Random Spanning Tree in Random Environment

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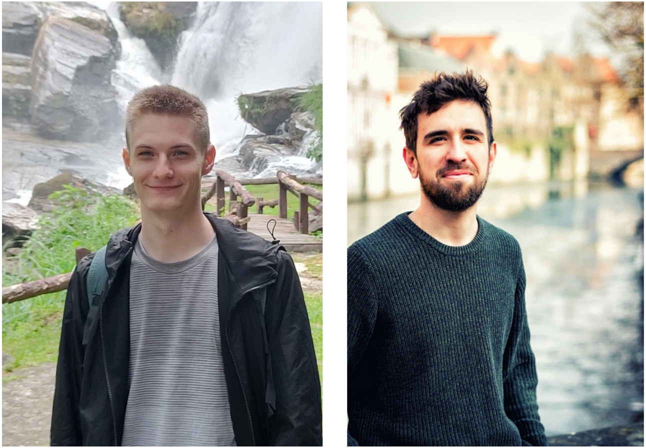

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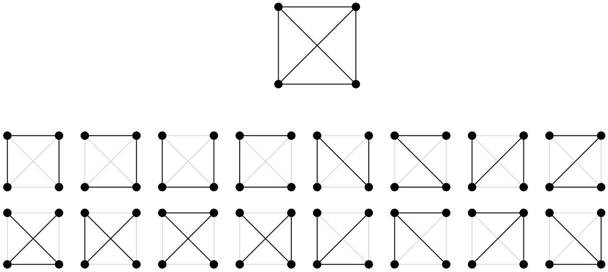
Outline

1. Uniform Spanning Tree (UST) and Minimum Spanning Tree (MST)
2. Random Spanning in Random Environment (RSTRE)
3. Diameter of RSTRE – From Low Disorder to High Disorder
4. Proof Ideas
5. Results for Local Observables
6. Open Questions

Uniform Spanning Tree (UST)

Given a graph $G = (V, E)$:



A spanning tree T of G is a tree subgraph of G with vertex set V. Denote by $\mathbb{T} = \mathbb{T}(G)$ the set of spanning trees of G.

Uniform Spanning Tree (UST)

A random spanning tree $\mathcal{T}$ chosen with equal probability among all spanning trees of G is called a Uniform Spanning Tree.

Uniform Spanning Tree on Weighted Graphs

UST on a weighted graph (conductance network)

Given a graph $G = (V, E)$ with edge weights $w : E \to (0, \infty)$, the UST on the weighted graph (G, w) (conductance network) is the random spanning tree $\mathcal{T}$ with

$$\mathbf{P}^w(\mathcal{T} = T) = \frac{1}{Z_w} \prod_{e \in T} w_e, \qquad T \in \mathbb{T}(G).$$

where Z_w is the normalizing constant.

Random Walk on a weighted graph (conductance network)

The random walk on the weighted graph (G, w) has transition kernel

$$\Pi^w(x, y) = \frac{w_{\{x,y\}}}{\sum_{z \sim x} w_{\{x,z\}}}, \qquad x, y \in V,$$

where $z \sim x$ means $\{z, x\}$ is an edge in E.

Sampling Algorithms for UST

There are two classic algorithms to sample the UST $\mathcal{T}$ on (G, w).

Aldous-Broder Algorithm

Run a random walk X on (G, w) until all vertices have been visited. Whenever a vertex is visited for the first time, add that edge to $\mathcal{T}$.

Wilson's Algorithm

Fix any ordering of vertices in V. Start with the tree $\mathcal{T}_0$ consisting of a single vertex v_0. Having constructed $\mathcal{T}_i$ for $i \geq 0$, pick the next vertex v_k not in $\mathcal{T}_i$ and run a loop erased random walk (LERW) from v_k until it hits $\mathcal{T}_i$. Define $\mathcal{T}_{i+1}$ to be the union of $\mathcal{T}_i$ and the LERW path from v_k to $\mathcal{T}_i$.

Minimum Spanning Tree (MST)

Minimum Spanning Tree Model

Given $G = (V, E)$, let $\omega := (\omega_e)_{e \in E}$ be i.i.d. continuous r.v.'s. The minimum spanning tree on G is the spanning tree $\mathcal{T} = \mathcal{T}(\omega) \in \mathbb{T}(G)$ that minimizes $H^\omega(\mathcal{T}) := \sum_{e \in \mathcal{T}} \omega_e$.

- $\mathcal{T}(\omega)$ is completely determined by the order among $(\omega_e)_{e \in E}$. This can be seen from the following greedy algorithms.

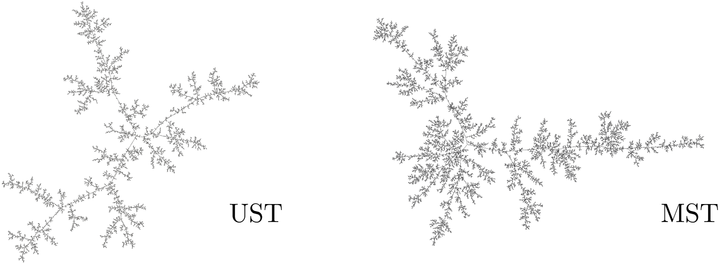
Prim's Algorithm

Start with $\mathcal{T} = \{v\}$ for any $v \in V$. Among all edges connecting $\mathcal{T}$ with vertices outside $\mathcal{T}$, add the edge e with the smallest ω_e. Iterate.

Kruskal's Algorithm

Start from a forest with $|V|$ isolated vertices. Add the edge e with the smallest ω_e that connects two disjoint components of the forest. Iterate.

Are the UST and MST different?



UST

MST

Simulations on a random 3-regular graph with $|V| = 100000$ (by Luca Makowiec)

Uniform Spanning Tree on Graphs with n Vertices

	Diameter	Scaling Limit
Complete graph	$\sqrt{n}$	Aldous' CRT [1]
dense graphs	$\sqrt{n}$ [2]	Aldous' CRT [3]
d-dim Torus: $d \geq 5$	$\sqrt{n}$ [4,5]	Aldous' CRT [4,6]
d-dim Torus: $d = 4$	$n^{1/2}(\log n)^{1/6}$	Aldous' CRT [7]
d-dim Box: $d = 3$	$n^{\beta/3}, \ \beta \in (1, 5/3]$ [8]	$\mathcal{U}$ [9]
d-dim Box: $d = 2$	$n^{4/5}$ [10]	$\mathcal{T}$ [11]

1.[Aldous'91]; 2.[Alon-Nachmias-Shalev'22]; 3.[Archer-Shalev'23]; 4.[Peres-Revelle'05];
5.[Michaeli-Nachmias-Shalev'21]; 6.[Archer-Nachmias-Shalev'22]; 7.[Schweinsberg'09];
8.[Shiraishi'18]; 9.[Angel-Croydon-Hernandez Torres-Shiraishi'21]; 10.[Kenyon'00];
11.[Lawler-Schramm-Werner'04].

Minimum Spanning Tree on Graphs with n Vertices

	Diameter	Scaling Limit
Complete graph	$n^{1/3}$ [1]	$\mathcal{M}$ [2]
d-dim Torus: $d \gg 1$	$n^{1/3}$??	$\mathcal{M}$??
d-dim Torus: $d = 2$	$n^{?}$	$\mathcal{X}$ [3] (triangular lattice)

1.[Addario Berry-Broutin-Reed'09]; 2.[Addario Berry-Broutin-Goldschmidt-Miermont'17];
3.[Garban-Pete-Schramm'18].

Random Spanning Tree in Random Environment

Inspired by the Directed Polymer in Random Environment (DPRM), we introduce a model that interpolates between the UST and MST [Makowiec-Salvi-Sun'24] (independently by [Kúsz'24]).

Random Spanning Tree in Random Environment (RSTRE)

Let $G = (V, E)$ be a connected graph. Let $\omega := (\omega_e)_{e \in E}$ be i.i.d. $U(0, 1)$ r.v.'s, playing the role of disorder/random environment.

Given ω and $\beta \in [0, \infty]$ (inverse temperature/disorder strength), the Random Spanning in Random Environment (RSTRE) is defined via the Gibbs measure

$$\mathbf{P}_\beta^\omega(\mathcal{T} = T) = \frac{1}{Z_\beta^\omega} \prod_{e \in T} e^{-\beta \omega_e}, \qquad T \in \mathbb{T}(G),$$

where Z_β^ω is the normalizing constant (partition function).

- $\mathbf{P}_0^\omega$ is the law of the UST on G.
- $\mathbf{P}_\infty^\omega$ is the law of the MST on G.
- $\mathbf{P}_\beta^\omega$ is the law of the UST on the weighted graph (G, w) with weights $w_e = e^{-\beta \omega_e}$.

Transition from UST to MST?

Consider UST and MST on a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \uparrow \infty$, e.g., boxes/toruses on $\mathbb{Z}^d$, the complete graph K_n, random d-regular graphs, etc.

Questions of interest

- As $n \to \infty$, for which values of disorder strength $\beta = \beta_n$ do we see the transition between UST and MST? Is there a transition window where we see a continuous interpolation?
- What observable/order parameter should we use to distinguish between UST and MST? Diameter? Overlap? Hausdorff dimension?
- What scaling limits can we obtain in this transition window?

We will focus mainly on the complete graph K_n and study the transition in the diameter as $n \uparrow \infty$, because UST and MST on K_n and their scaling limits are well-understood.

Diameter of RSTRE at Fixed/Low Disorder

Theorem 1 (Diameter at Fixed/Low Disorder [MSS'24, Kúsz'24])

Let $\mathcal{T}$ be the RSTRE on the complete graph K_n, with law $\mathbf{P}^{\omega}_{n,\beta}$. Let $\mathbb{E}$ be expectation w.r.t. the disorder ω. Then for any $\beta \in [0, \infty)$, with $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$ probability tending to 1 as $n \to \infty$,

$$\mathrm{Diam}(\mathcal{T}) = \Theta(\sqrt{n}).$$

The same holds if $\beta = \beta_n \leq C_0 n / \log n$.

The transition from UST to MST must occur at $\beta = \beta_n \geq C_0 \frac{n}{\log n}$!

- We call the regime of β_n where $\mathrm{Diam}(\mathcal{T}) = \Theta(\sqrt{n})$ the low disorder regime.

Weighted Graphs with Fixed Weight Distribution

On more general weighted graphs with a given weight distribution (e.g., fixed β in RSTRE), we also have $\mathrm{Diam}(\mathcal{T}) = n^{1/2+o(1)}$.

Theorem 2 (Fixed Weight Distribution [MSS'23])

Let $G = (V, E)$ with $|V| = n$ be either:

(1) expander with max degree $\Delta < \infty$;

(2) torus in $d \geq 5$.

Let $w := (w_e)_{e \in E}$ be i.i.d. with law μ s.t. $\mu(0, \infty) = 1$. Let $\mathbf{P}^w$ be the law of a UST on (G, w) with $\mathbf{P}^w(\mathcal{T} = T) = \frac{1}{Z_w} \prod_{e \in T} w_e$. Then $\exists\, c > 0$ s.t. $\forall\, \varepsilon > 0$ and $\forall\, n$ large (depending on μ)

$$\mathbb{E}\mathbf{P}^w \left[\frac{n^{1/2}}{(\varepsilon^{-1} \log n)^c} \leq \mathrm{Diam}(\mathcal{T}) \leq (\varepsilon^{-1} \log n)^c n^{1/2} \right] \geq 1 - \varepsilon \, .$$

- Proof of both Theorems 1 & 2 are based on extensions of criteria formulated in [Michaeli-Nachmias-Shalev'21] for graphs (G, w) with $w \equiv 1$.

Diameter of the RSTRE at High Disorder

Theorem 3 (Diameter at High Disorder [MSS'24, Kúsz'24])

Let $\mathcal{T}$ be the RSTRE on the complete graph K_n, with law $\mathbf{P}^{\omega}_{n,\beta}$. Let $\mathbb{E}$ be expectation w.r.t. the disorder ω. Then for $\beta = \beta_n \geq n^{4/3} \log n$, with $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$ probability tending to 1 as $n \to \infty$,

$$\mathrm{Diam}(\mathcal{T}) = \Theta(n^{1/3}).$$

- We call the regime of β_n where $\mathrm{Diam}(\mathcal{T}) = \Theta(n^{1/3})$ the high disorder regime.

Diameter of the RSTRE at Intermediate Disorder?

Conjecture

Let $\mathcal{T}$ be the RSTRE on the complete graph K_n, with law $\mathbf{P}^{\omega}_{n,\beta}$. Then with $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$ probability tending to 1 as $n \to \infty$,

$$\mathrm{Diam}(\mathcal{T}) \approx \begin{cases} n^{1/2}, & \beta_n \ll n, \\ n^{(1-\gamma)/2}, & \beta_n = n^{1+\gamma}, \quad 0 \leq \gamma \leq 1/3, \\ n^{1/3}, & \beta_n \gg n^{4/3}. \end{cases}$$

Proof Ideas for $\mathrm{Diam}(\mathcal{T}) = n^{1/2+o(1)}$ (Theorem 2)

Proposition 1 (Extension of [Michaeli-Nachmias-Shalev'21])

Let t_{mix} and p_t be the mixing time and transition kernel of the lazy random walk on (G, w). Assume (G, w) satisfies the three conditions:

- balanced:

$$\frac{\max_{u \in V} \sum_v w_{\{u,v\}}}{\min_{u \in V} \sum_v w_{\{u,v\}}} \leq D,$$

- mixing: $$t_{\mathrm{mix}}(G, w) \leq n^{\frac{1}{2}-\alpha},$$
- escaping: $$\sum_{t=0}^{t_{\mathrm{mix}}} (t+1) \sup_{v \in V} p_t(v, v) \leq \theta\,,$$

where $D, \alpha, \theta \in (0, \infty)$. Then $\forall \varepsilon > 0$, $\exists\, C = C(\varepsilon, D, \alpha, \theta)$ such that

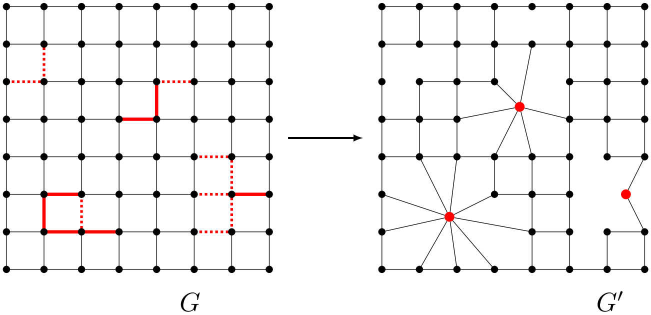
$$\mathbf{P}^w\big(C^{-1} n^{1/2} \leq \mathrm{Diam}(\mathcal{T}) \leq C n^{1/2}\big) \geq 1 - \varepsilon.$$

For high-dim graphs, e.g. dense graphs, expanders, d-dim torus with $d \geq 5$, D, α, θ are uniform if w is bounded away from 0 and ∞.

Proof Ideas for $\mathrm{Diam}(\mathcal{T}) = n^{1/2+o(1)}$ (Theorem 2)

For (G, w) with i.i.d. w_e and general weight distribution μ on $(0, \infty)$:

1. Fix a large A and condition on the realisation of the UST $\mathcal{T}$ on the edge set $K := \{e \in E : w_e \notin (1/A, A)\}$. Define G' by
   - Deleting all $e \in K \setminus \mathcal{T}$, and Contracting all $e \in K \cap \mathcal{T}$.



2. Show (G', w') satisfies the 3 conditions in Proposition 1 with $D(n), \alpha(n), \theta(n)$ (use isoperimetric constant/profile) to obtain

$$\mathrm{Diam}(G') \approx (\log n)^c |V'|^{1/2} \approx (\log n)^c n^{1/2}.$$

3. Uncontract the contracted edges, which introduces another factor of $\log n$ in the bound for $\mathrm{Diam}(\mathcal{T})$.

Proof Ideas for $\mathrm{Diam}(\mathcal{T}) = \Theta(n^{1/3})$ (Theorem 3)

Recall i.i.d. disorder $\omega_e \sim U(0,1)$ on the edges of K_n. For $p \in (0,1)$, $\{e \in E_n : \omega_e \leq p\}$ defines a percolation with parameter p. Let $\mathcal{C}_1(p)$ be the largest open cluster.

Key Lemma

There exists $C > 0$ such that with $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$ probability tending to 1,

$$\mathrm{Diam}(\mathcal{T}) \approx \mathrm{Diam}\big(\mathcal{T} \text{ on } \mathcal{C}_1(p_2)\big), \quad \text{where} \quad p_2 = p_1 + \frac{C \log n}{\beta_n},$$

and $p_1 = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ for some $\lambda > 0$.

1. Let $u, v \in \mathcal{C}_1(p_1)$ with $\omega_{\{u,v\}} > p_2$. Then

$$\mathbf{P}^{\omega}_{n,\beta_n}\big(\{u,v\} \in \mathcal{T}\big) = e^{-\beta_n \omega_{\{u,v\}}} R^{\omega}_{\mathrm{eff}}(u \leftrightarrow v)$$
$$\leq e^{-\beta_n p_2} \cdot n\, e^{\beta_n p_1} = n\, e^{-\beta_n (p_2 - p_1)}.$$

   If $p_2 - p_1 \geq C \frac{\log n}{\beta_n}$, this probability becomes polynomially small. This argument can be extended to show that

$$\mathcal{T}|_{V(\mathcal{C}_1(p_1))} \subseteq \mathcal{C}_1(p_2)$$

2. Vertices outside $\mathcal{C}_1(p_2)$ "hit $\mathcal{C}_1(p_2)$ fast" and do not add much to the diameter.

Proof Ideas for $\mathrm{Diam}(\mathcal{T}) = \Theta(n^{1/3})$ (Theorem 3)

Note that

$$p_1 = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$$

is chosen to be in the critical window of percolation for Erdös-Rényi random graph, and w.h.p.,

- $\mathcal{C}_1(p_1)$ is tree like (with a finite number of excess edges);
- $|\mathcal{C}_1(p_1)| = O(n^{2/3})$;
- $\mathrm{Diam}(\mathcal{C}_1(p_1)) \approx |\mathcal{C}_1(p)|^{1/2} = O(n^{1/3})$.

For $\beta_n \geq n^{4/3} \log n$,

$p_2 = p_1 + \dfrac{C \log n}{\beta_n}$ also in the critical window $\Longrightarrow$ $\mathrm{Diam}(\mathcal{T}) = O(n^{1/3})$.

Remark. If $\beta_n \ll n^{4/3} \log n$, then $\mathcal{C}_1(p_2)$ is no longer tree-like!

Sharp Transition for Local Observables

For RSTRE on the complete graph K_n, local observables undergo a relatively sharp transition around $\beta_n \approx n(\log n)^c$, in contrast to the conjecture on the diameter.

Let $\mathcal{T}$ be the RSTRE on the complete graph K_n, with law $\mathbf{P}^{\omega}_{n,\beta}$. Let $\mathbb{E}$ be expectation w.r.t. ω.

Theorem 4 [Makowiec'24, Kúsz'25]

If $\beta_n \ll n/\log n$, then

(i) $\mathbf{E}^{\omega}_{n,\beta_n}\left[\sum_{e\in\mathcal{T}} \omega_e\right] = (1+o(1))\frac{n}{\beta_n}$ w.h.p. w.r.t. $\mathbb{P}$;

(ii) $\sum_{e\in K_n} \mathbf{P}^{\omega}_{n,\beta_n}(e\in\mathcal{T})^2 = (1+o(1))\beta_n$ w.h.p. w.r.t. $\mathbb{P}$;

(iii) Under $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$, $\mathcal{T}$ has the same local limit as UST on K_n.

If $\beta_n \gg n(\log n)^c$ for some suitable c, then

(i) $\mathbf{E}^{\omega}_{n,\beta_n}\left[\sum_{e\in\mathcal{T}} \omega_e\right] = \zeta(3) + o(1)$ w.h.p. w.r.t. $\mathbb{P}$;

(ii) $\sum_{e\in K_n} \mathbf{P}^{\omega}_{n,\beta_n}(e\in\mathcal{T})^2 = (1+o(1))n$ w.h.p. w.r.t. $\mathbb{P}$;

(iii) Under $\mathbb{E}\mathbf{P}^{\omega}_{n,\beta}$, $\mathcal{T}$ has the same local limit as MST on K_n.

Open Questions

- For the RSTRE on K_n, find a sequence of $\beta_n \uparrow \infty$ s.t.

  $$\mathrm{Diam}(\mathcal{T}) = n^{\gamma + o(1)} \quad \text{for some} \quad \gamma \in (1/3, 1/2).$$

  How does the volume of the metric ball $B_{\mathcal{T}}(o, r)$ grow with r, and what is the scaling limit of $\mathcal{T}$ and its Hausdorff dimension?

- Study the point process generated by $H(\mathcal{T}) := \sum_{e \in \mathcal{T}} \omega_e$ on K_n, especially near the edge, and how the associated $\mathrm{Diam}(\mathcal{T})$ varies.

- Study RSTRE on other graphs, such as random d-regular graphs.

- Could a better understanding of RSTRE on $\mathbb{T}^d \subset \mathbb{Z}^d$ lead to a better understanding of MST on $\mathbb{T}^d$?


Thank you