Skipless Chain Decompositions & Improved Poset Saturation Bounds

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The Boolean lattice of dimension $n$:

- elements: $2^{[n]} = \mathcal{P}([1, \ldots, n])$
- relation: $\subseteq$
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A chain is a set system where every pair of elements is comparable.
An antichain is a set system where every pair of elements is incomparable.
The Boolean lattice of dimension $n$:

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A **chain** is a set system where every pair of elements is comparable.

An **antichain** is a set system where every pair of elements is incomparable.
A chain $C = \{C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq C_k\} \subseteq P$ is **skipless** in $P$ if for all $i \in [k - 1]$, there is no $X \in P$ with $C_i \subsetneq X \subsetneq C_{i+1}$.
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Chains in the hypercube

Theorem (Dilworth 1950)

For a family poset $\mathcal{P}$, the size of the largest antichain is equal to the size of smallest chain disjoint chain decomposition of $\mathcal{P}$. 
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Can you ask for Dilworth theorem to use disjoint skipless chains?
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Can you ask for Dilworth theorem to use disjoint skipless chains? \textbf{NO}

What if we view this poset embedded in the Boolean lattice...

True for every poset, and every way to embed it.
Structural Theorem [B., Groenland, Jacob, Johnston, 2022+]

Any subposet $\mathcal{P}$ of $2^{[n]}$ with largest antichain of size $k$ can be covered by a family of $k$ disjoint skipless chains in $2^{[n]}$.

"Any family of $k$ chains in $2^{[n]}$ can be covered by a family of $k$ disjoint skipless chains in $2^{[n]}."
Cover chains with skipless chains

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"Any family of $k$ chains in $2^{[n]}$ can be covered by a family of $k$ disjoint skipless chains in $2^{[n]}."$

We generalise a result of Lehman and Ron (2001) who proved the special case where all chains of the family are of size 2 and all top (resp. bottom) elements of the chain have the same size.

We generalise a result from Duffus, Howard and Leader (2019) who proved the special case where the family is convex\(^1\).

\(^1\) $\mathcal{F} \subseteq 2^{[n]}$ is convex if for all $X, Z \in \mathcal{F}$ and $X \subset Y \subset Z$, $Y \in \mathcal{F}$. 
Any family of $k$ chains in $2^n$ can be **covered** by a family of $k$ **disjoint skipless** chains in $2^n$. 

**Structural Theorem [B., Groenland, Jacob, Johnston, 2022+]**

![Diagram showing the covering of chains with disjoint skipless chains](image)
Structural Theorem [B., Groenland, Jacob, Johnston, 2022+] 

Any family of $k$ chains in $2^{[n]}$ can be **covered** by a family of $k$ **disjoint skipless** chains in $2^{[n]}$. 

Double counting + Menger
Sketch of the sketch of the proof

Any family of \( k \) chains in \( 2^{[n]} \) can be covered by a family of \( k \) disjoint skipless chains in \( 2^{[n]} \).
Antichain saturation

\[ F \subseteq 2^n, \text{ is } k\text{-saturated if:} \]

1. \( F \) has no antichain of size \( k \);
2. \( F \cup \{x\} \) has an antichain of size \( k \) for any \( x \in 2^n \setminus F \).

\[ \text{sat}^*(n, k) = \text{minimum } |F| \text{ over all } k\text{-saturated families } F \in 2^n. \]

Red sets form an \( 2\)-saturated family for the hypercube \( 2^3 \): \( \text{sat}^*(3, 2) \leq 4 \).

Can we extend this construction to \( k\)-saturated?
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Antichain saturation

Construction: \( \text{sat}^*(n, k) \leq (n - 1)(k - 1) + 2. \)

Conjecture (FKKMRSS): \( \forall k \geq 2, \text{sat}^*(n, k) \sim n(k - 1) \) as \( n \to \infty. \)

Conjecture (Danković and Ivan): \( \forall k \geq 2, \text{sat}^*(n, k) \geq n(k - 1) - C_k. \)
Antichain saturation

Construction: \( \text{sat}^*(n, k) \leq (n - 1)(k - 1) + 2. \)


\[
\begin{array}{c|ccc}
  k & 2 & 3 & 4 \\
  \text{sat}^*(k, n) & n + 1 & 2n & 3n - 1 \\
\end{array}
\]

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![Diagram of an antichain]

Construction: $\text{sat}^*(n, k) \leq (n - 1)(k - 1) + 2.$

Đanković and Ivan (2022+)

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{sat}^*(k, n)$</td>
<td>$n + 1$</td>
<td>$2n$</td>
<td>$3n - 1$</td>
<td>$4n - 2$</td>
<td>$5n - 5$</td>
</tr>
</tbody>
</table>

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Conjecture (Đanković and Ivan): $\forall k \geq 2, \text{sat}^*(n, k) \geq n(k - 1) - C_k.$
Quick application

Consider $\mathcal{F}$ \textbf{k-saturated}. Consider a chain decomposition (using Dilworth's Theorem) of $\mathcal{F}$. 
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For any element $Y \notin \mathcal{F}$, $Y$ can not be “added” to one of the chain (by Dilworth).
Consider $F$ \textbf{k-saturated}. Consider a chain decomposition (using Dilworth’s Theorem) of $F$.
For any element $Y \notin F$, $Y$ can not be “added” to one of the chain (by Dilworth).

\textbf{Claim.} For any $\ell$ such that $k \leq \left(\frac{\ell}{\lfloor \ell/2 \rfloor}\right)$, each chain contains an element of size at most $\ell$. They also all contains an element of size $n - \ell$. 

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Theorem [B., Groenland, Jacob, Johnston, 2022+]

Any family of \( k - 1 \) chains in \( 2^{[n]} \) can be covered by a family of \( k - 1 \) disjoint skipless chains in \( 2^{[n]} \).

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Dilworth $\implies \mathcal{F}$ decompose in $C_1, C_2, \ldots, C_{k-1}$ chains.

Claim $\implies$ all these chains start in layer $O(\log k)$ and end in layer $n - O(\log k)$.
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Th. $\implies$ $F$ coverable with $k - 1$ skipless disjoint chains.
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Th. \( \Rightarrow \) \( \mathcal{F} \) coverable with \( k-1 \) **skipless disjoint** chains.

\( k \)-saturated \( \Rightarrow \) \( \mathcal{F} \) partitioned into \( k-1 \) **skipless** chains.
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Every chain contains at least $n - \Theta(\log k)$ elements.
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$k$-saturated $\implies$ $\mathcal{F}$ partitioned into $k - 1$ skipless chains.

Every chain contains at least $n - \Theta(\log k)$ elements.

$\implies$ $|\mathcal{F}| \geq (n - 2\ell)(k - 1) = n(k - 1) - \Theta(k \log k)$
We now know that any $F_k$-saturated looks like this. To get an exact value, we need to improve both the upper bound and the lower bound. In the case $k - 1 = \ell \lfloor \frac{\ell}{2} \rfloor$, FKKMRSS (2017) improved the upper bound using the initial segment of colex.
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We now know that any $\mathcal{F}$ $k$-saturated looks like this. To get **exact** value, need to improve both the **upper bound** and the **lower bound**.

In the case $k - 1 = \binom{\ell}{\lceil \ell/2 \rceil}$ FKKMRSS (2017) improved the upper bound. Using the initial segment of colex.
Let $\mathcal{F} \subseteq \binom{[n]}{t}$. Its **shadow** is

$$\partial \mathcal{F} = \left\{ X \in \binom{[n]}{t-1} : X \subseteq Y \in \mathcal{F} \right\}.$$ 

Let $\mathcal{C}(m, t)$ denote the initial segment of colex of size $m$ on layer $t$, e.g.

$$\mathcal{C}(3, 6) = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{2, 3, 5\}.$$
Lower bound

\[ F \cup \{1, 2, 7\} = \partial F = \{6\} \cup \{2, 7\} \cup \{2, 7\} \]

\[ F \]

\[ \mathcal{F} \]

Initial segments of colex minimise the size of the shadow.

Lemma (B., Groenland, Jacob, Johnston, 2023+): The initial segment of colex minimise the matching to the shadow.
Kruskal-Katona (1963)

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Kruskal-Katona (1963)
Initial segments of colex minimise the size of the shadow.

Lemma (B., Groenland, Jacob, Johnston, 2023+)
*The initial segment of colex minimise the matching to the shadow.*

\[
\binom{[6]}{3} \cup \{1, 2, 7\} = \mathcal{F}
\]

\[
\binom{[6]}{2} \cup \{\{1, 7\}, \{2, 7\}\} = \partial \mathcal{F}
\]
Exact values

\( \nu(F) \rightarrow \) the size of the maximum matching from \( F \) to its shadow \( \partial F \).

\( C(m, t) \rightarrow \) initial segment of colex of size \( m \) on layer \( t \).

Define the sequence \( c_{\lfloor \ell/2 \rfloor} = k - 1 \), and for \( 0 \leq t < \lfloor \ell/2 \rfloor \), let \( c_t = \nu(C(c_{t+1}, t + 1)) \).

B, Groenland, Jacob and Johnston (2023+)

For \( n \geq 2\ell + 1 \),

\[
\text{sat}^*(n, k) = 2 \sum_{t=0}^{\lfloor \ell/2 \rfloor} c_t + (k - 1)(n - 1 - 2 \lfloor \ell/2 \rfloor).
\]

The lower bound still holds for \( n \geq \ell \) (and \( \text{sat}^*(n, k) = 2^n \) for \( n < \ell \)).
\( \nu(\mathcal{F}) \rightarrow \) the size of the maximum matching from \( \mathcal{F} \) to its shadow \( \partial \mathcal{F} \).

\( \mathcal{C}(m, t) \rightarrow \) initial segment of colex of size \( m \) on layer \( t \).

Define the sequence \( c_{[\ell/2]} = k - 1 \), and for \( 0 \leq t < [\ell/2] \), let \( c_t = \nu(\mathcal{C}(c_{t+1}, t + 1)) \).

**B, Groenland, Jacob and Johnston (2023+)**

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The lower bound still holds for \( n \geq \ell \) (and \( \text{sat}^*(n, k) = 2^n \) for \( n < \ell \)).

Open question: What happens when \( n \leq 2\ell \)? Finding a matching between the top and the bottom is harder.
Lemma

There exist a “canonical” way to decompose any integer \( k \) in the following way:

\[
k - 1 = \binom{a_{r_1}}{r_1} + \cdots + \binom{a_{r_s}}{r_s},
\]

In particular if \( k - 1 = \binom{\ell}{\lfloor \ell/2 \rfloor} \),

\[
s = 1, \quad r_1 = \ell/2, \quad a_{r_1} = \ell
\]
Lemma

There exist a “canonical” way to decompose any integer $k$ in the following way:

$$k - 1 = \left( \frac{a_{r_1}}{r_1} \right) + \cdots + \left( \frac{a_{r_s}}{r_s} \right),$$

satisfying the following conditions,

- $r_1 > \cdots > r_s \geq 1$;
- $a_{r_1} > \cdots > a_{r_s} \geq 1$;
- for all $i \in [s]$, we have $r_i \leq \lceil a_{r_i}/2 \rceil$.

In particular if $k - 1 = \left( \frac{\ell}{\lfloor \ell/2 \rfloor} \right)$, 
$s = 1$, $r_1 = \ell/2$, $a_{r_1} = \ell$.
General saturation

A set system $F \subseteq \mathcal{P}[2^n]$ is $P$-saturated if:

- $F$ has an induced copy of $P$;
- $F \cup \{x\}$ has an induced copy of $P$ for any $x \in 2^n \setminus P$.

Theorem (Morrison, Noel and Scott 2014; Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi, Patkós 2011)

$$\left( k - 3 \right) / 2 \leq \text{sat}^* (n, C_k) \leq 2^{0.98k^{17/24}}.$$
# General saturation

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F} \subseteq 2^{[n]}$ a set system is $P$-saturated if:</td>
</tr>
<tr>
<td>- $\mathcal{F}$ has induced copy of $P$;</td>
</tr>
<tr>
<td>- $\mathcal{F} \cup {x}$ has an induced copy of $P$ for any $x \in 2^{[n]} \setminus P$.</td>
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$$\frac{(k-3)}{2} \leq \text{sat}^*(n, C_k) \leq 2^{0.98k}$$
Definition

$F \subseteq 2^{[n]}$ a set system is $\mathcal{P}$-saturated if:

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$$2^{(k-3)/2} \leq \text{sat}^*(n, C_k) \leq 2^{0.98k}$$
<table>
<thead>
<tr>
<th>poset $P$</th>
<th>$\text{sat}(n, P)$</th>
<th>$\text{sat}^*(n, P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$, chain</td>
<td>$= 1$</td>
<td>$= 1$</td>
</tr>
<tr>
<td>$A_2$, antichain</td>
<td>$= 1$</td>
<td>$= n + 1$</td>
</tr>
<tr>
<td>$C_3$, chain</td>
<td>$= 2$</td>
<td>$= 2$</td>
</tr>
<tr>
<td>$C_2 + C_1$, chain and single</td>
<td>$= 2$</td>
<td>$= 4$</td>
</tr>
<tr>
<td>$\lor$ fork (or $\land$)</td>
<td>$= 2$</td>
<td>$= n + 1$</td>
</tr>
<tr>
<td>$A_3$, antichain</td>
<td>$= 2$</td>
<td>$= 3n - 1$</td>
</tr>
<tr>
<td>$C_4$, chain</td>
<td>$= 4$</td>
<td>$= 4$</td>
</tr>
<tr>
<td>$\lor_3$, fork with three tines</td>
<td>$= 3$</td>
<td>$\geq \log_2 n$</td>
</tr>
<tr>
<td>$\Diamond$, diamond</td>
<td>$= 3$</td>
<td>$\geq \sqrt{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq n + 1$</td>
</tr>
<tr>
<td>$\Diamond^-$, diamond minus an edge</td>
<td>$= 3$</td>
<td>$= 4$</td>
</tr>
<tr>
<td>$\bowtie$, butterfly</td>
<td>$= 4$</td>
<td>$\geq n + 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 6n - 10$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$= 3$</td>
<td>$\geq \log_2 n$</td>
</tr>
<tr>
<td>$N$</td>
<td>$= 3$</td>
<td>$\geq \sqrt{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 2n$</td>
</tr>
<tr>
<td>$2C_2$</td>
<td>$= 3$</td>
<td>$\geq n + 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 2n$</td>
</tr>
</tbody>
</table>

**Figure 1:** Table from Keszegh, Lemons, Martin, Pálvölgyi and Patkós 2022
<table>
<thead>
<tr>
<th>Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3 + C_1$, chain and single</td>
</tr>
<tr>
<td>$\lor + 1$, fork and single</td>
</tr>
<tr>
<td>$C_2 + A_2$</td>
</tr>
<tr>
<td>$A_4$, antichain</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$C_5$, chain</td>
</tr>
<tr>
<td>$C_6$, chain</td>
</tr>
<tr>
<td>$C_k$, chain ($k \geq 7$)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$A_k$, antichain</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$3C_2$</td>
</tr>
<tr>
<td>$5C_2$</td>
</tr>
<tr>
<td>$7C_2$</td>
</tr>
<tr>
<td>any poset on $k$ elements</td>
</tr>
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**Figure 2:** Table from Keszegh, Lemons, Martin, Pálvölgyi and Patkós 2022
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For any poset $P$ either $\text{sat}^*(n, P) \geq 2\sqrt{n} - 2$ or $\text{sat}^*(n, P) = O_P(1)$.

What about a general upper bound? Can we hope to have $\text{sat}^*(n, P) \leq 2\sqrt{n}$ for every poset $P$?

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For any poset $P$, $\text{sat}^*(n, P) \leq n|P|^2$. 
For a poset $\mathcal{P}$, we define the **cube-height** $h^*(\mathcal{P})$ to be the minimum $h^* \in \mathbb{N}$ for which there exists $n \in \mathbb{N}$ such that $\left(\begin{bmatrix} n \\ \leq h^* \end{bmatrix}\right)$ contains an induced copy of $\mathcal{P}$. 
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Theorem (B., Groenland, Ivan, Johnston, 2023+)

For any poset $P$, $\text{sat}^*(n, P) \leq n^{|P|^2}$.

We give a constructive proof.

- $\mathcal{F}_0$: first $h^*(P)$ layers.
- $\mathcal{F}_1$: Any completion.

Key lemma: $\mathcal{F}_1$ has bounded VC-dimension.

Main idea: if we shatter a large enough set, we can find a copy of $P \setminus \max(P)$ in the first $h^*(P)$ layers such that we have, in $\mathcal{F}_0$, all possible relations to this copy.
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Remark

For every $P$, $h^*(P) \leq |P|$, $w^*(P) \leq |P| \cdot h^*(P) \leq |P|^2$. 

With a bit more effort we proved:

Lemma (B., Groenland, Ivan, Johnston, 2023+)

For every $P$, $w^*(P) \leq |P|^2/4 + 1$. 

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\[
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\text{sat}^*(2C_2, n) \geq n \\
\text{sat}^*(3C_2, n) \leq 14
\]

Thank you!
<table>
<thead>
<tr>
<th>poset $P$</th>
<th>$\text{sat}(n, P)$</th>
<th>$\text{sat}^*(n, P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$, chain</td>
<td>= 1</td>
<td>= 1</td>
</tr>
<tr>
<td>$A_2$, antichain</td>
<td>= 1</td>
<td>= $n + 1$</td>
</tr>
<tr>
<td>$C_3$, chain</td>
<td>= 2</td>
<td>= 2</td>
</tr>
<tr>
<td>$C_2 + C_1$, chain and single</td>
<td>= 2</td>
<td>= 4</td>
</tr>
<tr>
<td>$\lor$ fork (or $\land$)</td>
<td>= 2</td>
<td>= $n + 1$</td>
</tr>
<tr>
<td>$A_3$, antichain</td>
<td>= 2</td>
<td>= $3n - 1$</td>
</tr>
<tr>
<td>$C_4$, chain</td>
<td>= 4</td>
<td>= 4</td>
</tr>
<tr>
<td>$\lor_3$, fork with three tines</td>
<td>= 3</td>
<td>$\geq \log_2 n$</td>
</tr>
<tr>
<td>$\lozenge$, diamond</td>
<td>= 3</td>
<td>$\geq \sqrt{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq n + 1$</td>
</tr>
<tr>
<td>$\lozenge^-$, diamond minus an edge</td>
<td>= 3</td>
<td>= 4</td>
</tr>
<tr>
<td>$\Join$, butterfly</td>
<td>= 4</td>
<td>$\geq n + 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 6n - 10$</td>
</tr>
<tr>
<td>$Y$</td>
<td>= 3</td>
<td>$\geq \log_2 n$</td>
</tr>
<tr>
<td>$N$</td>
<td>= 3</td>
<td>$\geq \sqrt{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 2n$</td>
</tr>
<tr>
<td>$2C_2$</td>
<td>= 3</td>
<td>$\geq n + 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 2n$</td>
</tr>
</tbody>
</table>

**Figure 3:** Table from [?]
<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
<th>Bound</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3 + C_1$, chain and single</td>
<td>3</td>
<td>$\leq 8$</td>
<td>[Prop. 3.18]</td>
</tr>
<tr>
<td>$\lor + 1$, fork and single</td>
<td>3</td>
<td>$\geq \log_2 n$</td>
<td>[F7]</td>
</tr>
<tr>
<td>$C_2 + A_2$</td>
<td>3</td>
<td>$\leq 8$</td>
<td>[Prop. 3.18]</td>
</tr>
<tr>
<td>$A_4$, antichain</td>
<td>3</td>
<td>$\geq 3n - 1$</td>
<td>[F7]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 4n + 2$</td>
<td>[F7]</td>
</tr>
<tr>
<td>$C_5$, chain</td>
<td>8</td>
<td>8</td>
<td>[G6]+[MNS]</td>
</tr>
<tr>
<td>$C_6$, chain</td>
<td>16</td>
<td>16</td>
<td>[G6]+[MNS]</td>
</tr>
<tr>
<td>$C_k$, chain ($k \geq 7$)</td>
<td></td>
<td>$\geq 2^{(k-3)/2}$</td>
<td>[G6]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq 2^{0.98k}$</td>
<td>[MNS]</td>
</tr>
<tr>
<td>$A_k$, antichain</td>
<td>$k - 1$</td>
<td>$\geq \left(1 - \frac{1}{\log_2 k}\right) \frac{k}{\log_2 k} n$</td>
<td>[MSW]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\leq kn - k - \frac{1}{2} \log_2 k + O(1)$</td>
<td>[F7]</td>
</tr>
<tr>
<td>$3C_2$</td>
<td>5</td>
<td>14</td>
<td>[Prop. 3.13]</td>
</tr>
<tr>
<td>$5C_2$</td>
<td>9</td>
<td>42</td>
<td>[Prop. 3.18]</td>
</tr>
<tr>
<td>$7C_2$</td>
<td>13</td>
<td>60</td>
<td>[Prop. 3.18]</td>
</tr>
<tr>
<td>any poset on $k$ elements</td>
<td></td>
<td>$\leq 2^{k-2}$</td>
<td>[Thm. 1.1]</td>
</tr>
<tr>
<td>UCTP (def. in Section 3.2)</td>
<td>$O(1)$</td>
<td>$\geq \log_2 n$</td>
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<td>$\geq \log_2 n$</td>
<td>[Thm. 3.6]</td>
</tr>
<tr>
<td>chain + shallower</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>[Thm. 3.8]</td>
</tr>
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</table>

**Figure 4:** Table from [?]