Let $G = (V,E)$ be a graph. \[ \text{[countable, locally finite, connected]} \]

Let $p \in (0,1)$.

Bernoulli bond percolation $\mathbb{P}_p^G := \left\{ \text{law of random spanning subgraph } w : E \to \{0,1\} \right\}$

\[
\begin{align*}
(w(e))_{e \in E} & \sim \text{iid Bernoulli}(p).
\end{align*}
\]

Connected components of $w$ are called clusters.
Transitive means $\forall u, v \in V \exists \phi \in \text{Aut } G: \phi(u) = v$.

Eg:

- $\mathbb{Z}^d$
- Regular tree

Fact: $\exists p_c \in [0, 1]$ such that $P_p^G(\exists \infty \text{ cluster}) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \\ ? & p = p_c \end{cases}$

Questions:
- When is $p_c \in (0, 1)$?
- How many $\infty$ clusters?
- Is $P_{p_c}^G(\exists \infty \text{ cluster}) = 0$?
Questions: When is $P_c \in (0,1)$?

**Thm:** (Duminil-Copin, Goswami, Raoufi, Severo, Kadin 2018)

$P_c \in (0,1)$ if and only if $G$ is not 1-dimensional.

How many $\infty$ clusters?

Say $G$ is amenable if

$$\inf_{w, v \in \mathbb{V}} \frac{|wv|}{|w|} = 0.$$  

**Thm:** (Aizenman, Kesten, Newman '87) (Burtman, Keane '89)

If $G$ is amenable, then $P_p$: "$\exists \infty \text{ cluster} \Rightarrow \exists! \infty \text{ cluster}$" $P_p$-a.s.

Converse remains open.

Is $P_c^G(\exists \infty \text{ cluster}) = 0$?

Conjectured to hold if $G$ is not 1-dimensional.

Open for $\mathbb{Z}^3$!
\[ G_n = \text{complete graph on } n \text{ vertices} \]  

\[
\text{Enumerate clusters } K_1, K_2, \ldots \text{ with } |K_1| > |K_2| > \ldots \quad (|K| = \# \text{ vertices in } K).
\]

\[ \|K\| := \frac{|K|}{|V(G)|} \]

\[ \text{(G_n) has a percolation threshold at } \langle \lambda_n \rangle: \]

\[
\text{(P_n) supercritical } \quad \left[ \text{i.e. } \liminf_{n \to \infty} \frac{\lambda_n}{n} > 1 \right] \quad \Rightarrow \quad \exists \xi > 0 : \lim_{n \to \infty} P_{G_n}^{G_n}(\|K\| \geq \xi) = 1.
\]

\[
\text{(P_n) subcritical } \quad \left[ \text{i.e. } \limsup_{n \to \infty} \frac{\lambda_n}{n} < 1 \right] \quad \Rightarrow \quad \forall \xi > 0 : \lim_{n \to \infty} P_{P_n}^{G_n}(\|K\| \geq \xi) = 0.
\]

\[ \text{(G_n) has the supercritical uniqueness property:} \]

\[
\forall \text{ supercritical } (P_n) : \quad \|K_2\| \overset{p}{\to} 0 \quad \text{under } P_{P_n}\]

\[ \text{(G_n) has the supercritical concentration property:} \]

\[
\forall \text{ supercritical } (P_n) : \quad \|K\| - E_{P_n}^{G_n} \|K\| \overset{p}{\to} 0 \quad \text{under } P_{P_n}\]
$G_n$ finite transitive, $|V(G_n)| \to \infty$

Questions:

- When does $(G_n)$ have... percolation threshold?
- supercritical uniqueness property?
- supercritical concentration property?

⚠️ Supercritical uniqueness comes first!

Say $(P_n)$ supercritical if $\exists \epsilon > 0$ such that $\mathbb{P}^{G_n}_{(1-\epsilon)P_n}(|K| \geq \epsilon) \geq \delta n$.
**Warm-up:** \((G_n)\) has bounded vertex degrees

**Conjecture:** (Benjamini '01)

\((G_n)\) has the supercritical uniqueness property.

(Previously known for tori and (not necessarily transitive) expanders)

**Thm:** (E., Hutchcroft)

\((G_n)\) has...

- supercritical uniqueness property
- percolation threshold
- supercritical concentration property

**Thm:** (Hutchcroft, Tointon '21)

If \((G_n)\) has "at least \((1+\varepsilon)\)-dimensional growth" then \(\exists \delta > 0\): \(P_{1-\varepsilon}^{G_n}(||K|| \geq \delta) \geq \varepsilon.\)
Say \((G_n) \to G\) locally if \(\forall R \in \mathbb{N} \forall n \in \mathbb{N} : B_{R}^{G_n} \cong B_{R}^{G}\).

If \((P.P.P. \ldots)\) is supercritical and \(G_n \to G\) locally, then

\[ \mathbb{P}_{P}^{G_n}(0 \in K_i) \to \mathbb{P}_{P}^{G}(0 \in \text{infinite cluster}) \]

**Theorem (E. Hutchcroft '22+)**

If \((P.P.P. \ldots)\) is supercritical and \(G_n \to G\) locally, then

\[ \mathbb{P}_{P}^{G_n}(0 \in K_i) \to \mathbb{P}_{P}^{G}(0 \in \text{infinite cluster}) \]

**Conjecture (Schramm)**

Let \((G_n)\) be a sequence of infinite transitive graphs with \(\limsup_{n \to \infty} P_c(G_n) < 1\).

If \(G_n \to G\) locally, then \(P_c(G_n) \to P_c(G)\).

The analogue for finite \(G_n\) should hold for "nice" sequences. Can not 1-dimensional?
A curious case...

Suppose

- $(G_n)$ is "nice"
- $G$ is nonamenable
  \[ \exists p > p_c(G) \text{ st. } P_p^G(\exists \text{multiple } \infty \text{-clusters}) = 1. \]
- $G_n \rightarrow G$ locally.

How are \( \{ \text{uniqueness of the giant cluster in } P_p^{G_n} \} \) consistent? \( \{ \text{non-uniqueness of the } \infty \text{ cluster in } P_p^G \} \)
General case: \((G_n)\) may have unbounded vertex degrees does not have the supercritical uniqueness property.

At \(P_n = \frac{2}{n}\):

\begin{equation}
\text{Supercritical } \quad \text{Poisson } \quad \text{Supercritical}
\end{equation}

\begin{equation}
\text{Erdős–Rényi} \quad \text{process} \quad \text{Erdős–Rényi}
\end{equation}

Same argument works for similar constructions, called molecules.

Thus: (E. Hutchcroft '21, 22, 22+)

These are the only obstacles to:
- existence of percolation threshold
- supercritical uniqueness property
- supercritical concentration property
Sketch of proof of Benjamini's conjecture

Fix \( (G_n) \) with bounded vertex degrees and \( |V(G_n)| \to \infty \).

**Con**: (Talagrand '94)
Every increasing and \( Aut(G_n) \)-invariant event has threshold width \( O\left(\frac{1}{\log |V(G_n)|}\right) \).

Fix \( (P_n) \) supercritical.

Pick \( \Delta > 0 \) s.t. \( (P_n - \Delta) \) supercritical.

**WTS**: \( \|K_2\|_P \to 0 \) under \( P_{P_n}^{G_n} \).
Claim: \( \exists (q_n) \) such that

- \( \forall n: \ P_n - \Delta \leq q_n \leq P_n \)
- \( \|K_i\| - \left( E_{q_n}^G \|K_i\| \right) \xrightarrow[p]{} 0 \) under \( P_{q_n}^G \)

Proof: Let \( n \geq 1 \).

Define \( M(\rho) = \text{median of } \|K_i\| \text{ under } P_{\rho}^G \).

\[ \forall \rho \exists q \in (p_\rho - \Delta, p_\rho): \quad |M(q + \frac{\Delta}{N}) - M(q - \frac{\Delta}{N})| \leq \frac{2}{N}. \]

Note that:
- \( P_{q - \frac{\Delta}{N}}^G (\|K_i\| \geq M(q) - \frac{2}{N}) \geq \frac{1}{2} \)
- \( P_{q + \frac{\Delta}{N}}^G (\|K_i\| \leq M(q) + \frac{2}{N}) \geq \frac{1}{2} \)
\[ P_{q - \frac{\theta}{N}}^G (\| K \| \geq M(q) - \frac{2}{N}) \geq \frac{1}{2} \]

\[ P_{q + \frac{\theta}{N}}^G (\| K \| \leq M(q) + \frac{2}{N}) \geq \frac{1}{2} \]

Set \( N = \sqrt{\log |V(G_n)|} \) so that

\[
\text{[threshold width]} \leq \frac{1}{\log |V(G_n)|} \ll \frac{\theta}{N}.
\]

Therefore:

\[ P_{q}^G (\| K \| \geq M(a) - \frac{1}{N}) = 1 - o(1) \]

\[ P_{q}^G (\| K \| \leq M(a) + \frac{1}{N}) = 1 - o(1) \]
Claim: \( \| K_2 \| \to 0 \) under \( P_{\mathcal{G}_n}^{G_n} \).

Proof: "concentration \( \Rightarrow \) uniqueness"

Assume for contradiction:

- \( \exists \varepsilon > 0 \quad \forall n: \quad P_{\mathcal{G}_n}^{G_n} (\| K_2 \| \geq \varepsilon) > \varepsilon \)
- \( \| K_1 \| \stackrel{p}{\to} \alpha \in (0, 1) \) under \( P_{\mathcal{G}_n}^{G_n} \).

Def: A subgraph \( S \subseteq G \) is called a sandcastle if \( |S| \geq \varepsilon \) and

\[
P_{\mathcal{G}_n/p_n}^S (\| K_1 \| \leq \frac{\alpha}{2} \cdot |V(G_n)|) \geq \frac{1}{2}
\]
Claim: \( \exists N \forall n \geq N: P^n_{\text{P}}(K_i \text{ or } K_2 \text{ is a sandcastle}) \geq \frac{\epsilon}{2} \).

Proof: Suppose false at \( n \).

Then \( P^n_{\text{P}}(\|K_1\|,\|K_2\| \geq \epsilon \text{ but } K_1, K_2 \text{ not sandcastles}) \geq \frac{\epsilon}{2} \).

So \( P^n_{\text{P}}(\|K_2\| \geq \frac{\alpha}{2}) \geq \frac{\epsilon}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \). \( \star \) Contradiction for \( n \) large.
Claim: \( J(s_n) \) such that

- \( \forall n: s_n \in G_n \) and \( \|s_n\| \leq \varepsilon \)
- \( \|K_i(w^j)\| \overset{p}{\to} \alpha \) under \( \mathbb{P}_{G_n} \).

Proof: Otherwise \( \exists s \geq 0 \) s.t. with good probability:

\[ \|s\| \leq \frac{1}{2} \]

\[ \|s\| - \alpha \geq \delta \]

So \( \|K_i(w^j)\| - \alpha \geq \delta \). *Contradiction for large \( n \)
Claim: \( \exists N \forall n \geq N: \quad P_{q_n}^{G_n} \left( \|K\| \geq \alpha + \frac{\alpha \epsilon}{3} \right) \geq \frac{\alpha}{4}. \)

Proof: By transitivity,

\[
P_{q_n}^{G_n} \left( \left| \{ v \in S_n : \|K_v\| \geq \frac{\alpha}{2} \} \right| \geq \frac{\alpha}{3} |S_n| \right) \geq \frac{\alpha}{3}.
\]

This claim contradicts concentration of \( \|K\| \) under \( P_{q_n}^{G_n} \)!
An open problem

Let \((G_n)\) be sequence of finite transitive graphs with \(|V(G_n)| \to \infty\).

Say \((G_n)\) has the general uniqueness property if

\[
A(G_n) : \|K_2\|^p \to 0 \quad \text{under } P^G_{p_n}.
\]

Fails for approximately 1-dimensional examples.

\[\mathbb{Z}/n\mathbb{Z}\]

\[(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/2^n\mathbb{Z})\]

**Conjecture** (Alon, Benjamini, Stacey '04)

If \(\text{diam } G_n = o\left(\frac{|V(G_n)|}{\log |V(G_n)|}\right)\), then \((G_n)\) has the general uniqueness property.