A loglog step towards the Erdős-Hajnal conjecture

Paul Seymour (Princeton)
Joint work with Matija Bucić, Tung Nguyen and Alex Scott.

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Conjecture (Erdős, Hajnal, 1977)

For every graph H, there exists $c > 0$ such that every H-free graph G has a clique or stable set of size at least $|G|^c$.

H has the EH-property if there exists $c > 0$ such that $\max(\alpha(G), \omega(G)) \geq |G|^c$ for every H-free graph G.

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If H_1, H_2 have the EH-property, and H is obtained by substituting H_1 for a vertex of H_2, then H has the EH-property.

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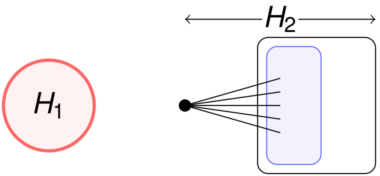

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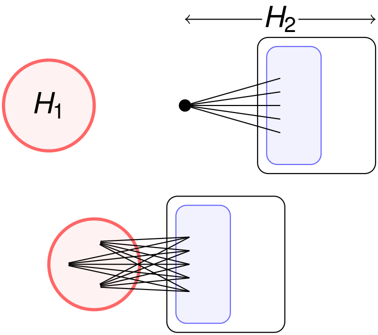

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Theorem (Blanco, Bucić, 2022)

There exists $c > 0$ such that

$$\max(\alpha(G), \omega(G)) \geq 2^{c(\log|G|)^{2/3}}.$$

for every P_5-free graph G.

Cograph: P_4-free graph. Equivalently, a graph that can be constructed starting from one-vertex graphs by repeatedly taking disjoint unions and complete joins.
Define $\mu(G)$ = size of largest induced cograph in G.

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For every graph H, there exists $c > 0$ such that $\mu(G) \geq |G|^c$ for every H-free graph G.

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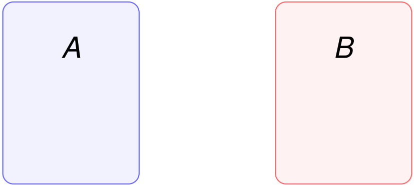
(A, B) is almost-pure if either every vertex in B has at most $|A|/(2\mu(G))$ neighbours in A, or every vertex in B has at most $|A|/(2\mu(G))$ non-neighbours in A.



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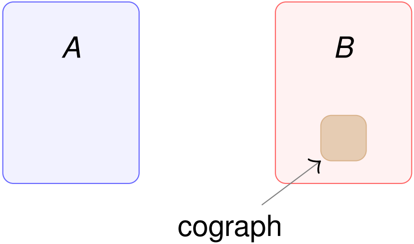

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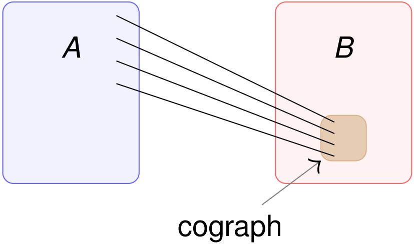

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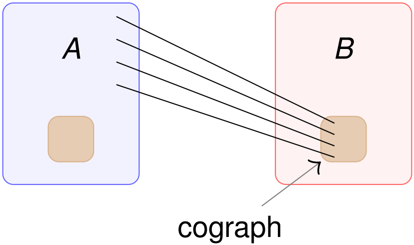

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For all H, there exist $k > 0$ such that for every H-free graph G and every x with $0 < x \leq \frac{1}{8|H|}$, there is a sequence $A_1, \ldots, A_n$ of disjoint subsets of $V(G)$ with $n \geq \log(1/x)$, and each of cardinality at least $\lfloor x^k |G| \rfloor$, such that for $1 \leq i \leq n$, either
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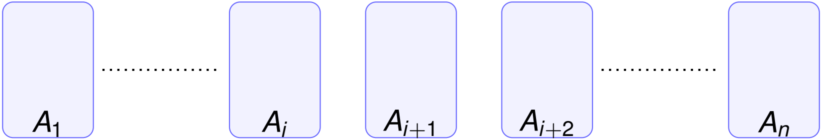

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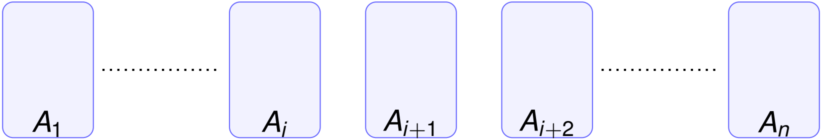

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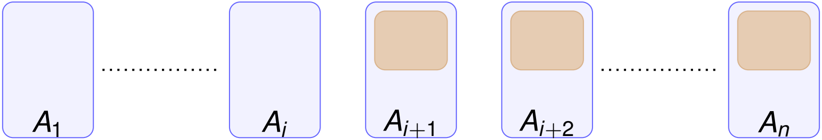

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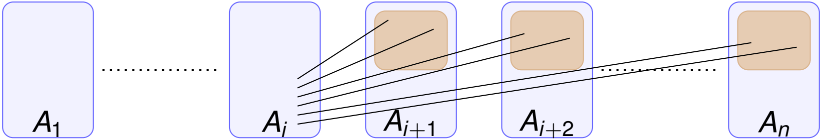

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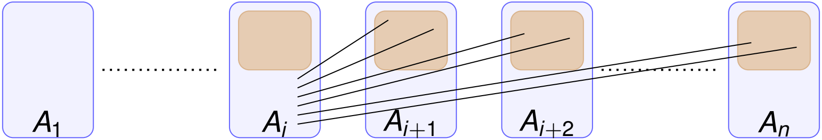

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For all H, there exist $k_1, k_2 > 0$ such that for every graph G and every x with $0 < x \leq \frac{1}{8|H|}$, if $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$, there is a sequence $A_1, \ldots, A_n$ of disjoint subsets of $V(G)$ with $n \geq \log(1/x)$, and each of cardinality at least $\lfloor x^{k_2}|G| \rfloor$, such that for $1 \leq i \leq n$, either every vertex of $A_{i+1} \cup \cdots \cup A_n$ has at most $x|A_i|$ neighbours in A_i, or every vertex of $A_{i+1} \cup \cdots \cup A_n$ has at most $x|A_i|$ non-neighbours in A_i.

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Let H be a graph and let $g \in V(H)$. Let $b, c > 0$, and let $a := b + (1 + c)|H|$. Let G be a graph, let A, B be disjoint subsets of $V(G)$, and let $0 < x \leq 1/2$. Suppose that every vertex in A has at least $x|B|$ non-neighbours in B. Then either:

- $\mathrm{ind}_H(G) \geq x^a|A| \cdot |B|^{|H|-1}$; *or*
- *there exists $B' \subseteq B$ with $|B'| \geq x|B|$ such that $\mathrm{ind}_{H \setminus g}(G[B']) < x^b|B'|^{|H|-1}$; or*
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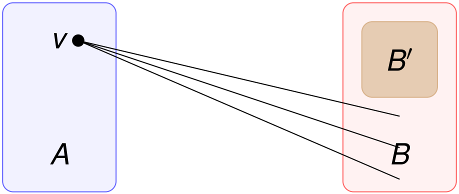

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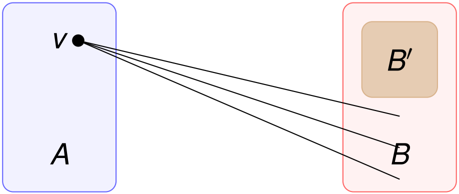

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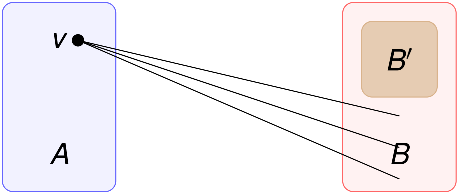

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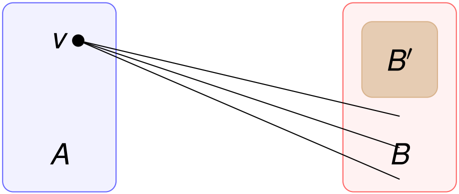

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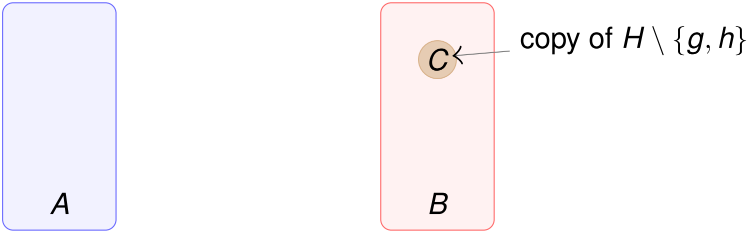

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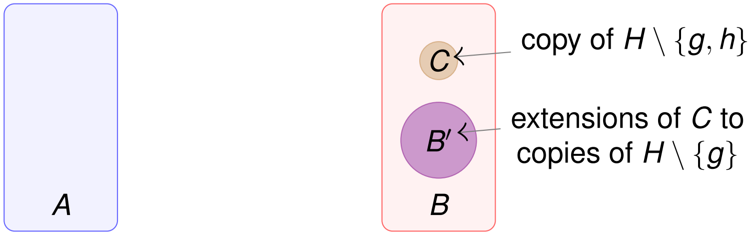

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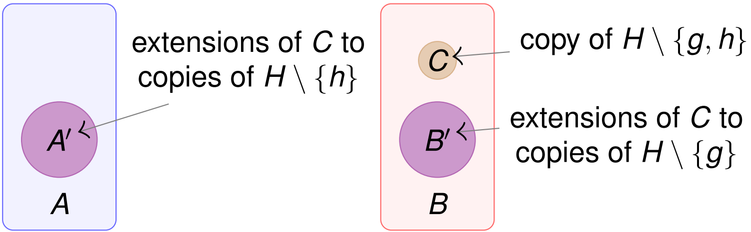

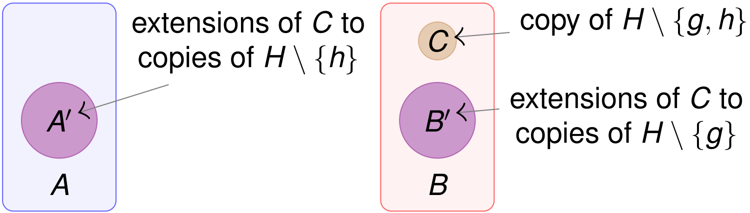

extensions of C to copies of $H \setminus \{h\}$
copy of $H \setminus \{g, h\}$
C
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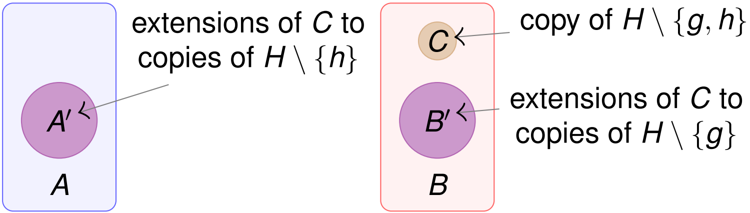
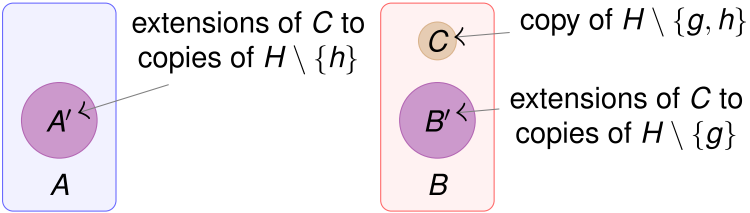
A'
B'
A
B



- On average (over the choices of C) there are at least $x^{|H|-1+b+c(d-1)}|A| \cdot |B|$ nonedges between A' and B'.



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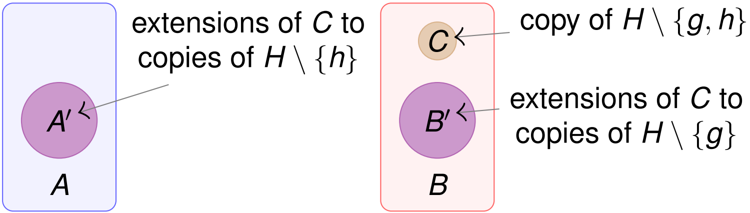

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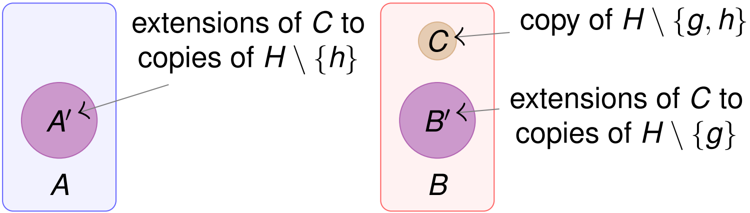

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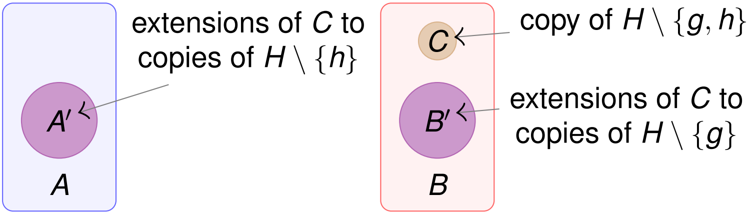

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- Otherwise, there are always at least $2x^c|A'| \cdot |B'|$ edges between A', B'; so the number of good copies of H is big and the first outcome holds.

Approximate blowups

J is a graph, $t > 0$ an integer, and $q \leq 1$ a real number. A (t, q)-blowup of J in G means a family A_j ($j \in V(J)$) of pairwise disjoint subsets of $V(G)$, all of size t, such that for all distinct $i, j \in V(J)$,

- if $ij \notin E(J)$ then every vertex in A_i has at most $q|A_j|$ neighbours in A_j and vice versa;
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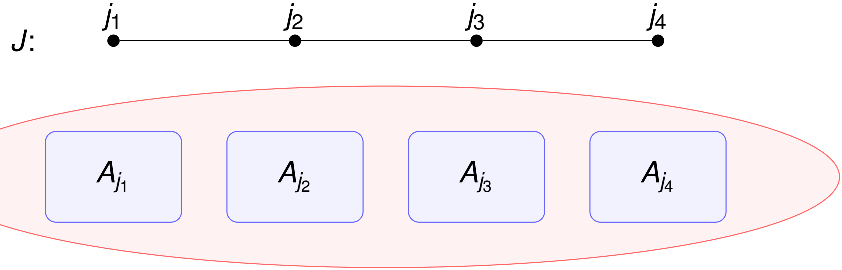

Proof of the main theorem

Theorem

For all H, there exist $k_1, k_2 > 0$ such that for every graph G and every x with $0 < x \leq \frac{1}{8|H|}$, if $\mathrm{ind}_H(G) < x^{k_1}|G|^{|H|}$, there is a sequence $A_1, \ldots, A_n$ of disjoint subsets of $V(G)$ with $n \geq \log(1/x)$, and each of cardinality at least $\lfloor x^{k_2}|G| \rfloor$, such that for $1 \leq i \leq n$, either every vertex of $A_{i+1} \cup \cdots \cup A_n$ has at most $x|A_i|$ neighbours in A_i, or every vertex of $A_{i+1} \cup \cdots \cup A_n$ has at most $x|A_i|$ non-neighbours in A_i.

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- Choose an induced subgraph J of H maximal such that there is an approximate blowup of J in G. (ie a (t, q)-blowup where $t = \lfloor x^{r_1}|G| \rfloor$ and $q = x^{r_2}$ for appropriate r_1, r_2 depending on J.)

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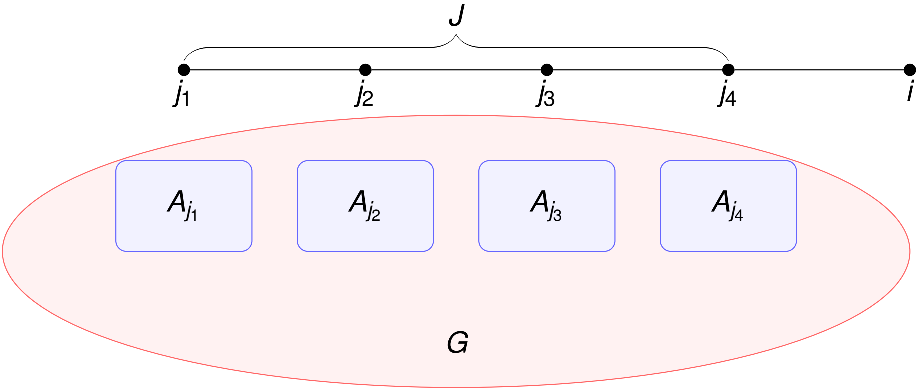
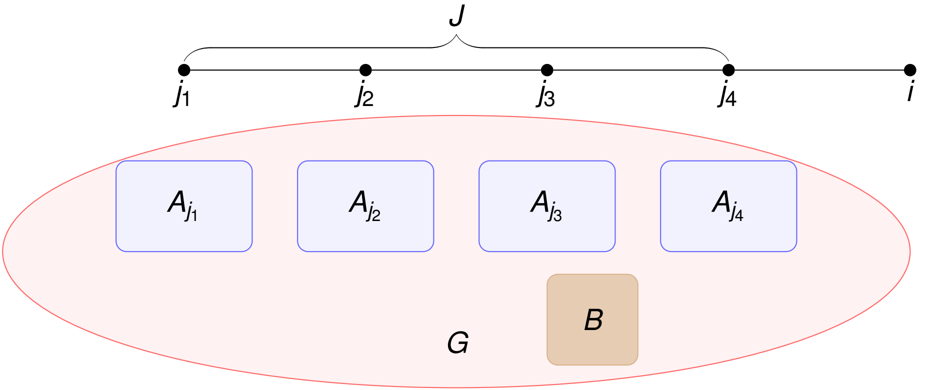
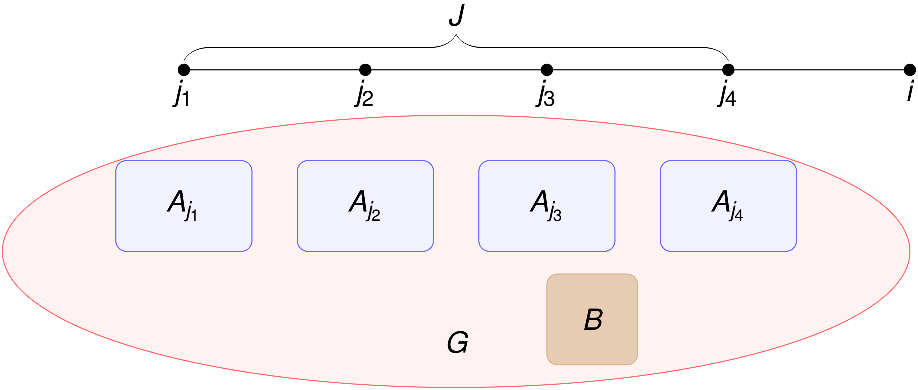
- $J \ne H$ since $\mathrm{ind}_H(G) < x^{k_1}|G|^{|H|}$. Choose $i \in V(H) \setminus V(J)$.


J
j1
j2
j3
j4
i
A j1
A j2
A j3
A j4
G



Case 1: there is a subset B disjoint from the A_j's, that is very sparse to some A_j, and has size $c|G|$.



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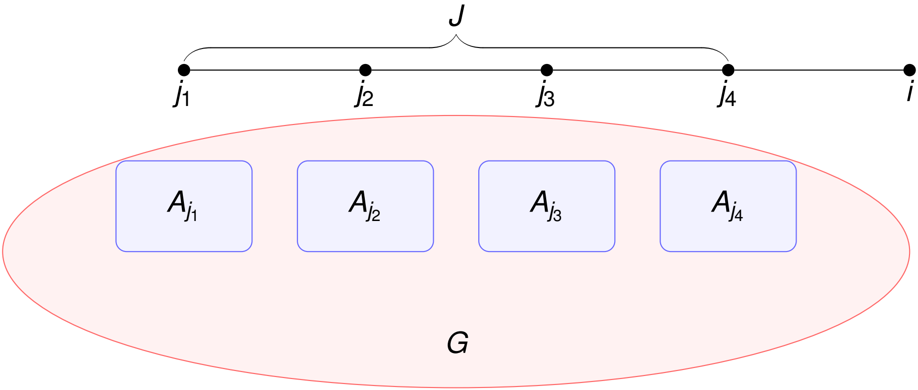
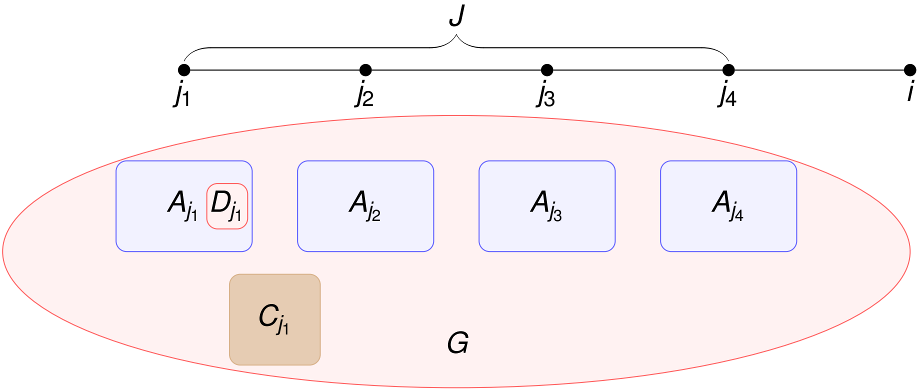
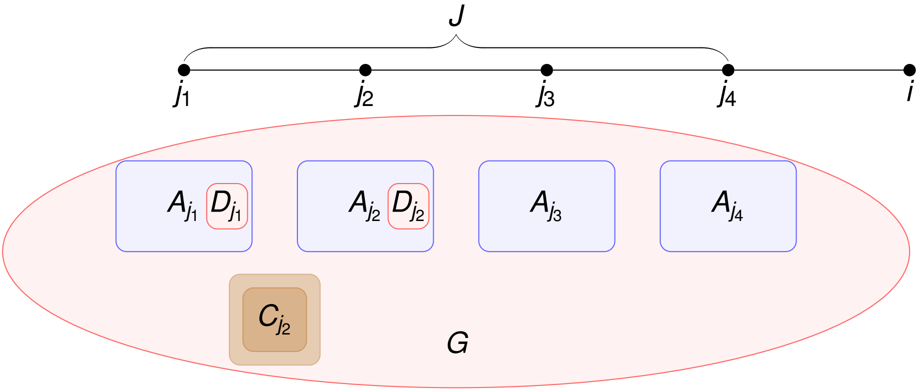


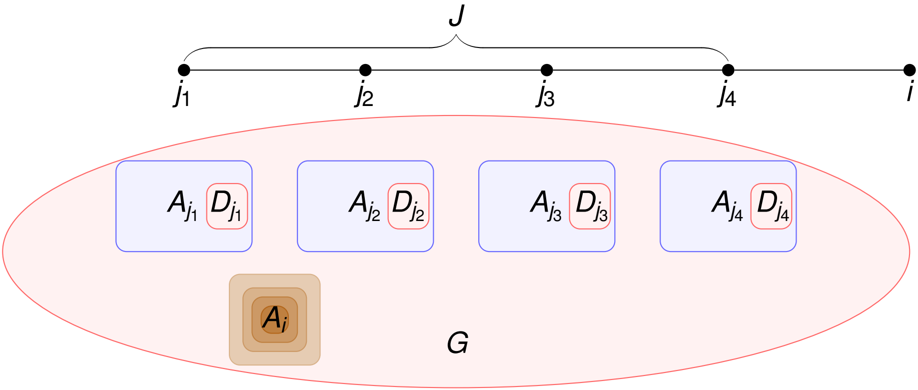
So most vertices in $V(G) \setminus \bigcup_{j \in V(J)} A_j$ are adjacent to at least a small fraction of each A_j, and also nonadjacent to at least a small fraction of each A_j.



So most vertices in $V(G) \setminus \bigcup_{j \in V(J)} A_j$ are adjacent to at least a small fraction of each A_j, and also nonadjacent to at least a small fraction of each A_j. Use the key lemma to get a subset C_{j_1} of $V(G) \setminus \bigcup_{j \in V(J)} A_j$, not too small, that is very dense or very sparse (whichever we want) to a subset $D_{j_1} \subseteq A_{j_1}$ that is not too small.



Repeat to get $C_{j_2} \subseteq C_{j_1}$ not too small, that is dense or sparse to a subset $D_{j_2} \subseteq A_{j_2}$ that is not too small.



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Repeat for all other A_j. This give an approximate blowup of J+i, contrary to the choice of J.