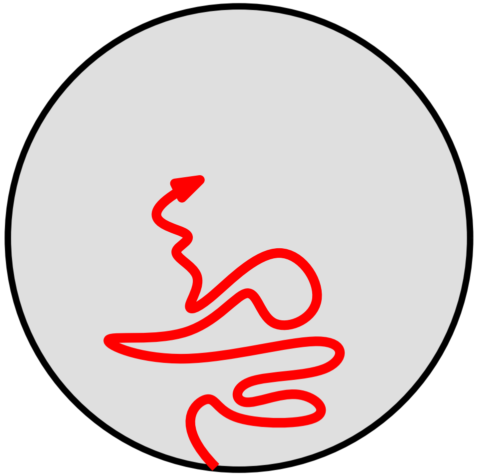
SLE$_\kappa$ is a random (non-crossing, parametrized) curve in a $\mathbb{C}$-domain.



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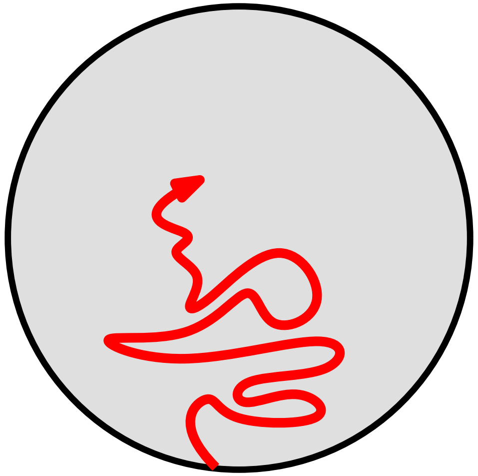


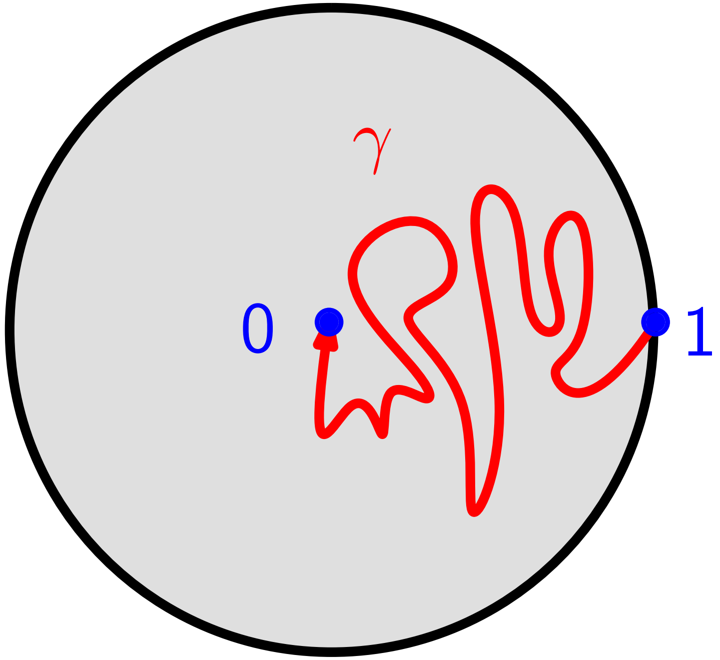
The parameter κ determines how much the curve "wiggles".

SLE$_\kappa$ were introduced to describe the scaling limit of curves from statistical mechanics.

What is... a SLE (Schramm–Loewner evolution)?

SLE are **characterized** by:

- Conformal invariance property
- Markov domain property



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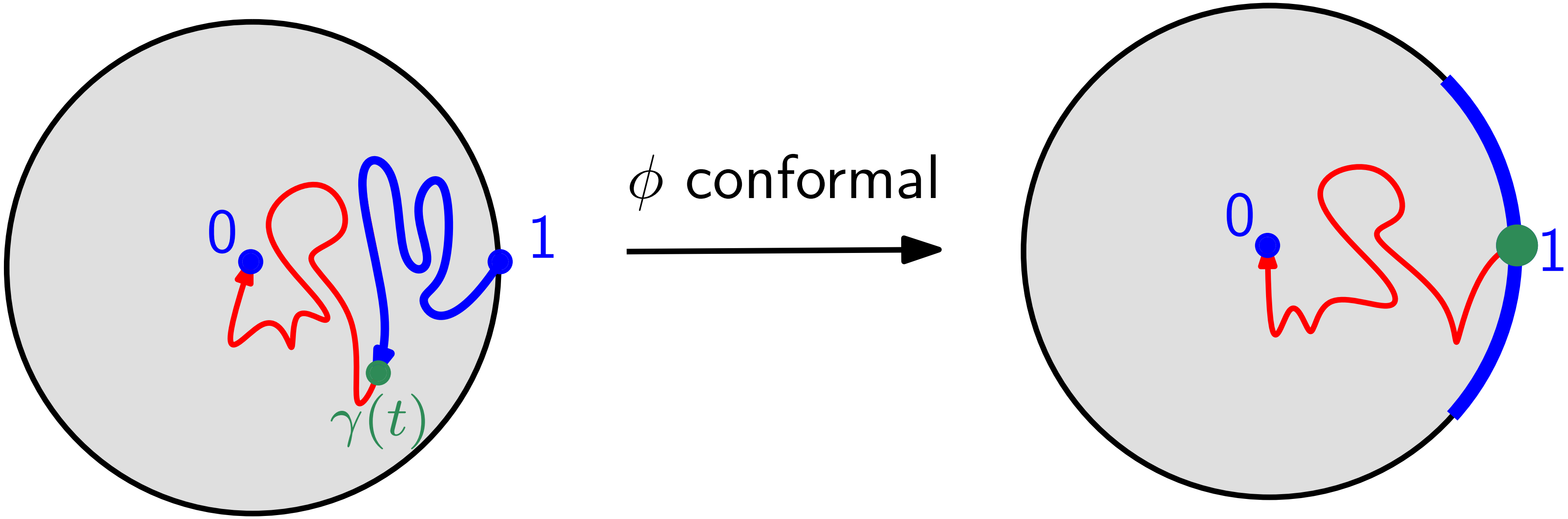

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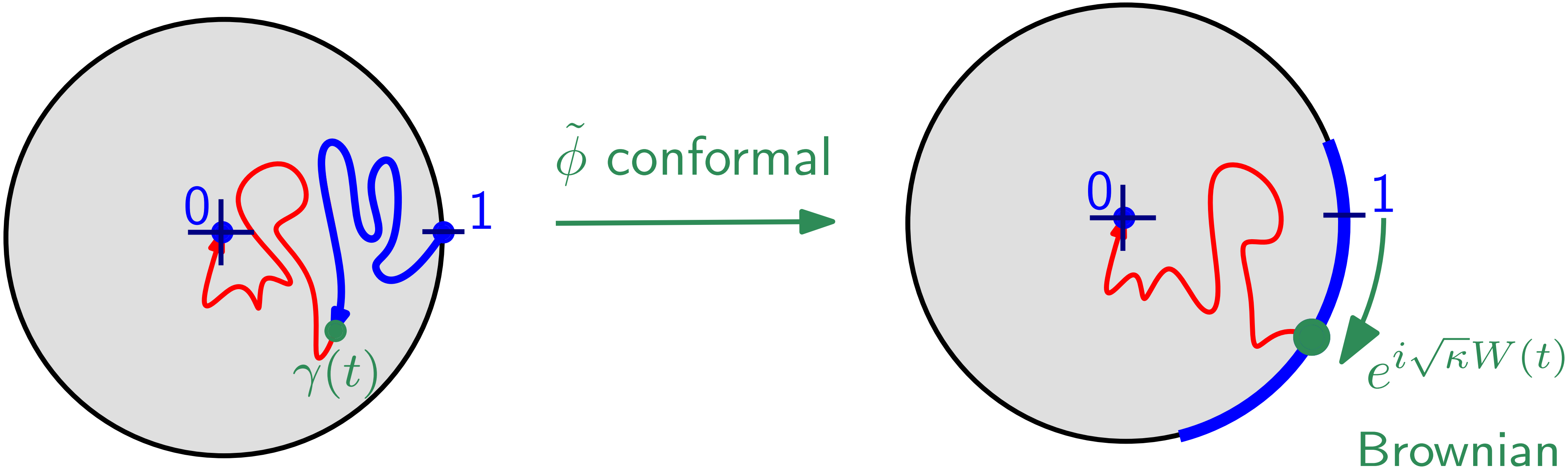

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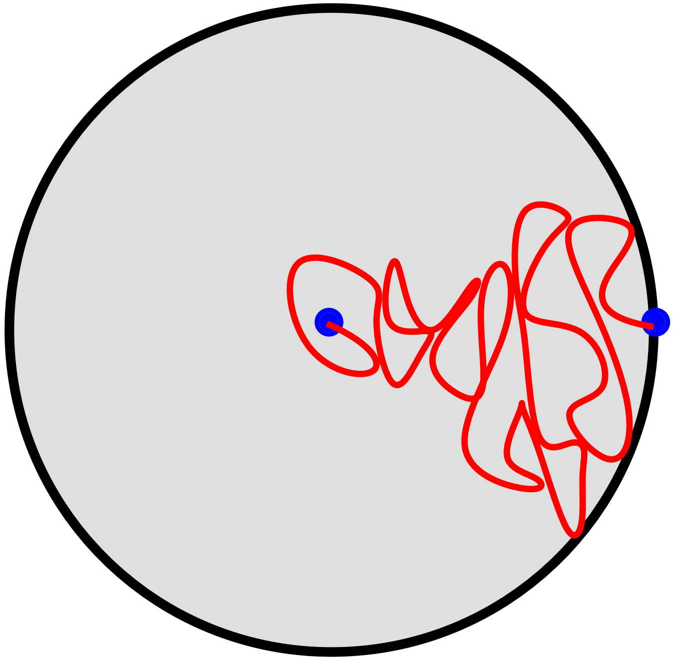
Today: $\kappa = 6$ (percolation)



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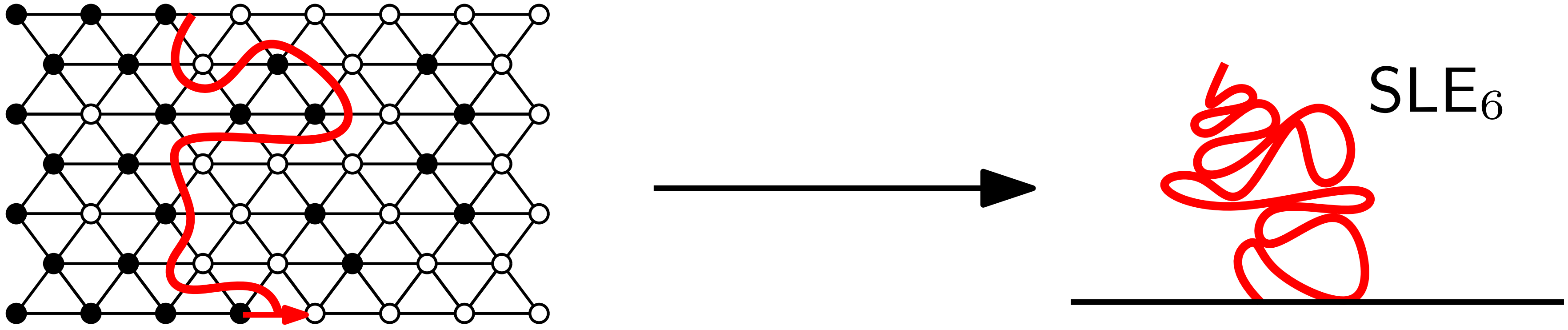
Theorem [Smirnov 01]: Convergence.



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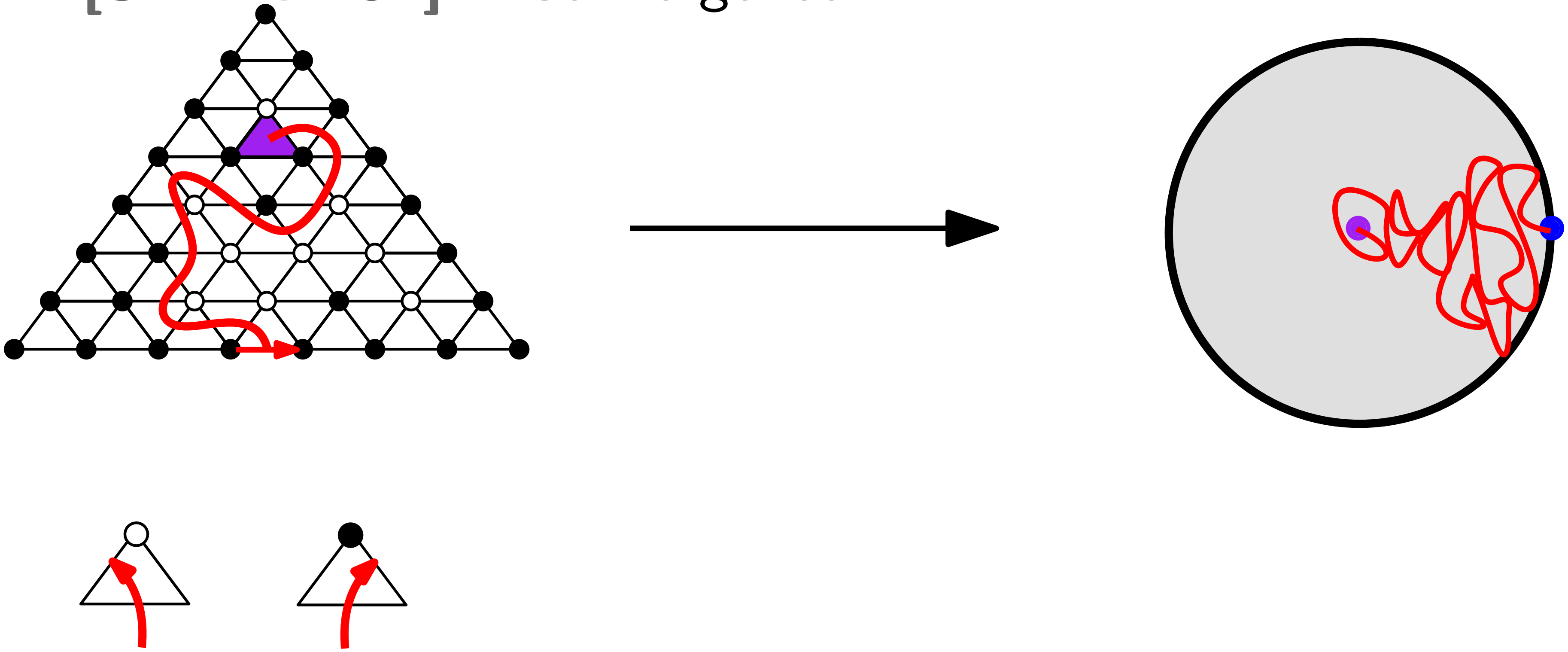

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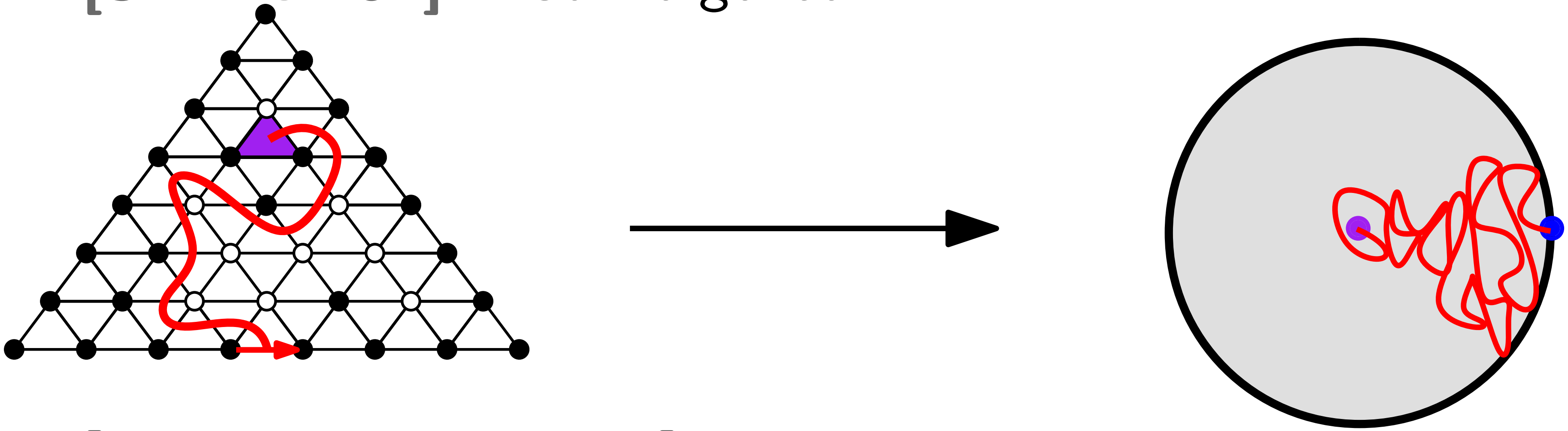
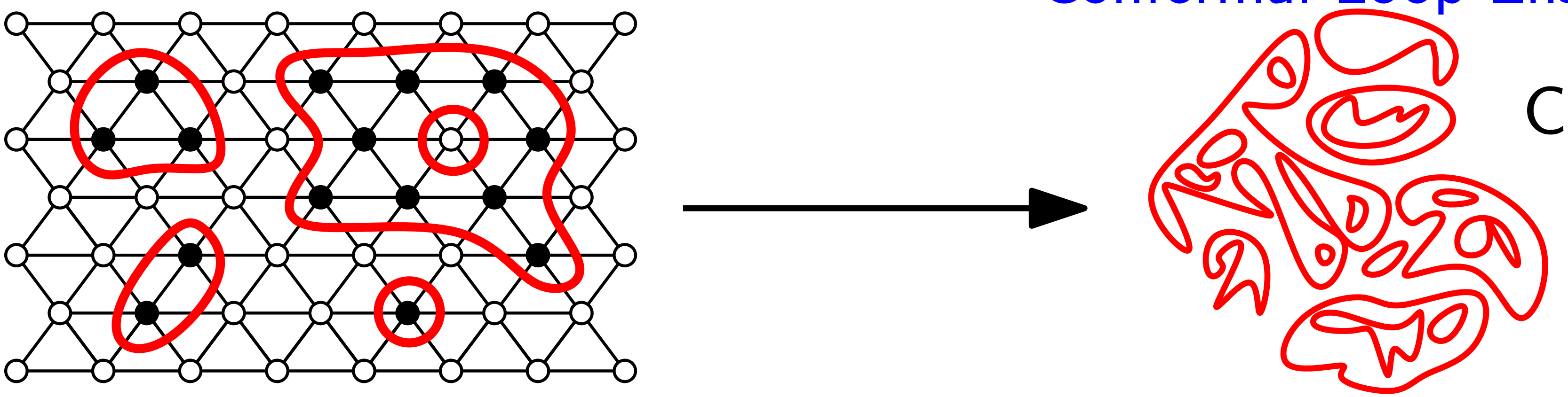


Theorem [Camia, Newman 09]: Convergence.

Conformal Loop Ensemble



CLE_6

Conjectural relation (1990s)

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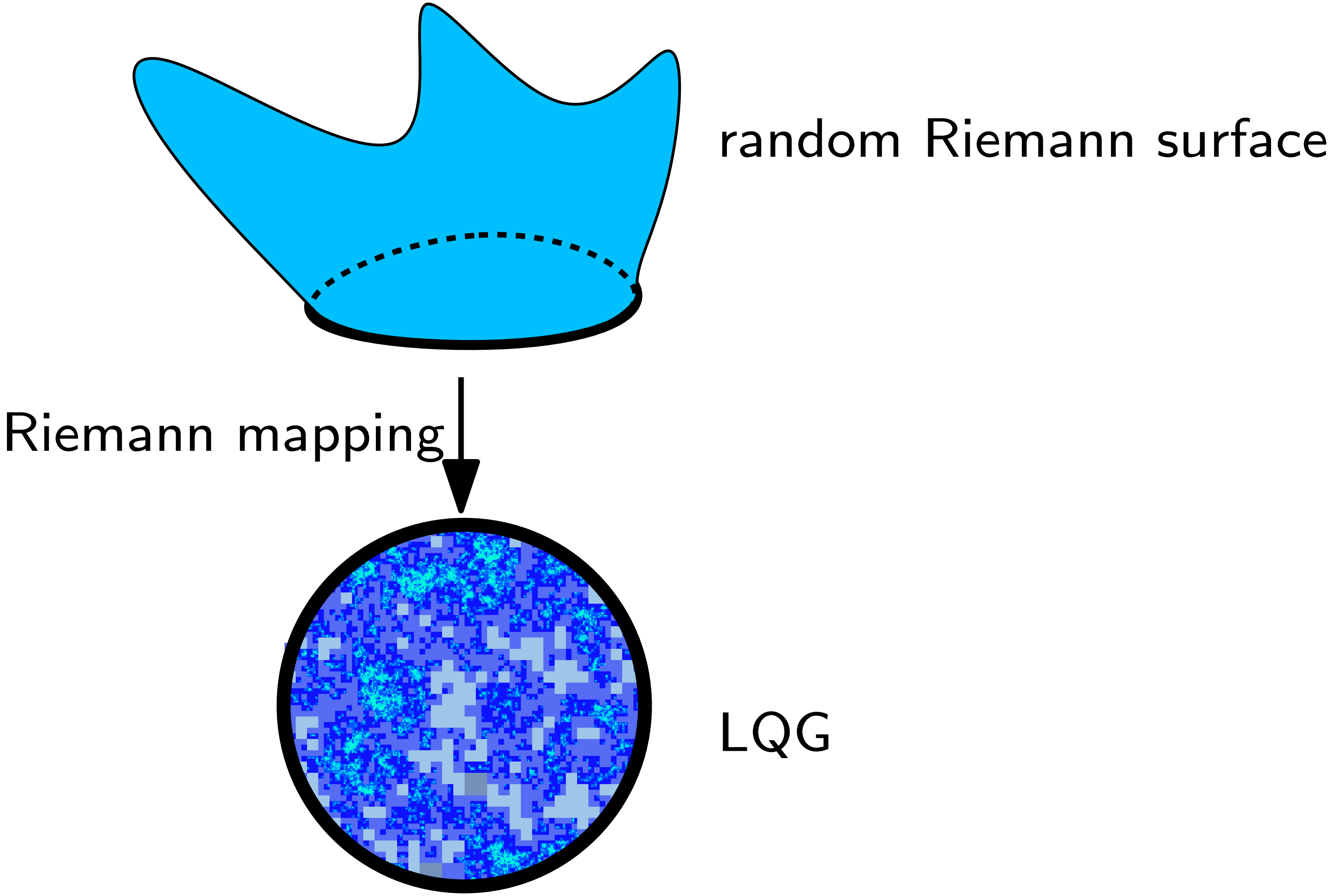
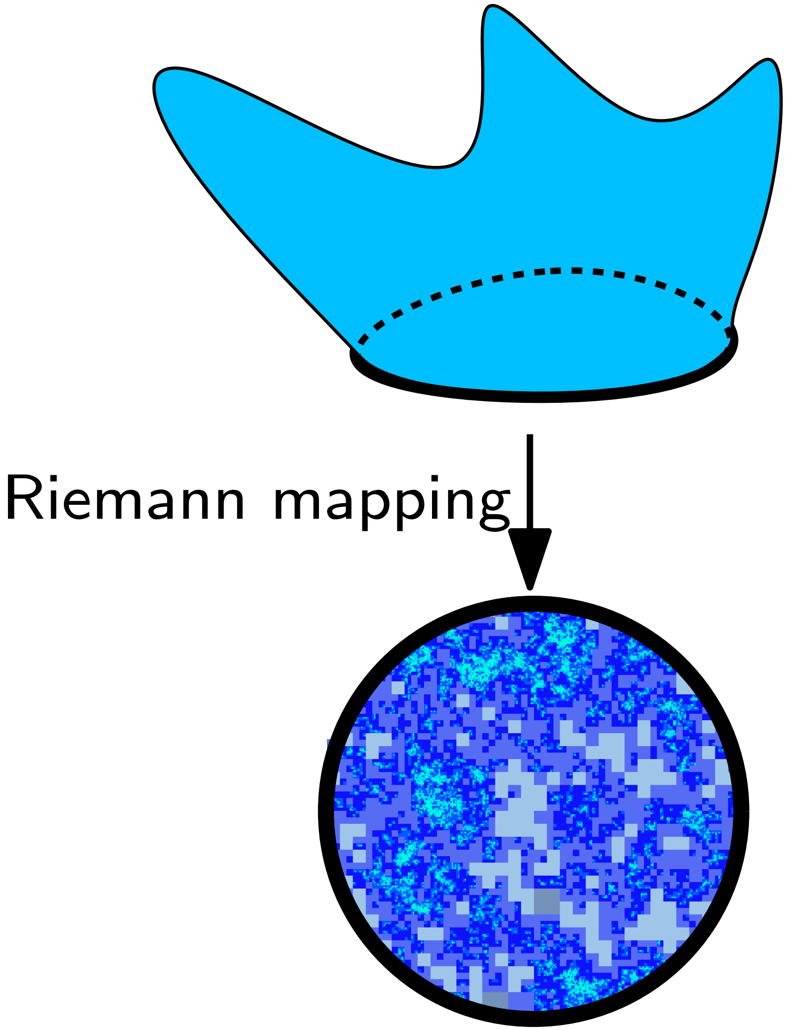
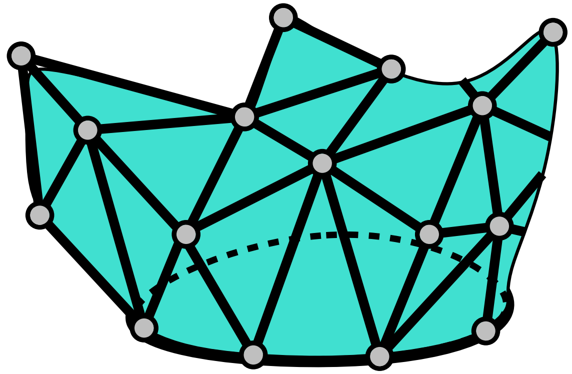


LQG was introduced in physics as a model of random surface describing space-time evolution of strings.

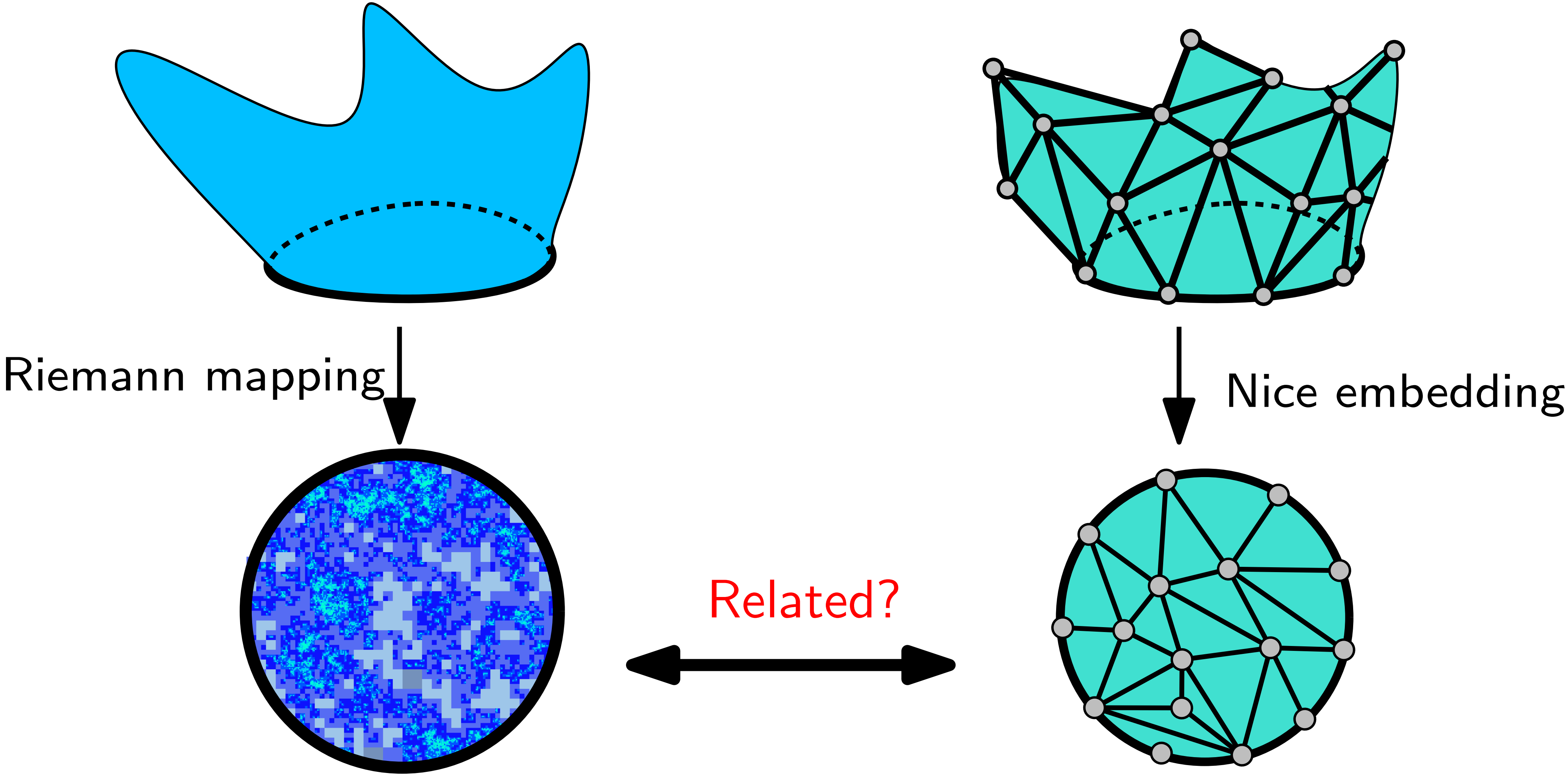
Conjectural relation (1990s)





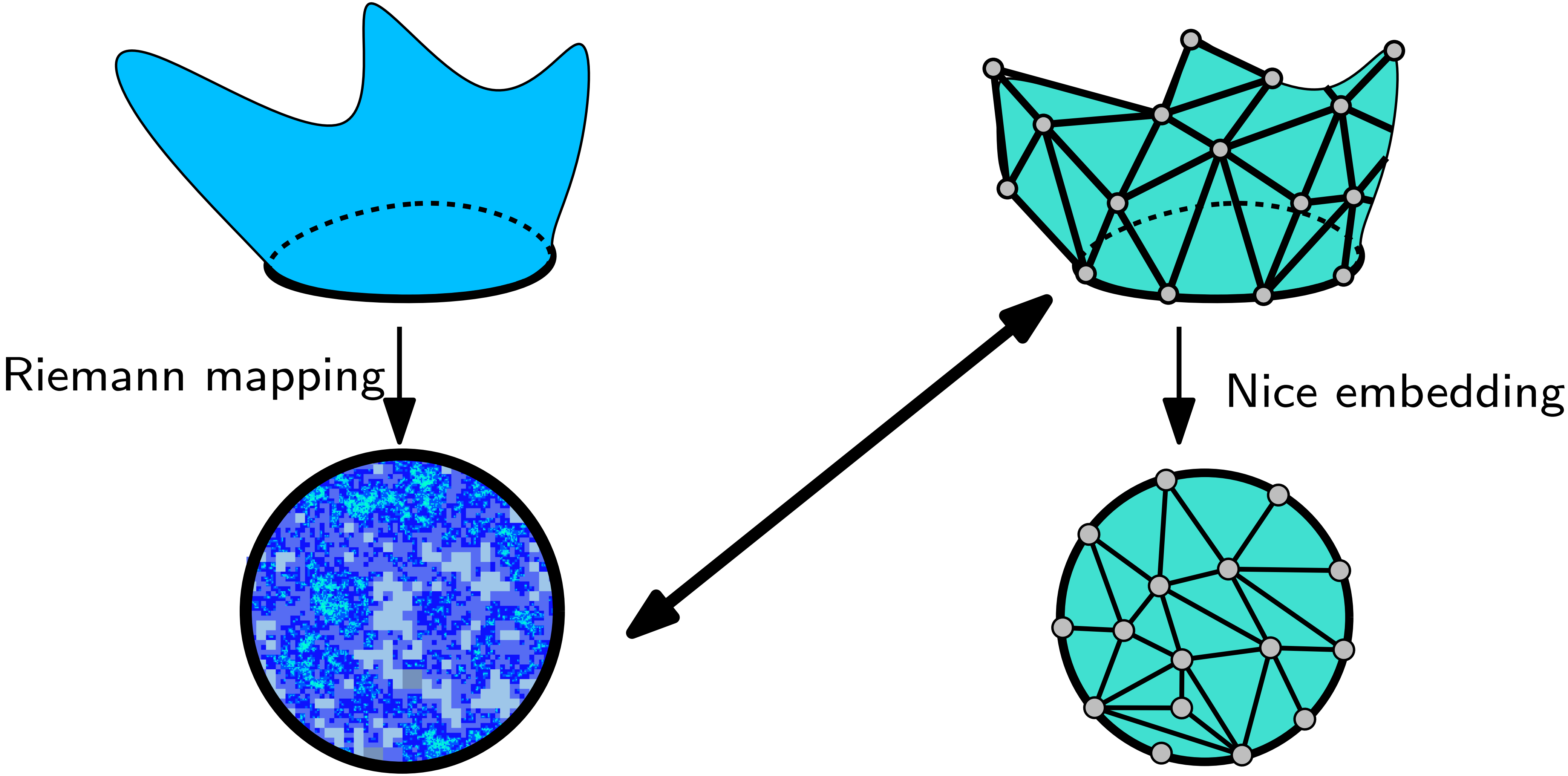
Random triangulations gives another natural model of random surfaces.

Conjectural relation (1990s)



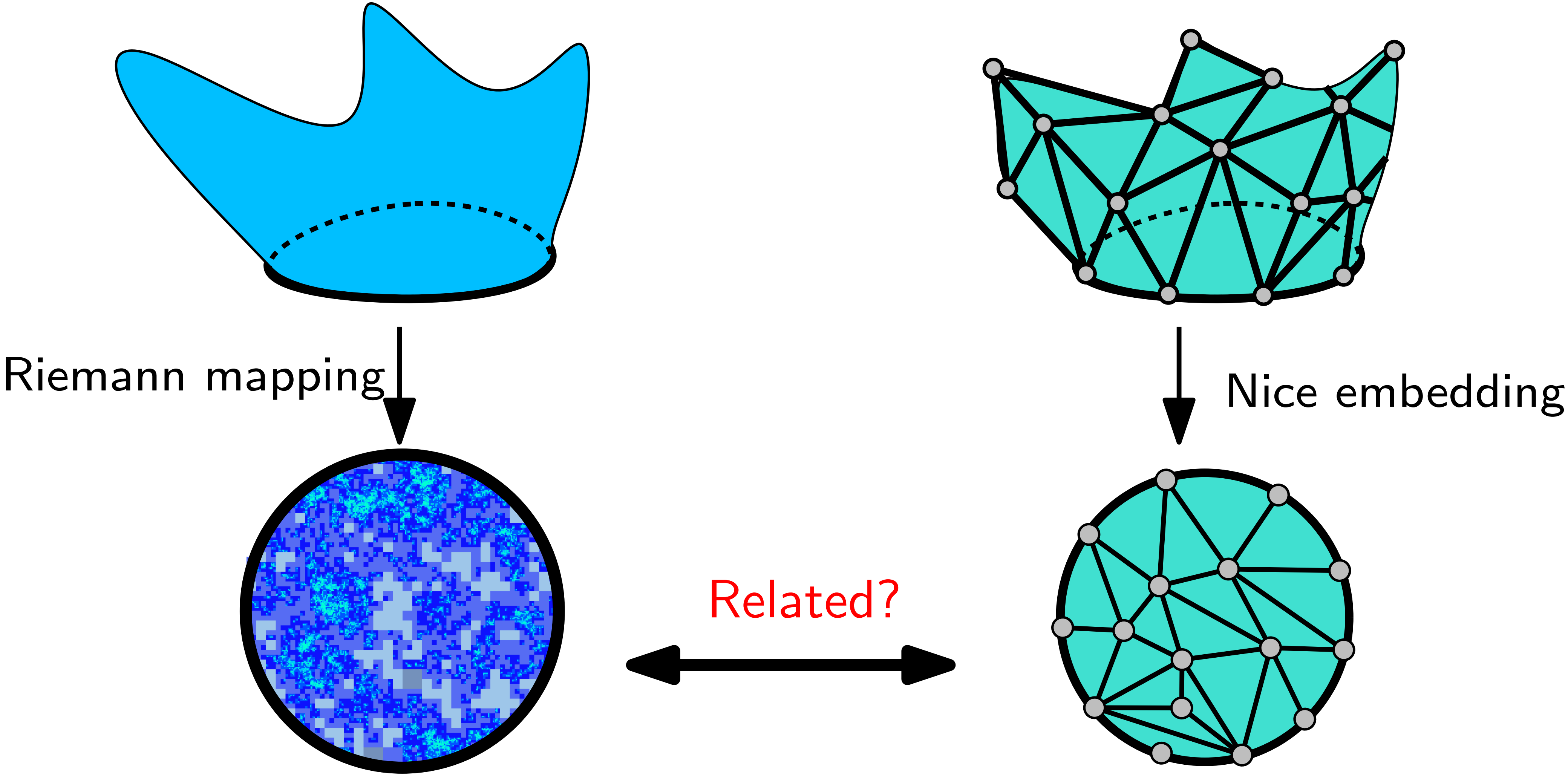
It was **conjectured that the two models were in fact exactly related**.

Conjectural relation (1990s)



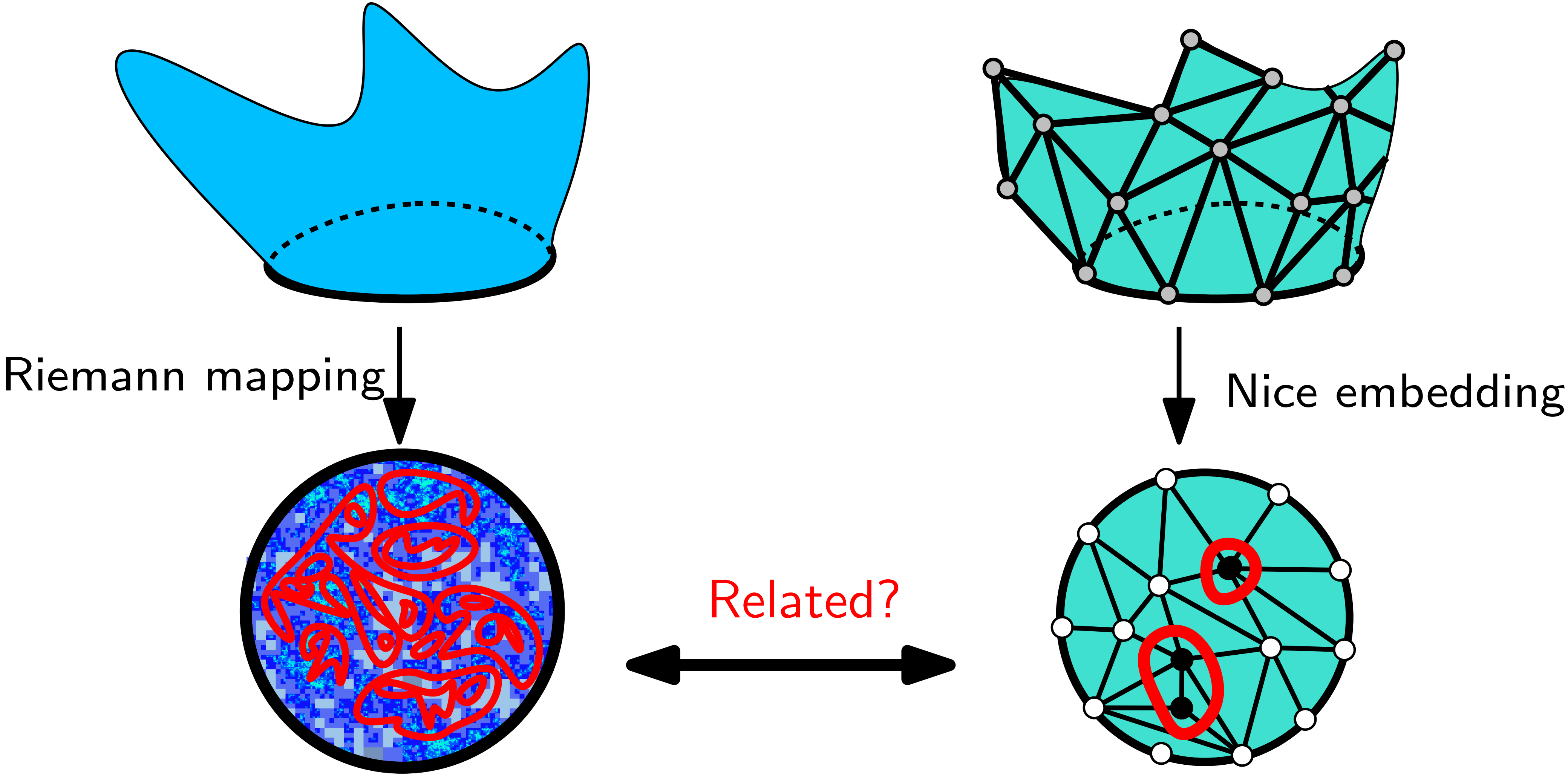
Thm [Miller, Sheffield 2016]: Equality as **metric spaces**.

Conjectural relation (1990s)



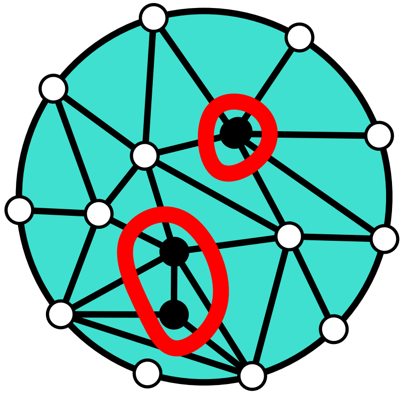
Goal 2': Establish a relation between LQG and "embedded" random triangulations.

Conjectural relation (1990s)



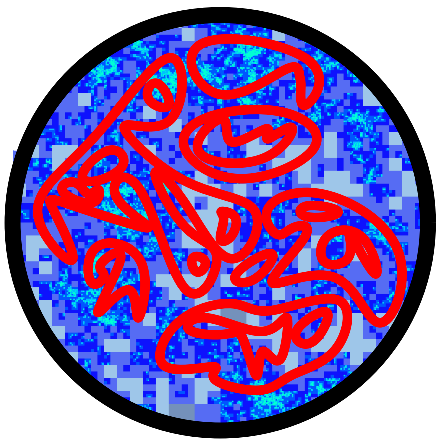
Goal 3: Establish a relation between **percolation interfaces** on random triangulations and CLE_6.

Convergence results



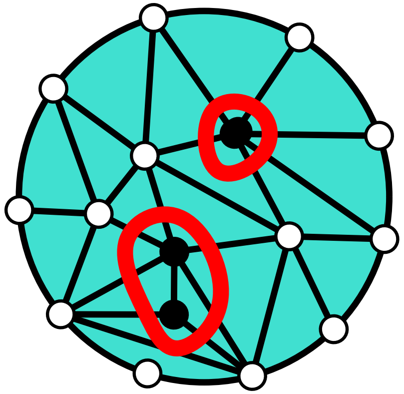
Percolation on random triangulations

convergence under nice embedding



CLE on Liouville Quantum Gravity

Convergence results

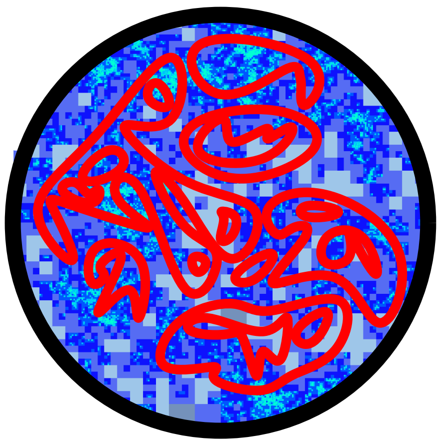


Percolation on random triangulations

convergence

some

under ~~nice~~ embedding



CLE on Liouville Quantum Gravity

Thm [Bernardi, Holden, Sun 18]:
Let (M_n, σ_n) uniformly random percolated triangulation of size n (n interior vertices, $\sqrt{n}$ exterior vertices).

There exist embeddings $\phi_n : M_n \to \mathbb{D}$ (and coupling) such that the following **converge jointly in probability**:

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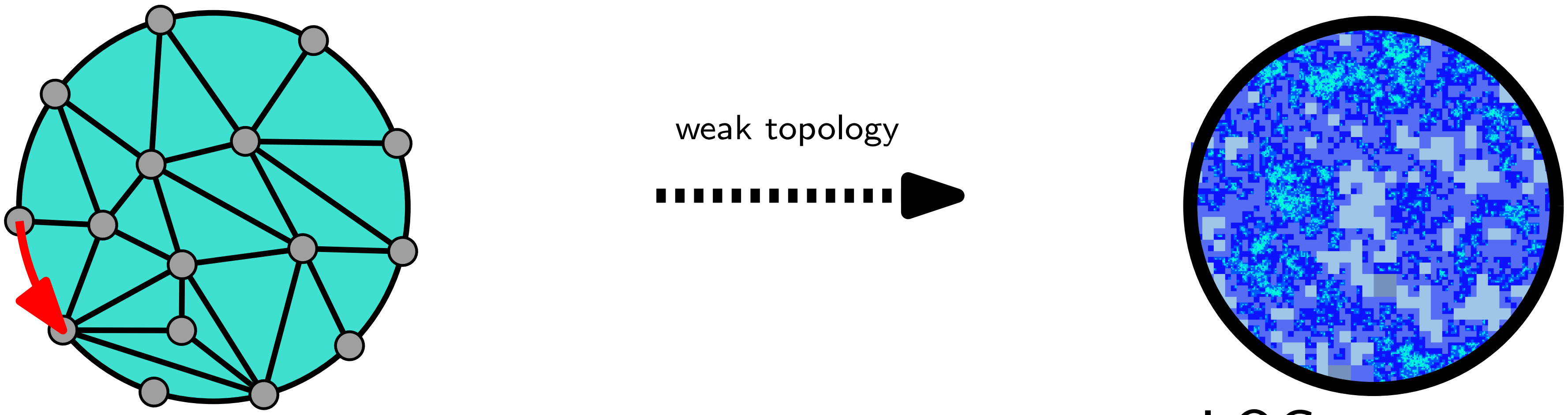

$\phi_n(M_n)$ LQG$_{\sqrt{8/3}}$

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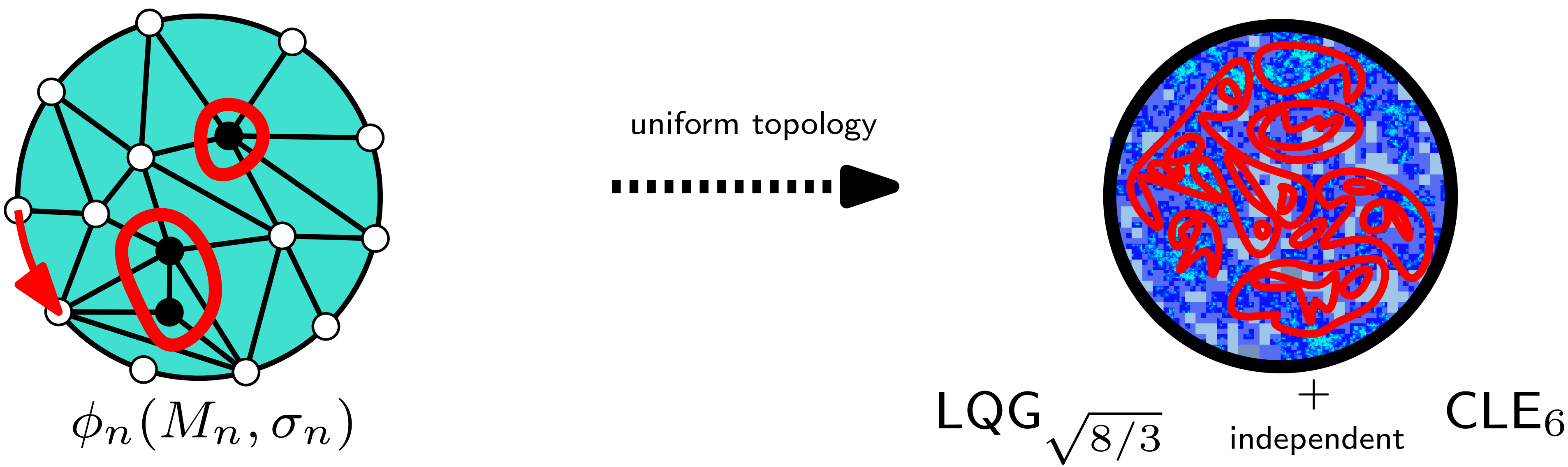

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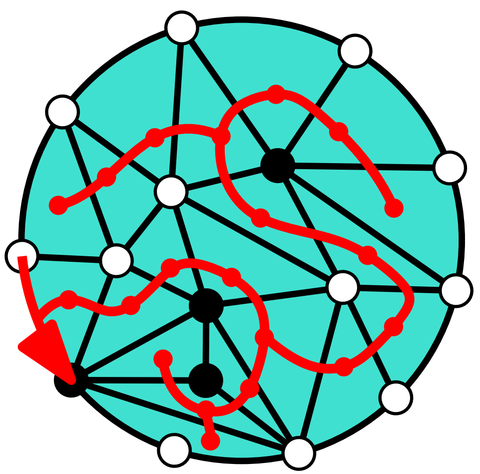
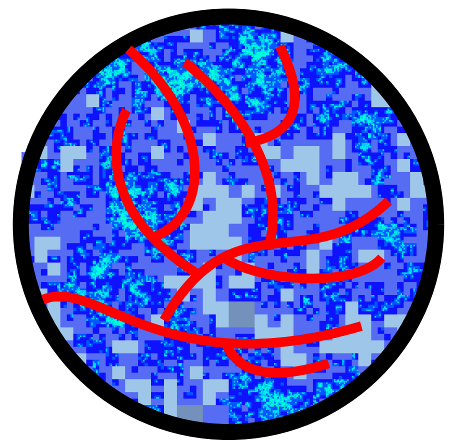


uniform topology on subtrees



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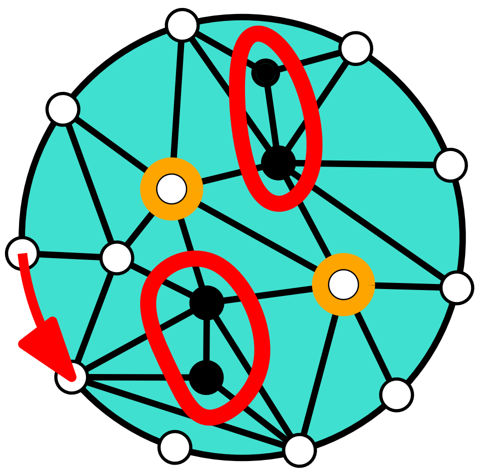

weak topology

Thm [Bernardi, Holden, Sun 18]:
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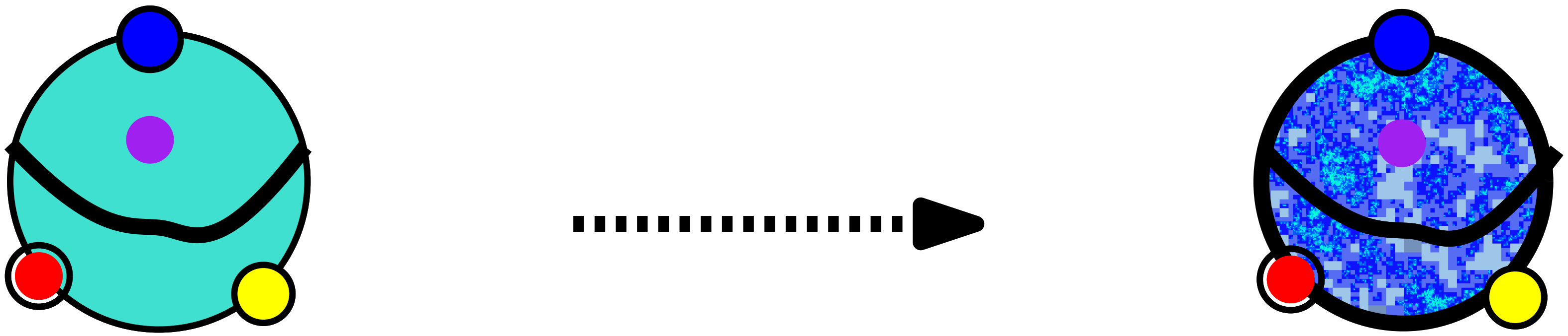
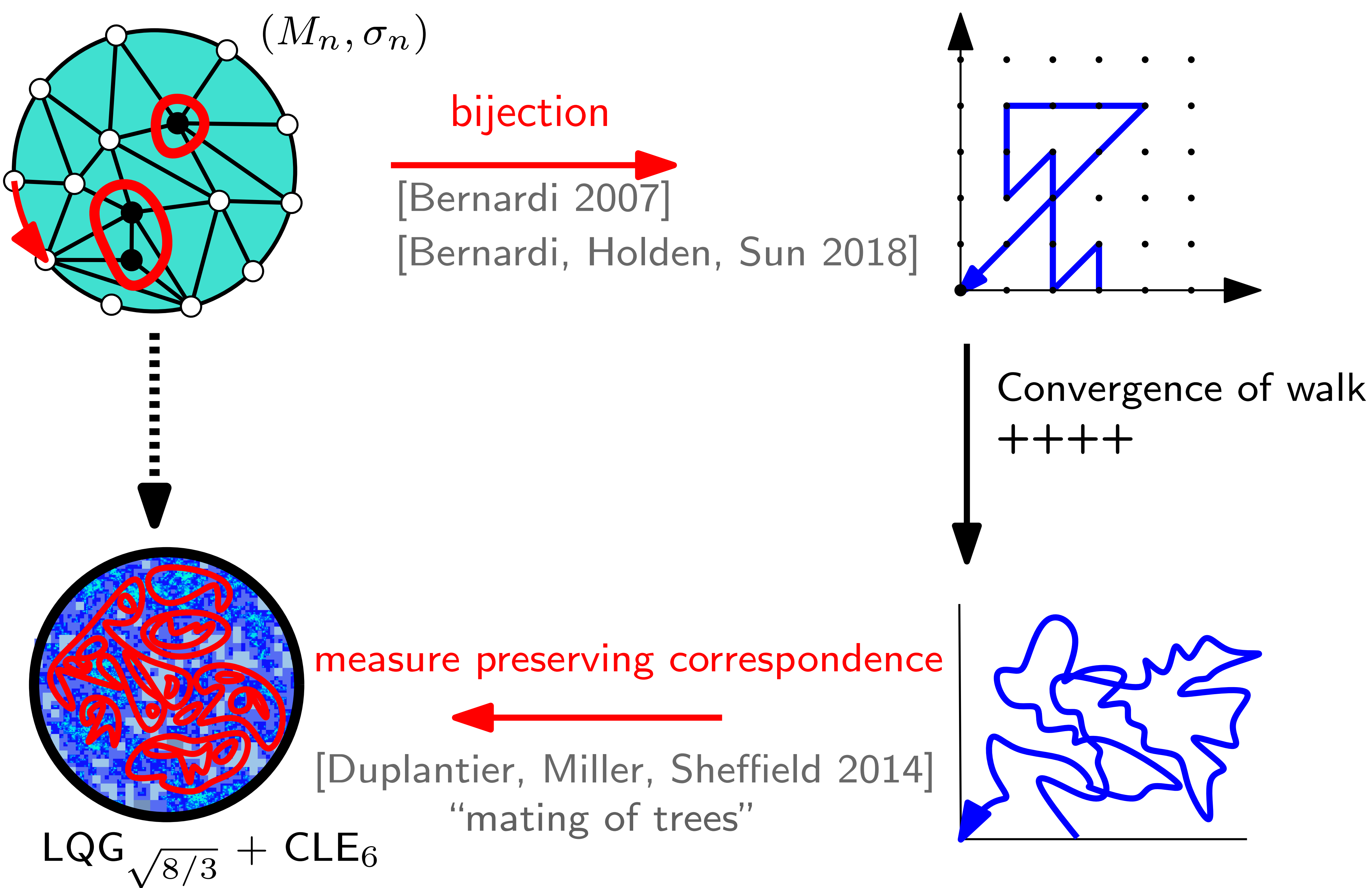
- **Pivotal measures:** $\forall \epsilon, i, j$, $\nu_{i,n}^{\epsilon} \longrightarrow \boldsymbol{\nu}_i^{\epsilon}$, and , $\nu_{i,j,n}^{\epsilon} \longrightarrow \boldsymbol{\nu}_{i,j}^{\epsilon}$.
- **Crossing events:** For random vertex v_n, $E_b(v_n) \longrightarrow \boldsymbol{E}_b(\boldsymbol{v})$.



Strategy of proof:



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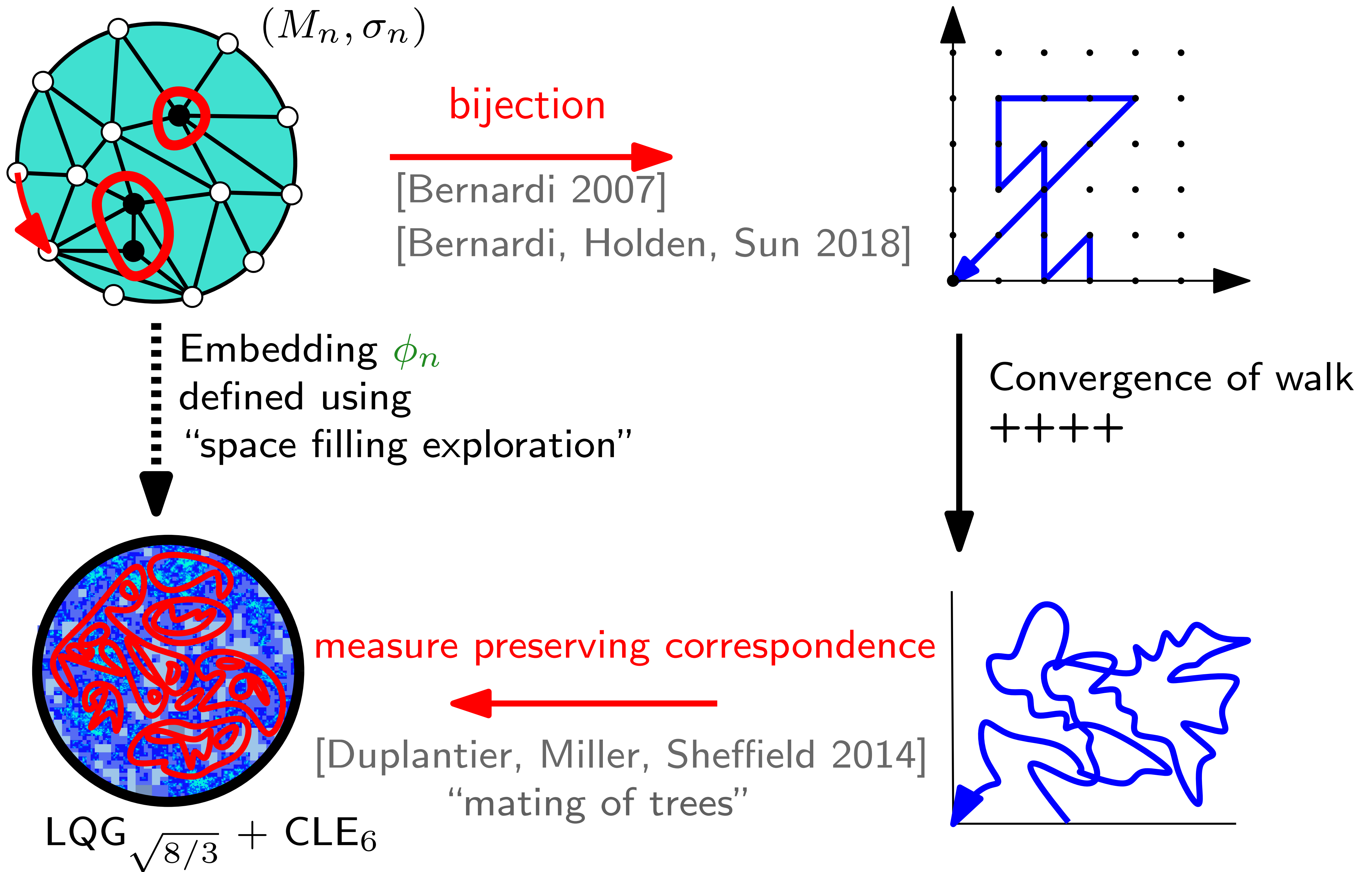
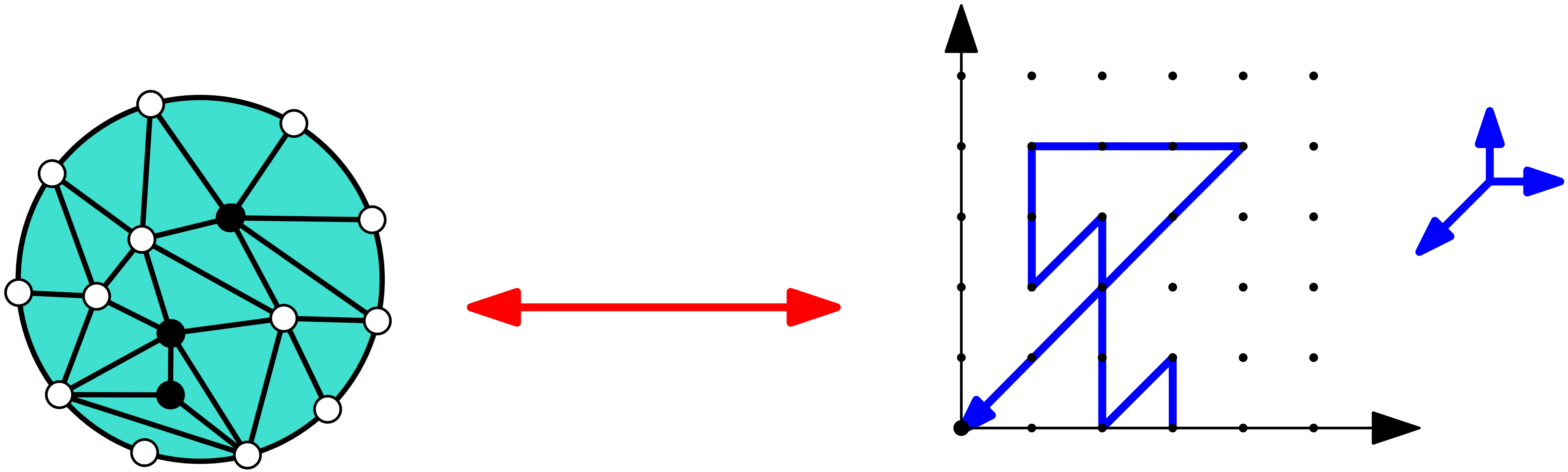
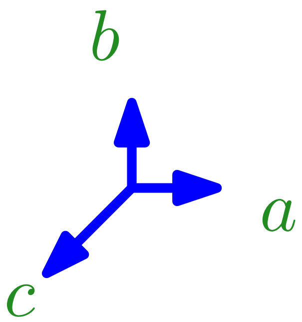


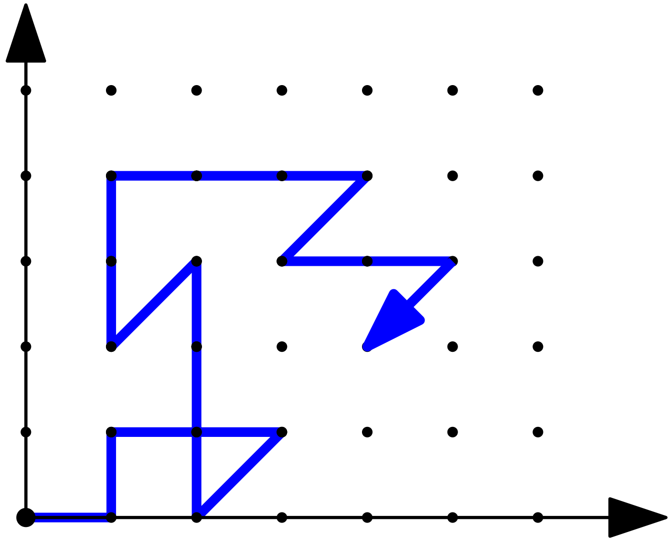
The bijection



Kreweras walks

Def. A **Kreweras walk** is a lattice walk on $\mathbb{Z}^2$ using the steps $a = (1, 0)$, $b = (0, 1)$ and $c = (-1, -1)$.

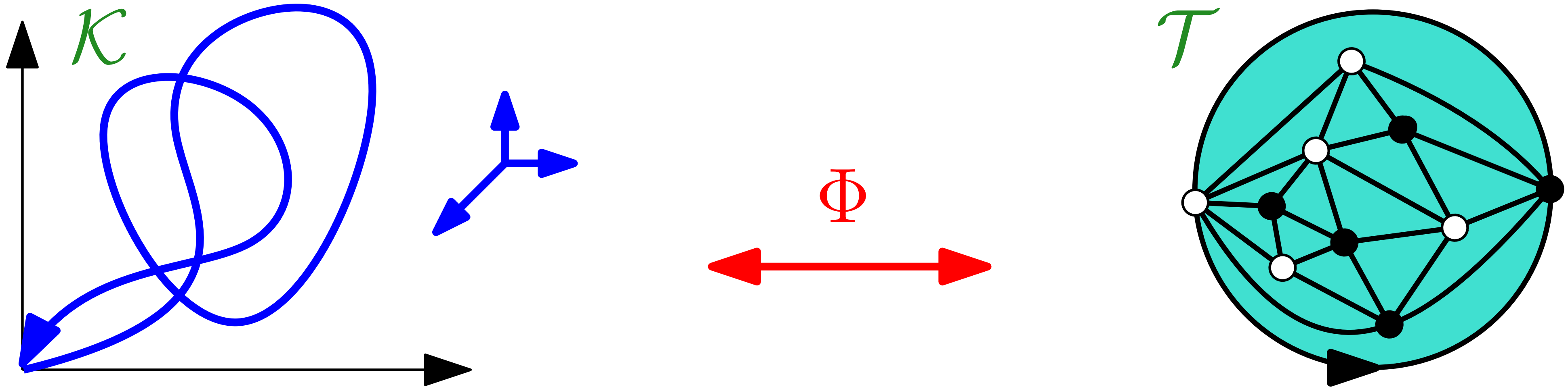




Thm [Bernardi 07/ Bernardi, Holden, Sun 18]:

There is a **bijection** between:

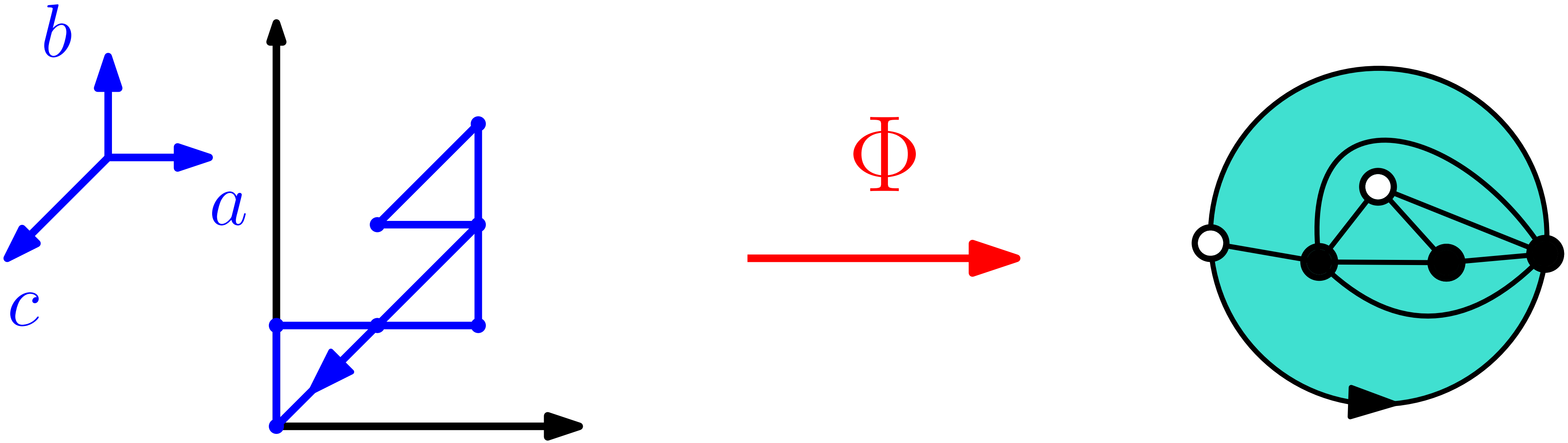
- $\mathcal{K}$ = set of Kreweras walks starting and ending at $(0,0)$ and staying in $\mathbb{N}^2$.
- $\mathcal{T}$ = set of percolated triangulations of the disk with 2 exterior vertices: one white and one black.



$3n$ steps

n interior vertices

Example: $w = baabbcacc$



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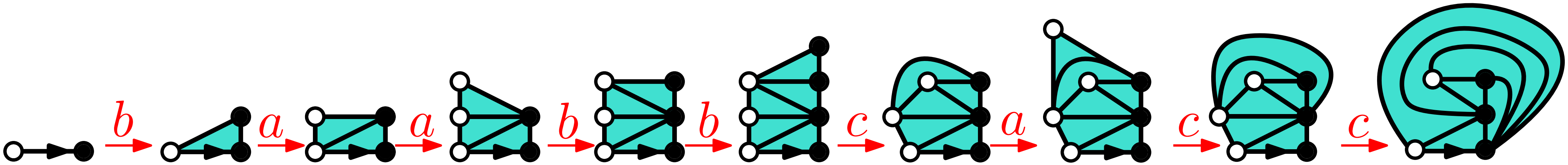

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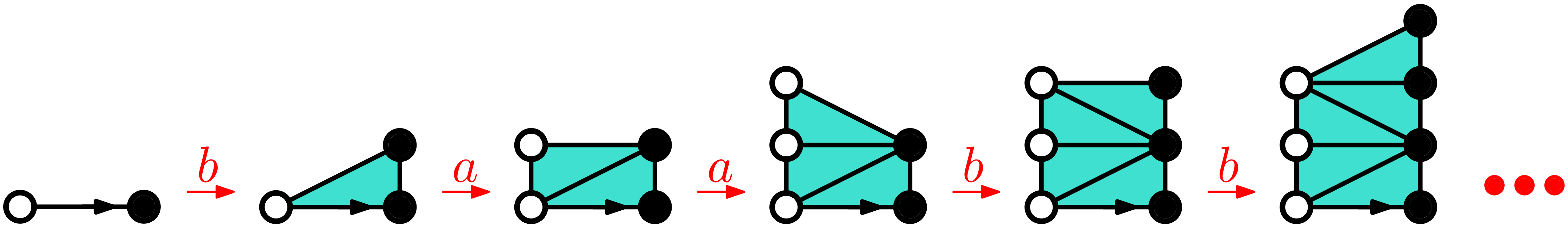
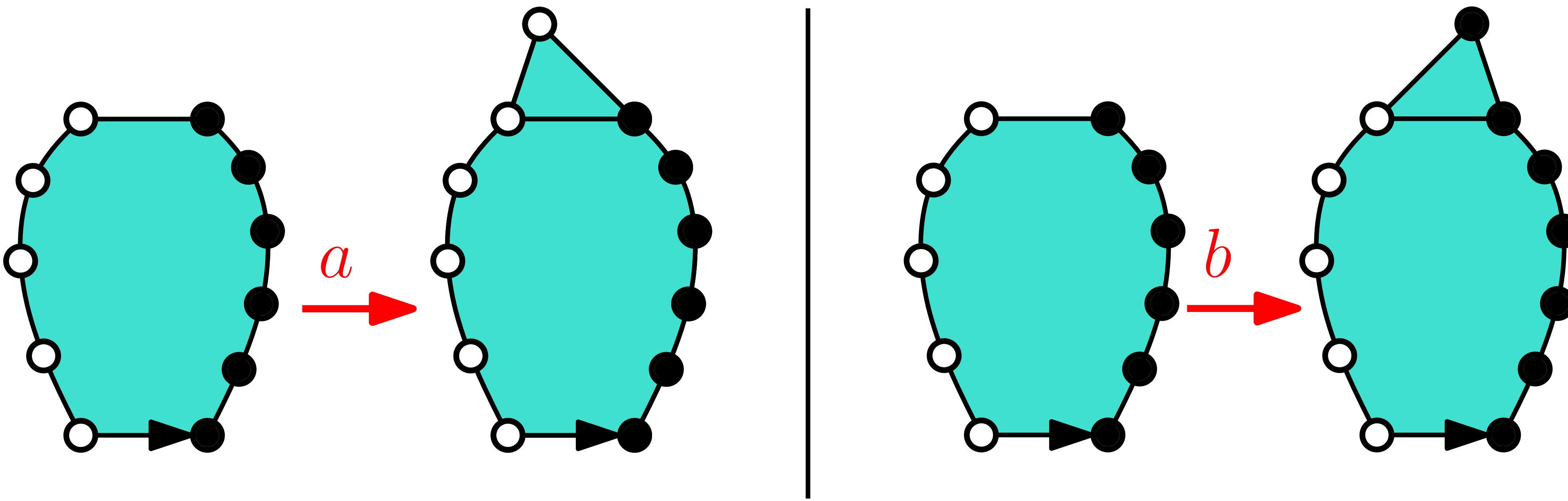


Definition:



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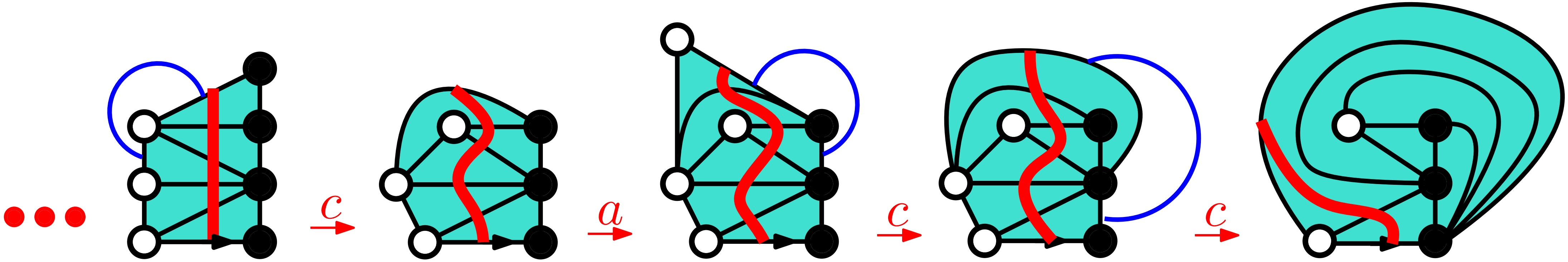
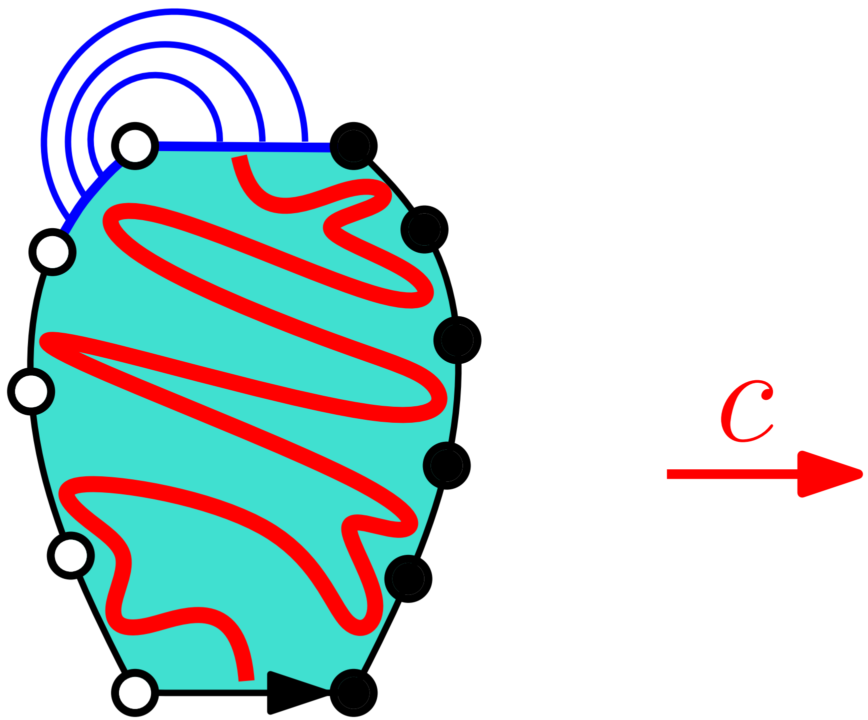
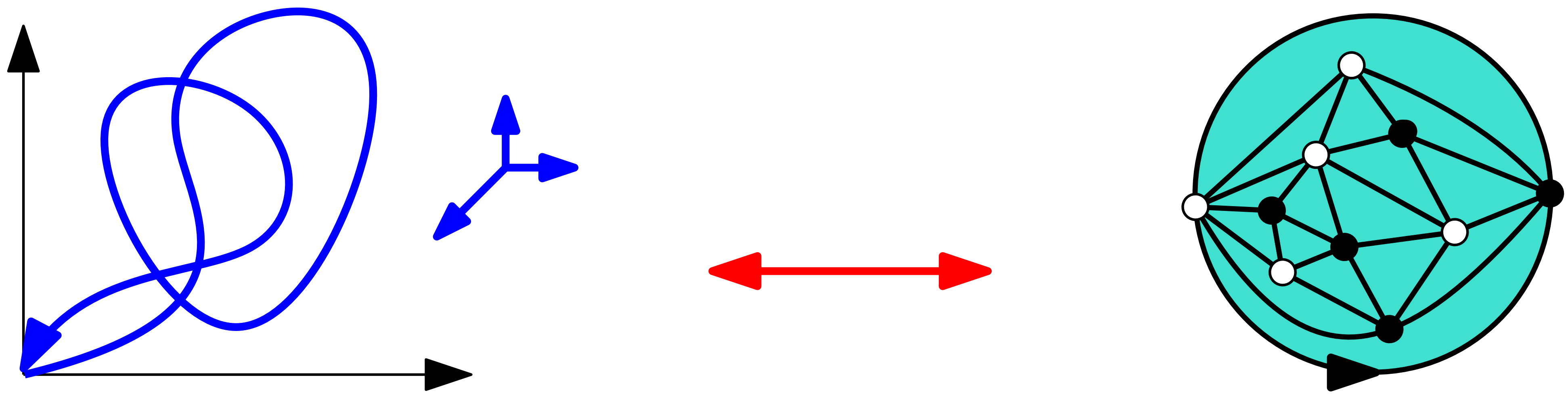


Definition:



Thm: This is a **bijection**.

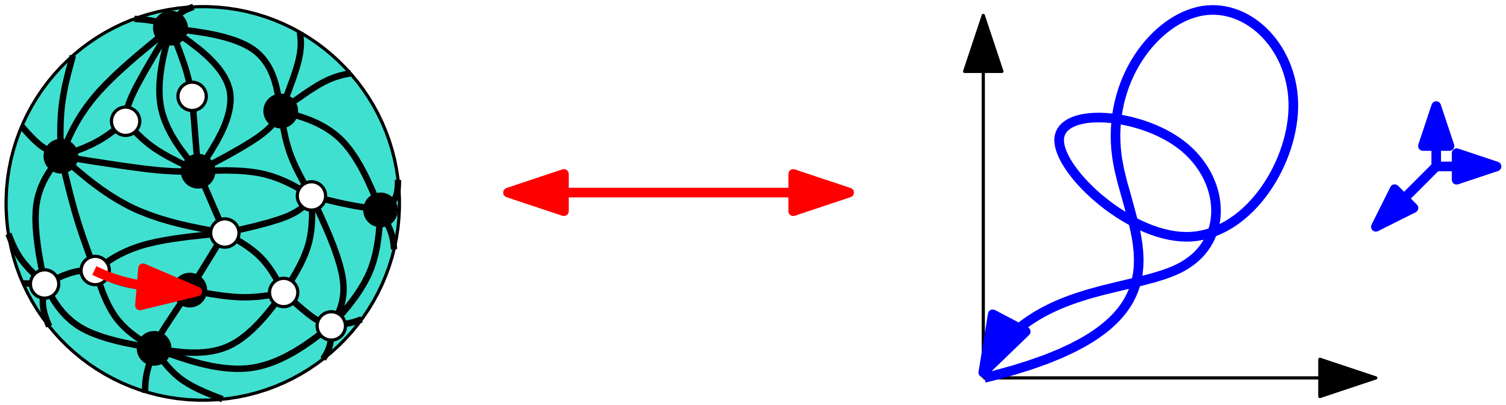


$3n$ steps

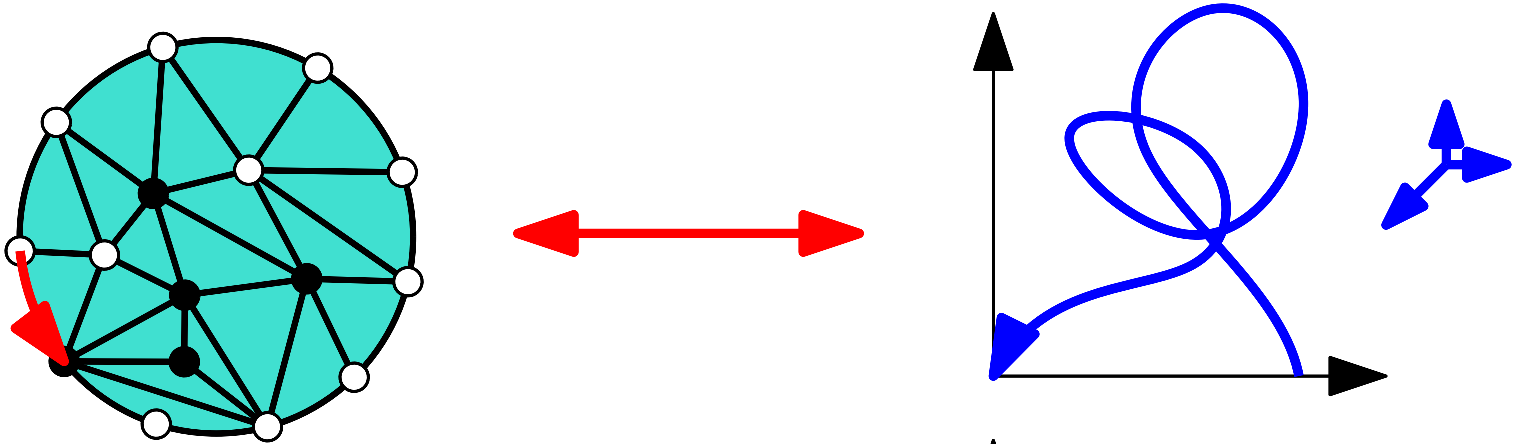
n interior vertices

Variants of the bijection

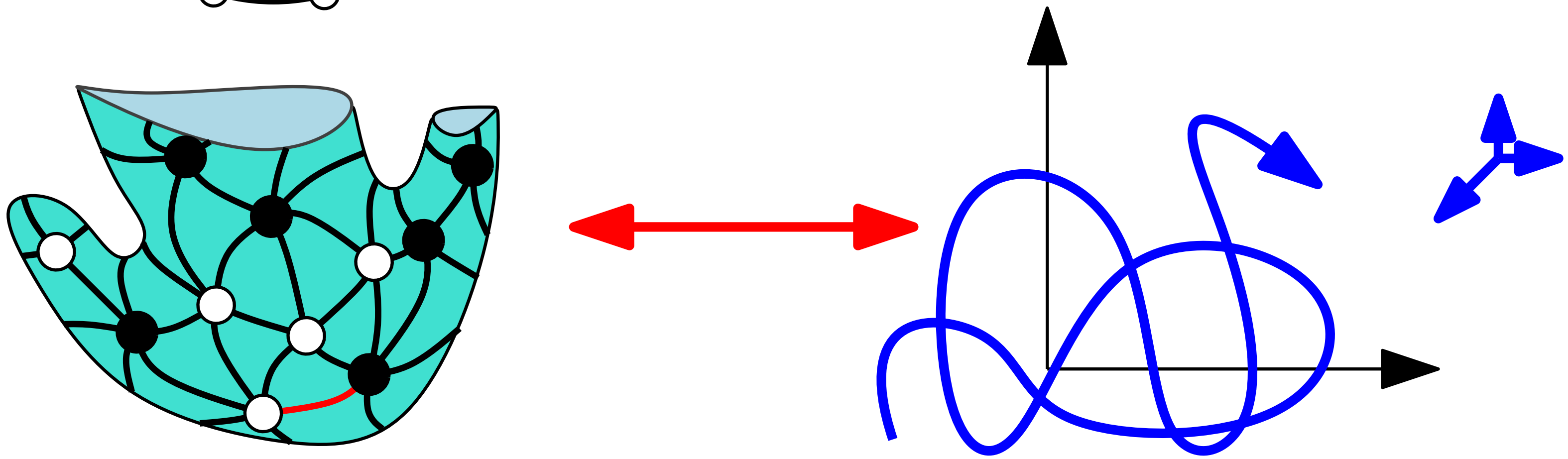
Spherical case



Disk case



UIPT case



Dictionary

percolated triangulation	walk
edges	steps
left-boundary length	x-coordinate of walk
vertices	c-steps
black vertices	c steps of type abc
⋮	⋮
perco-interface toward t	walk of excursions
⋮	⋮
clusters	envelope intervals
cluster's bubbles	cone intervals
⋮	⋮

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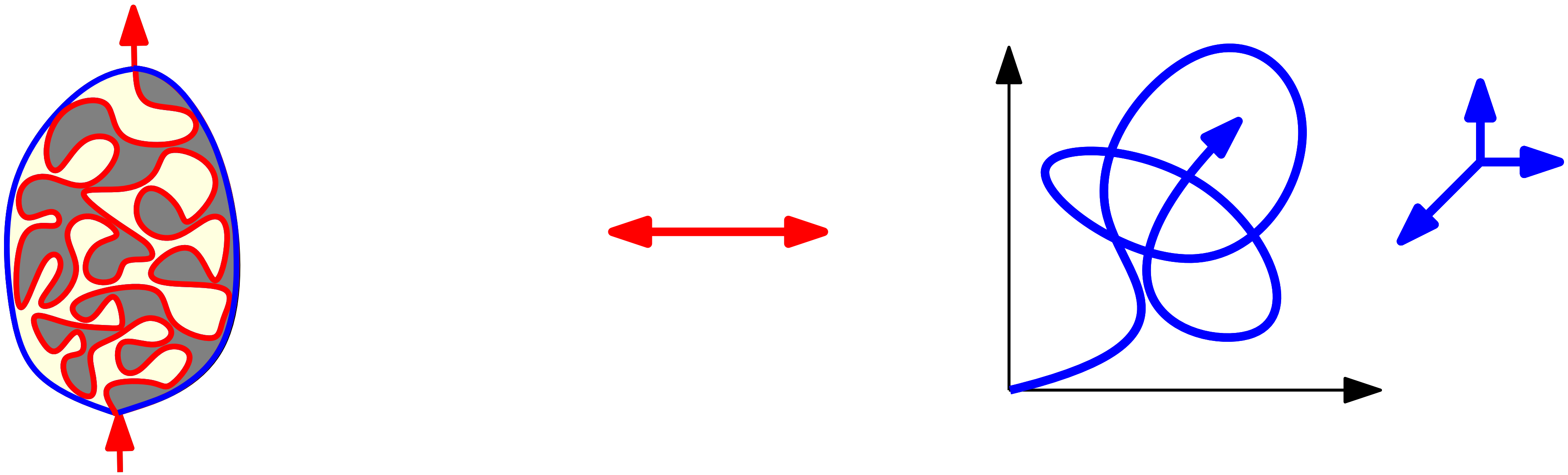
Dictionary: percolation-interface to $v \longleftrightarrow$ walk of excursions



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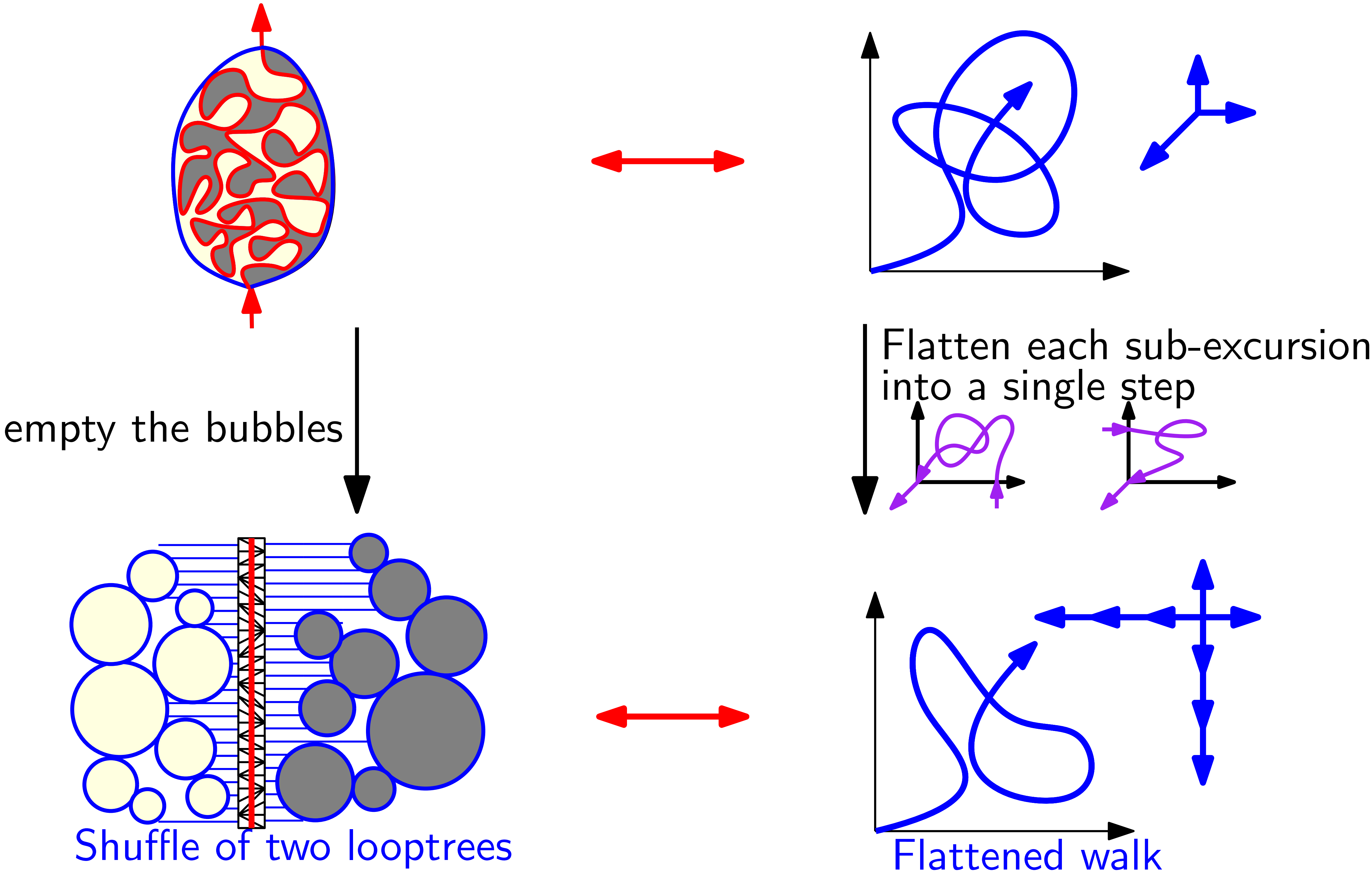
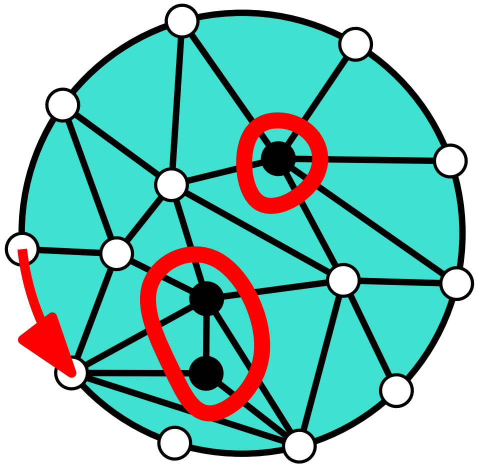
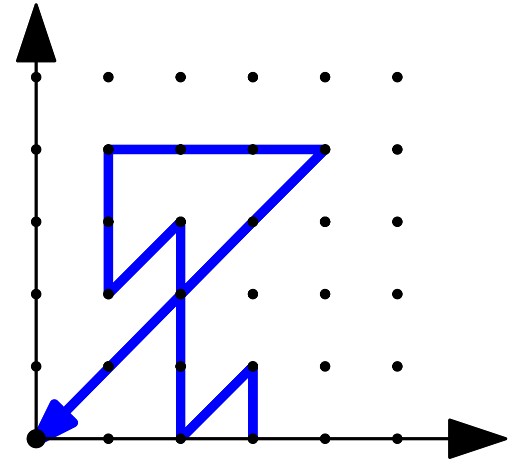
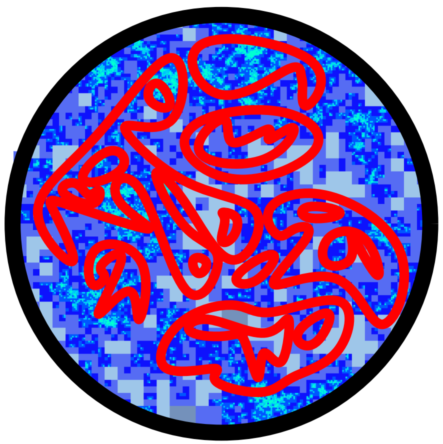




discrete dictionary

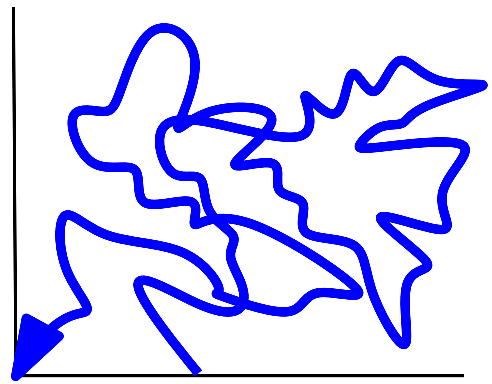
[Bernardi, Holden, Sun]

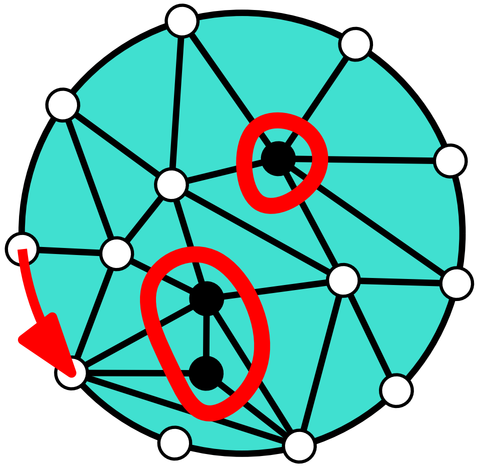


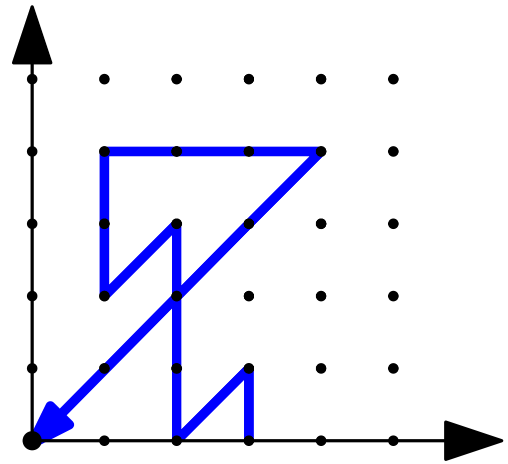


continuum dictionary

[Duplantier, Miller, Sheffield]



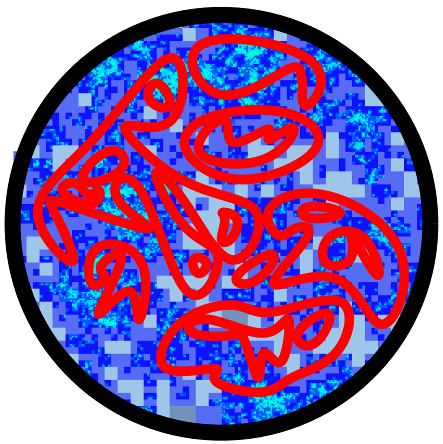


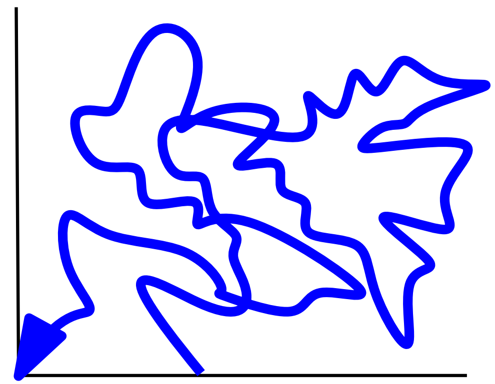


discrete dictionary

[Bernardi, Holden, Sun]

Perfect correspondence!



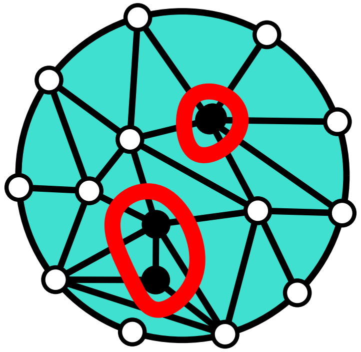


continuum dictionary

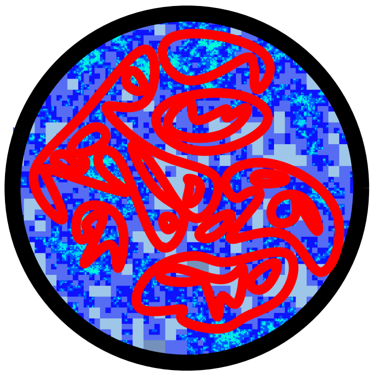
[Duplantier, Miller, Sheffield]

Strengthening the convergence results

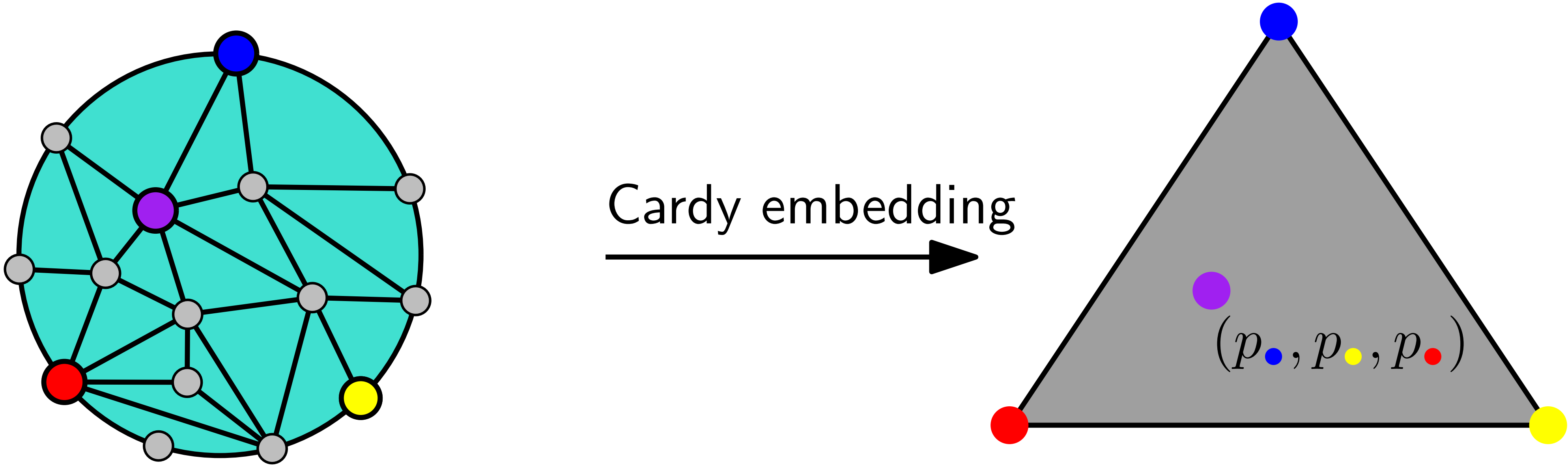
Holden, Sun + Albenque, Garban, Gwynne, Lawler, Li, Sepulveda
+ Miller, Sheffield

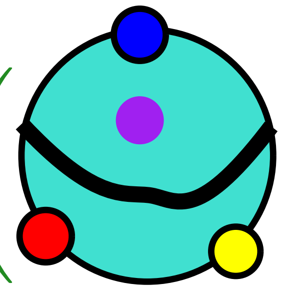


convergence ↓ under **nice** embedding



Cardy embedding of triangulations



where $p_{\bullet} = \mathbb{P}_{perco}\Big($  $\Big)$

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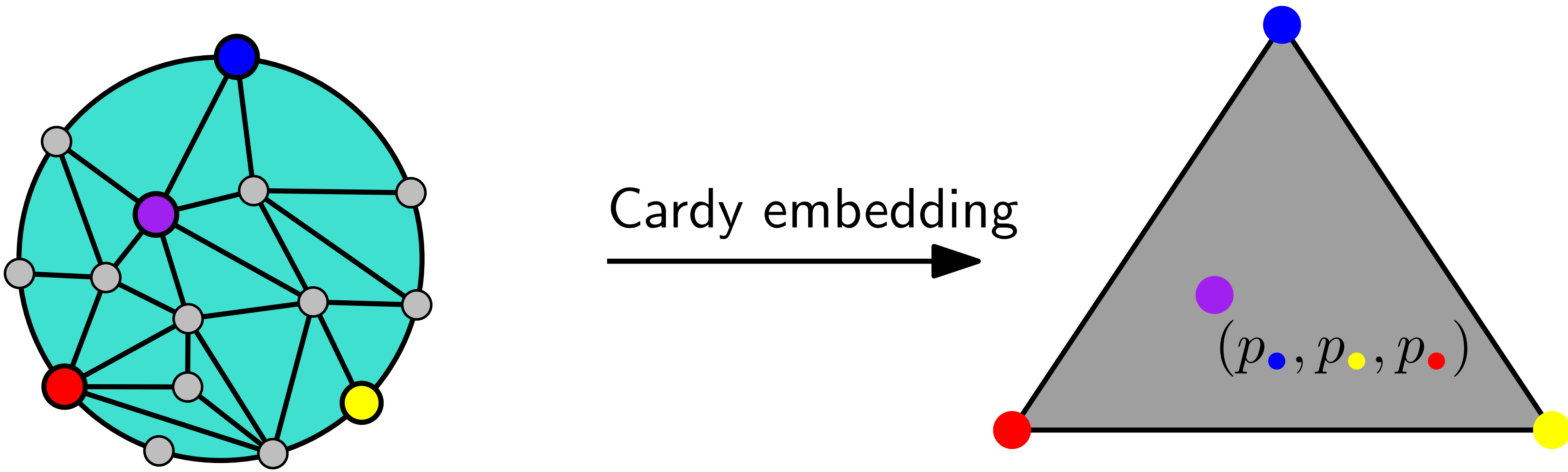
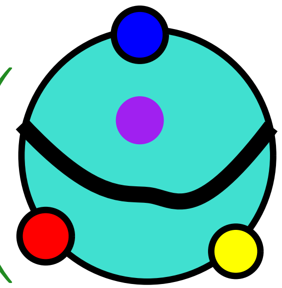
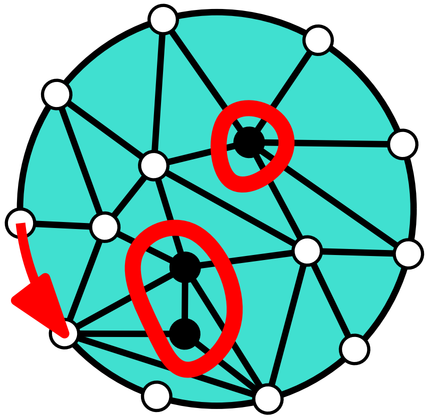


where $p_{\bullet} = \mathbb{P}_{perco}\Bigg($  $\Bigg)$

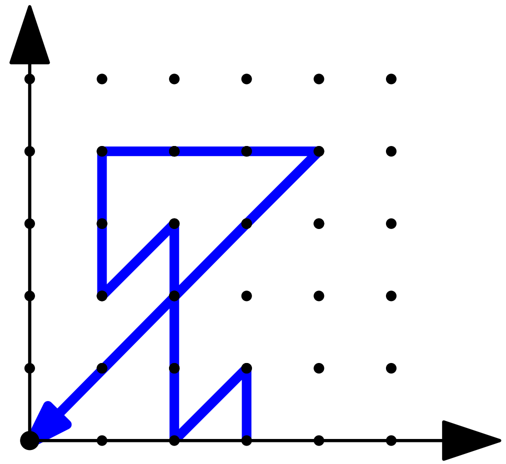
Thm [Holden, Sun]: Convergence holds for the **Cardy embedding**.
(because $\phi_n \approx$ Cardy embedding)

Key ingredient used: “convergence componentwise”



Same triangulation

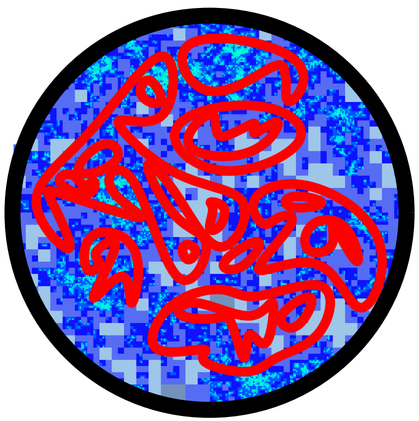
k independent percolations



k Kreweras walks

Same LQG

k independent CLE



k Brownian motions

Why useful?

To upgrade the “crossing event result” from an **annealed result** to a **quenched result**.
This implies (...) that $\phi_n \approx$ **Cardy embedding**.

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How is it proved?

- **LQG stay the same**: prove the previous convergence is joint with convergence in Gromov-Hausdorff-Prokhorov topology.
- **CLE are independent**: prove CLE mixes fast (using pivotal point result).

Thanks.