# A solution to Erdős and Hajnal's odd cycle problem

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Given the *average degree*, $d(G)$, of a graph $G$
what can we say about its different cycle lengths?

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Given the *chromatic number*, $\chi(G)$, of a graph $G$
what can we say about its different odd cycle lengths?

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$\mathcal{C}(G) = \{\ell : G \text{ contains an } \ell\text{-cycle}\}$ $\mathcal{C}_{\text{odd}}(G) = \{\ell \in \mathcal{C}(G) : \ \ell \text{ is odd}\}$

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| Density of $\mathcal{C}(G)$ | Density of $\mathcal{C}_{\text{odd}}(G)$ |
|---|---|
| Specific lengths in $\mathcal{C}(G)$ | Specific lengths in $\mathcal{C}_{\text{odd}}(G)$ |

**Density of $\mathcal{C}(G)$?** Easy to show $|\mathcal{C}(G)| = \Omega(d(G))$, if $d(G) \geq 2$.

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In 1966, Erdős and Hajnal asked whether

$$\sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell} \to \infty \quad \text{as} \quad \chi(G) \to \infty,$$

and (perhaps later) suggested that $d(G) \to \infty$ was sufficient.

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For the complete bipartite graphs $K_{d,d}$, we have

$$\sum_{\ell \in \mathcal{C}(K_{d,d})} \frac{1}{\ell} = \left(\frac{1}{2} - o_d(1)\right) \log d.$$

In 1975, Erdős conjectured this is an asymptotic lower-bound if $d(G) = d$.

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In 1984, Gyárfás, Komlós and Szemerédi proved that

$$\sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell} = \Omega(\log(d(G))).$$

Erdős (1959) : for fixed $k$, $\chi(G)$ can be arbitrarily large yet $k \notin \mathcal{C}(G)$.

Instead, can we say there must be one of $\sigma_1, \sigma_2, \sigma_3 \ldots$ in $\mathcal{C}(G)$?

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| Given $d(G)$, how dense is $\mathcal{C}(G)$? | Given $\chi(G)$, how dense is $\mathcal{C}_{\text{odd}}(G)$? |
| Given $d(G)$, does $\mathcal{C}(G)$ contain one of $\sigma_1, \sigma_2, \sigma_3 \ldots$? | Given $\chi(G)$, does $\mathcal{C}_{\text{odd}}(G)$ contain one of $\sigma_1, \sigma_2, \sigma_3 \ldots$? |

| **Even cycles** | **Odd cycles** |
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| Gyárfás, Komlós, Szemerédi ('84): $$\sum_{\ell\in\mathcal{C}(G)}\frac{1}{\ell}=\Omega(\log(d(G))).$$ Erdős ('75): Does $$\sum_{\ell\in\mathcal{C}(G)}\frac{1}{\ell}\geq\left(\frac{1}{2}-o(1)\right)\log(d(G))?$$ | Erdős, Hajnal ('81): Does $$\sum_{\ell\in\mathcal{C}_{\mathrm{odd}}(G)}\frac{1}{\ell}\to\infty$$ as $\chi(G)\to\infty$? |
| Erdős ('84): Is there some $d$ such that if $d(G)\geq d$ then $\mathcal{C}(G)$ contains a power of 2? | Is there some $k$ such that if $\chi(G)\geq k$ then $\mathcal{C}_{\mathrm{odd}}(G)$ contains a prime? |

# Average degree and even cycle lengths

A sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ is *unavoidable with high average degree* if there is some $d$ such that if $d(G) \geq d$ then $\sigma_i \in \mathcal{C}(G)$ for some $i$.

Is $1, 2, 4, 8, 16, \ldots$ unavoidable with high average degree?

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**Bollobás (1977):** if $\sigma_i$ is an arithmetic progression containing even numbers, then $\sigma_i$ is unavoidable with high average degree.

**Verstraëte (2005):** an increasing sequence with density 0 which is unavoidable with high average degree must exist.

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**Sudakov and Verstraëte (2008):** If an even increasing sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ is exponentially bounded, then it is unavoidable with high average degree except in (perhaps) graphs $G$ with $d(G) \leq \exp(\log^* |G|)$.

*Exp. bounded:* $\sigma_i \leq C\sigma_{i-1}$ for each $i \in \mathbb{N}$ and some fixed $C > 1$.

**Theorem 1** (Liu and M., 2021+).

There is some $d_0 > 0$ such that:

If $d(G) = d \geq d_0$, then, for some $r \geq d/(10 \log^{10} d)$, $\mathcal{C}(G)$ contains every even integer in $[\log^8 r, r]$.

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$$\sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell} \geq \left(\frac{1}{2} - o_r(1)\right) \log r \geq \left(\frac{1}{2} - o_d(1)\right) \log d.$$

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If an even increasing sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ eventually appears in each interval $[\log^8 r, r]$, $r \geq 1$, then it is unavoidable with high average degree.

If an even increasing sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ is exponentially bounded ($\sigma_i \leq C\sigma_{i-1}$ for each $i \in \mathbb{N}$ and some fixed $C > 1$), then it is unavoidable with high average degree.

# Chromatic number and odd cycle lengths

A sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ is *unavoidable with high chromatic number* if there is some $k$ such that if $\chi(G) \geq k$ then $\sigma_i \in \mathcal{C}(G)$ for some $i$.

Is $2, 3, 5, 7, 11, 13, \ldots$ unavoidable with high chromatic number?

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**Gyárfás (1992):** If $\chi(G) \geq 2k+1$, then $|\mathcal{C}_{\text{odd}}(G)| \geq k$.

**Sudakov and Verstraëte (2008):**
If $\chi(G) \geq k$, then $\mathcal{C}(G)$ contains $\Omega(k)$ consecutive integers.

**Sudakov and Verstraëte (2011):**
$\sum_{\ell \in \mathcal{C}_{\text{odd}}(G)} 1/\ell \to \infty$ if the independence ratio of $G \to \infty$
(a relaxation of $\chi(G)$) not extremely slowly compared to $|G|$.

**Theorem 2** (Liu and M., 2021+).

For each $\varepsilon > 0$, there is some $k_0 \in \mathbb{N}$ such that:

If $\chi(G) = k \geq k_0$, then, for some $r \geq 1$,
$\mathcal{C}(G)$ contains every odd integer in $[r, r \cdot k^{1-\varepsilon}]$.

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$$\sum_{\ell \in \mathcal{C}_{\mathrm{odd}}(G)} \frac{1}{\ell} \geq \left(\frac{1}{2} - o_k(1)\right) \log k.$$

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If an odd increasing sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ is exponentially bounded ($\sigma_i \leq C\sigma_{i-1}$ for each $i \in \mathbb{N}$ and some fixed $C > 1$), then it is unavoidable with high chromatic number.

| **Even cycles** | **Odd cycles** |
| --- | --- |
| Gyárfás, Komlós, Szemerédi ('84): $$\sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell} = \Omega(\log(d(G))).$$ Liu, M. ('21+): $$\sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell} \geq \left(\frac{1}{2} - o(1)\right) \log(d(G)).$$ | Liu, M. ('21+): $$\sum_{\ell \in \mathcal{C}_{\mathrm{odd}}(G)} \frac{1}{\ell} \geq \left(\frac{1}{2} - o(1)\right) \log(\chi(G)).$$ |
| Liu, M. ('21+): There is some $d$ such that if $d(G) \geq d$ then $\mathcal{C}(G)$ contains a power of 2. | Liu, M. ('21+): There some $k$ such that if $\chi(G) \geq k$ then $\mathcal{C}_{\mathrm{odd}}(G)$ contains a prime. |

# Proof Elements

Need to work in very sparse graphs:

Use a form of expansion introduced by Komlós and Szemerédi to find subdivisions in sparse graphs.

Use a constructive approach:

Connect pairs of vertices with short paths to create a structure which can adjust the length of cycles.

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Can assume $G$ is bipartite and has $\delta(G) \geq d$.

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Outline of the rest of the talk

1. Constructing even cycles in a simplified setting.
2. Komlós-Szemerédi expansion.
3. Finding odd cycle lengths.
4. The problem of sublinear expansion.

## 1. Constructing even cycles in a simplified setting.

**What we want to do:** Find a path with a specific length between any two vertices $x, y$ in $G$.

We could apply this with $xy \in E(G)$ to get a cycle with a specific length.

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**What we can do:** Find a path which is reasonably short between any two vertices $x, y$ in $G$.

Let's say we can do this vertex-disjointly between lots of vertex pairs.

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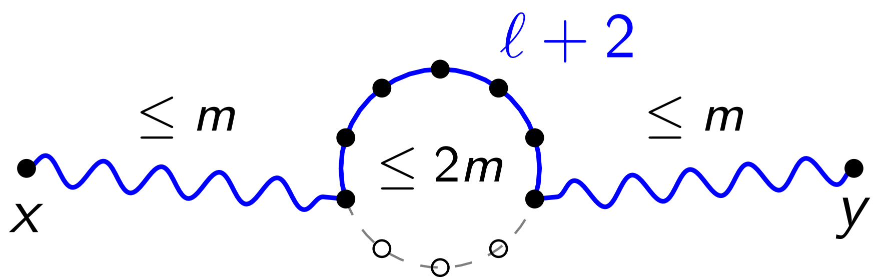
Let $n = |G|$ and say we can connect with paths length $\leq m = O(\log^3 n)$.




For some $\ell \leq 3m$ there is an $x, y$-path with length $\ell$ and with length $\ell + 2$.

**What we can do:** Find a path with length $\leq m = O(\log^3 n)$ between any two vertices $x, y$ in $G$ with $n = |G|$.

Let's say we can do this vertex-disjointly between lots of vertices pairs.

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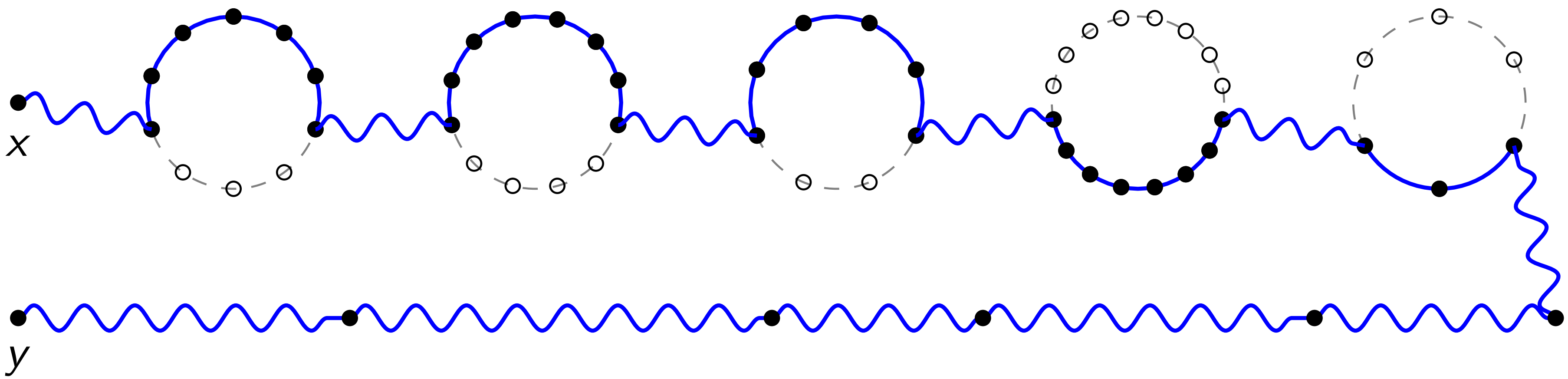
For $\ell = \log^{10} n$, can now find an $x, y$-path with length $\ell$ or $\ell + 1$:

- Take $m$ vertex disjoint cycles, and join them in a path from $x$.
- Lengthen until the path's (shortest) length is in $(\ell - 2m, \ell - m]$.
- Connect to $y$: path length is in $(\ell - 2m, \ell]$.
- Lengthen it in steps of 2 until it has length $\ell$ or $\ell + 1$.



## 2. Komlós-Szemerédi expansion.

**Komlós and Szemerédi (1984):**

Every graph $G$ contains some $H \subset G$ in which every pair of vertices is connected by a path with length at most $O(\log^3(|H|/d(G)))$.

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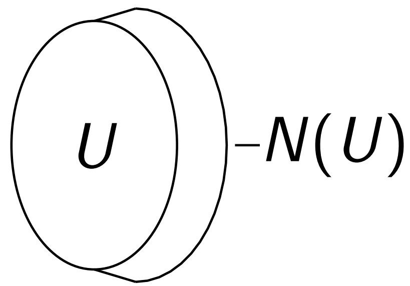
A graph $H$ is an $(\varepsilon, k)$-**expander** if, for each $U \subset V(G)$ with $k \leq |U| \leq |H|/2$,

$$|N(U)| \geq \frac{\varepsilon}{\log^2(15|U|/k)} \cdot |U|.$$

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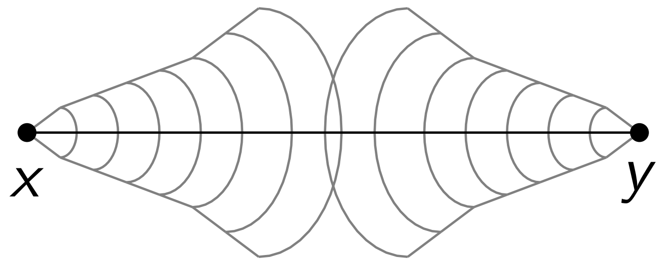
**Komlós and Szemerédi (1984):**

$\exists \varepsilon \in (0,1)$ such that every graph $G$ with $d(G) \geq 8d$ has a bipartite $(\varepsilon, \varepsilon d)$-expander subgraph $H$ with minimum degree $\delta(H) \geq d$.

If $H$ is an $n$-vertex expander with $\delta(H) \geq d$, then any pair of vertices $x, y$ are connected by a path with length $O(\log^3 n)$.

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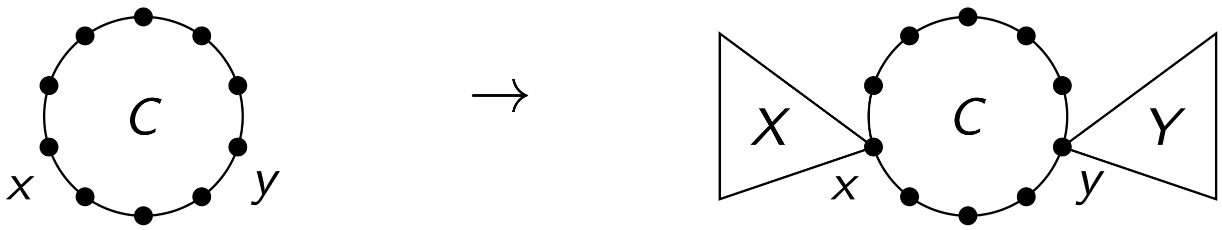
- Pairs of single vertices are hard to connect vertex-disjointly.
- Instead of finding cycles, we find *adjusters*:




- If $|X|, |Y| \gg |C|$, can expand $X$ or $Y$ while avoiding the rest of $V(C)$, and connect to other adjusters.
- Finding many vertex-disjoint adjusters is the main challenge.

Finding an expander subgraph $H \subset G$, and applying the construction using adjusters gives:

**Liu, M. (2021+):**

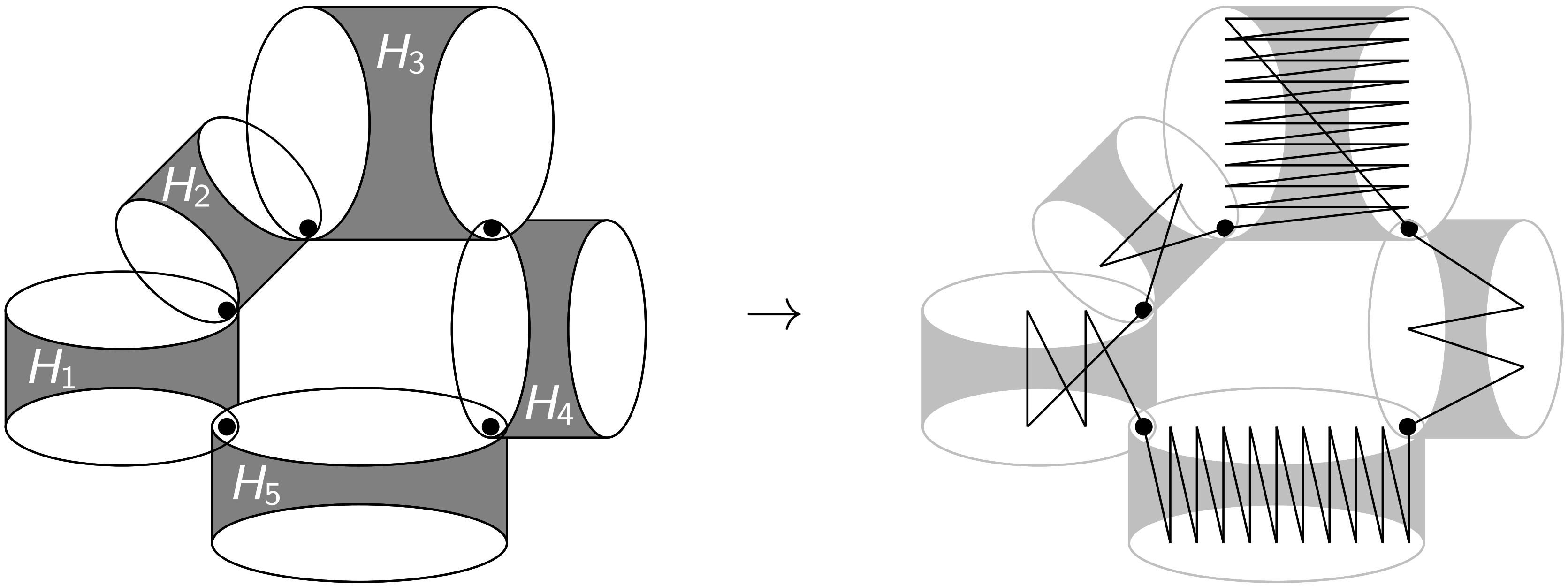
If $d(G) \geq d$, then there is a connected bipartite subgraph $H \subset G$ and some $\ell_H \geq 1$ such that:

> For each distinct $x, y \in V(H)$ and each $r \in [\ell_H, \ell_H \cdot d^{1-o_d(1)}]$, there is an $x, y$-path with length $r$ in $H$, if parity permits.

**3.** Finding odd cycle lengths.

Suppose we could find the following cycle of bipartite graphs $H_1, \ldots, H_5$, intersecting only on the vertices shown, and satisfying:

> For each distinct $x, y \in V(H_i)$ and each $r \in [\ell_i, \ell_i \cdot d^{1-o_d(1)}]$, there is an $x, y$-path with length $r$ in $H_i$, if parity permits.




Vary paths between the shared vertices to get many odd cycles.

**Liu, M. (2020+):**

If $d(G) \geq d$, then there is a connected bipartite subgraph $H \subset G$ with bipartition $A \cup B$ and some $\ell_H \geq 1$ such that:

> For each distinct $x, y \in V(H)$ and each $r \in [\ell_H, \ell_H \cdot d^{1-o_d(1)}]$, there is an $x, y$-path with length $r$ in $H$, if parity permits.

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Suppose $\chi(G) \geq 3d$. Iteratively find as many edge-disjoint connected bipartite graphs $H_1, \ldots, H_t$ satisfying the boxed property above.

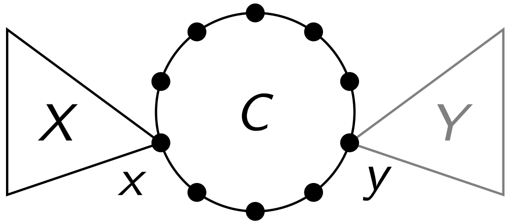
From the result above, we can deduce $\chi(H_1 \cup \ldots \cup H_t) \geq 3$, and thus $H_1 \cup \ldots \cup H_t$ contains an odd cycle.

Roughly: taking a minimal such cycle, and replacing some of the edges using the boxed property can create many odd cycles.

**4.** The problem of sublinear expansion.

**One adjuster in $H$:** relatively simple to find.

Take a minimal cycle $C$. Expanding from $x$, we are slow to encounter other vertices in $V(C)$, and get $X$ with $|X| \gg |C|$.

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**One adjuster in $H - W$:** much harder to find. Could try:

- Try the above construction in $H - W$.
- If it fails, then you find a *bad* set which does not expand in $H - W$.
- Repeat this, finding many disjoint bad sets.
- Put them together to get a big bad set ($\gg |W|$) which does not expand in $H - W$, and thus still does not expand well in $H$.

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**Problem:** Komlós-Szemerédi expansion is sublinear.

## The problem of sublinear expansion.

In an $(\varepsilon, k)$-**expander** $H$, for each $U \subset V(G)$ with $k \leq |U| \leq |H|/2$,

$$|N(U)| \geq \frac{\varepsilon}{\log^2(15|U|/k)} \cdot |U|.$$

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- The function $\frac{\varepsilon}{\log^2(15|U|/k)}$ decreases as $|U|$ increases.
- Therefore, bad sets may not expand in $H - W$ but their union might.

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**Strategy:**

- Make sure bad sets cannot expand sharply into $W$.
- Find many bad sets with the same size and same neighbourhood in $W$.
- Combine enough of these bad sets to be larger than their neighbourhood in $W$, but not enough to decrease $\frac{\varepsilon}{\log^2(15|U|/k)}$ a lot.
- This finds a contradiction.

# Limitations of our methods and open questions

Our construction can not hope for more than:

Any increasing even sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ with, for each $i$, $\sigma_{i+1} \leq \exp((\sigma_i)^{1/3})$, is unavoidable with high average degree.

**Question:** Can you replace $(\sigma_i)^{1/3}$ with $\varepsilon\sigma_i$ for some small $\varepsilon > 0$?

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**Conjecture: (Erdős and Gyárfás)** Every graph of minimum degree at least 3 contains a cycle whose length is a power of 2.

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**Problems: (Scott and Seymour)** Similar questions for the lengths of the *induced* cycles in a graph and $\chi$-boundedness.