An improvement on Łuczak's connected matchings method

Shoham Letzter

University College London

Oxford Combinatorics Seminar

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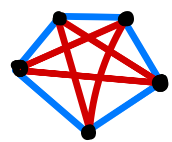
Write $\underline{G \to H}$ if in every red-blue edge-colouring of G there is a monochromatic (= red or blue) copy of H.

E.g.: $K_5 \not\to K_3$ , but $K_6 \to K_3$.

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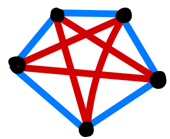
E.g.: $K_5 \not\to K_3$ , but $K_6 \to K_3$. So $r(K_3) = 6$.

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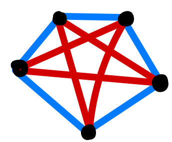
E.g.: $K_5 \not\to K_3$  , but $K_6 \to K_3$. So $r(K_3) = 6$.

A central question in Ramsey theory:

what is $r(K_n)$?

Gerencsér - Gyarfás '67: The Ramsey number of P_{n+1} is:

$$r(P_{n+1}) = \left\lfloor \frac{3n+1}{2} \right\rfloor.$$

Path of length n

An early Ramsey-type result

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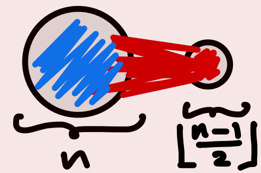

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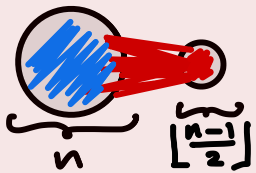

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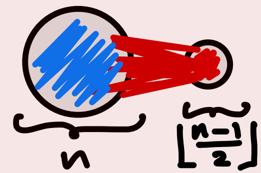

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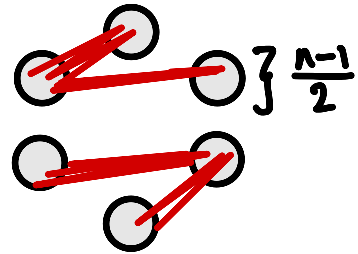

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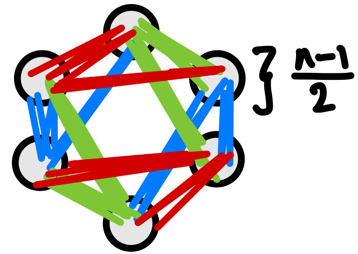

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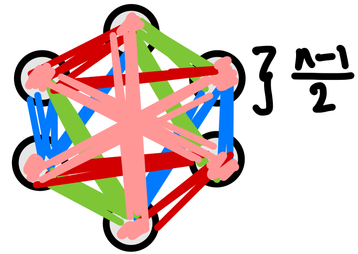

Yongqi – Yuansheng – Feng – Bingxi '06

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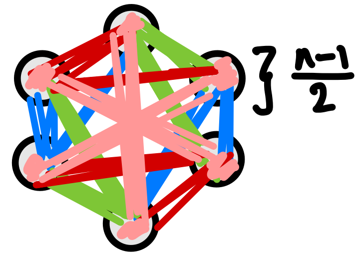

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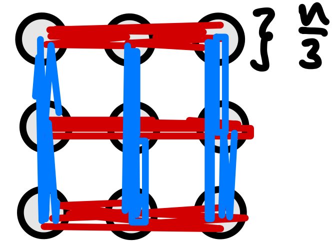

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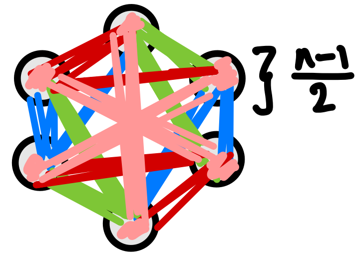

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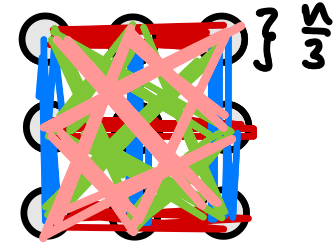

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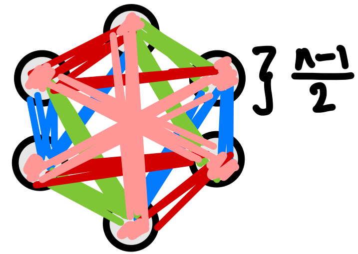

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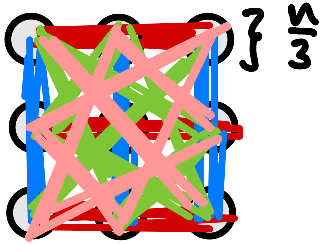

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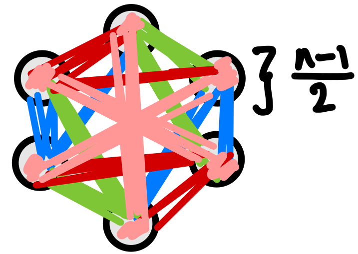

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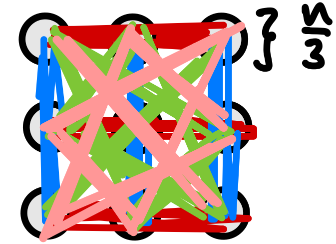

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Apply to majority colour.

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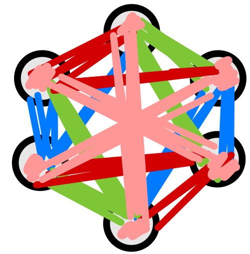

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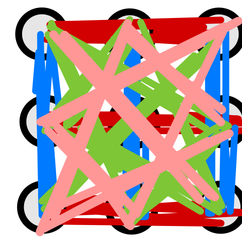

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It is believed that $r_s(P_n) \approx (s-1)n$.

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Gyárfás–Ruszinkó–Sárközy–Szemerédi '07:

$$r_3(P_{n+1}) = \begin{cases} 2n+1 & n \text{ even} \\ 2n & n \text{ odd} \end{cases}$$ for large n.

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Both papers used an idea of Łuczak '99, relating Ramsey numbers of paths/cycles to Ramsey-type problems about connected matchings.

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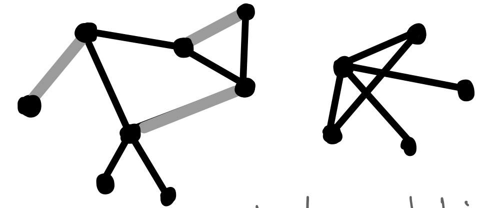

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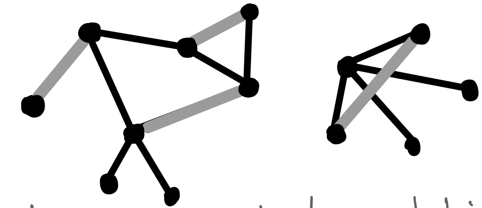

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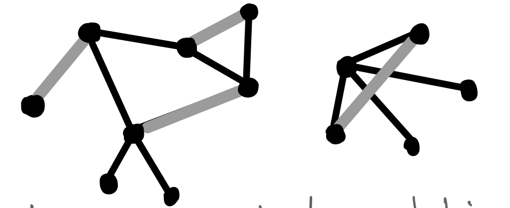
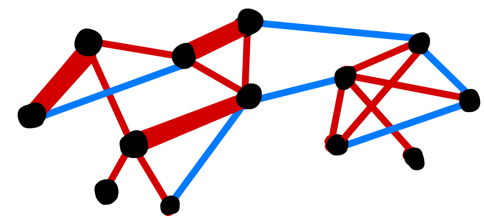


not a connected matching

A <u>mono connected matching</u> is a matching contained in a mono connected compt.



a red connected matching

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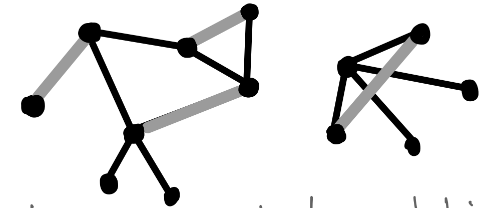

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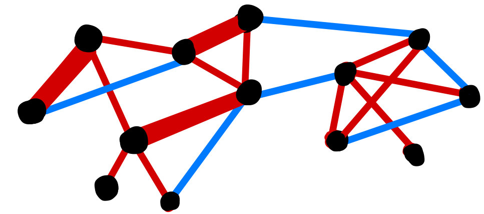

* The edges/vertices of a connected matching refer to the matching itself, not the component.

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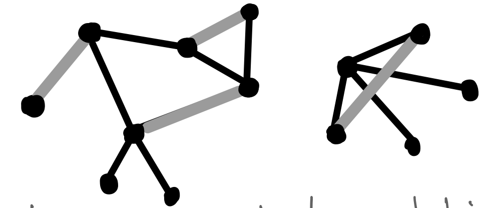

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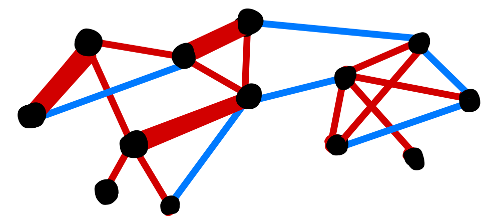

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Connected matchings

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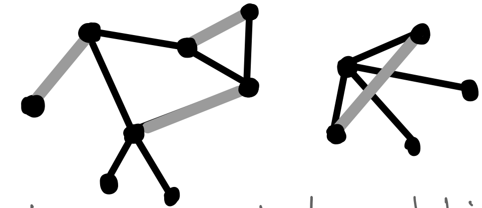

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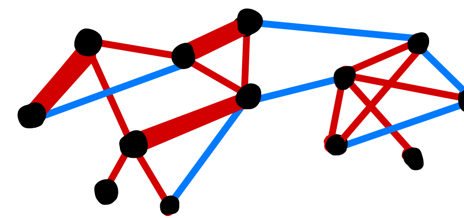

a red connected matching on 6 vertices

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CM(n) = family of connected matchings on $\geq n$ vs.

Łuczak's idea

P_{n+1} contains a $CM(n)$

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if $K_N \xrightarrow{s} P_{n+1}$ then $K_N \xrightarrow{s} CM(n)$.

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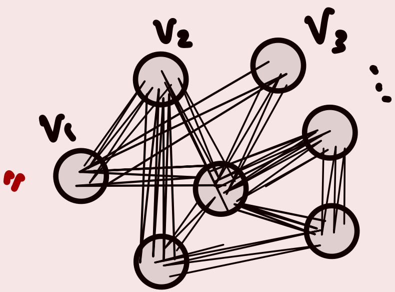
<u>Lemma</u> (Figaj–Łuczak '07).

<u>If</u> $\forall \varepsilon > 0$, large n: $\forall$ "almost-complete" G on $(\alpha+\varepsilon)n$ vs: $G \xrightarrow{s} CM(n)$,

<u>Then</u> $r_s(P_n) \leq (\alpha + o(1))n$.

<u>Regularity lemma</u> (Szemerédi '76):

For every large graph G, $V(G)$ has a partition into "not-too-few-or-too-many" parts of equal size $V_1, \_\_, V_k$ s.t. for almost every i,j $G[V_i, V_j]$ is "random-like".



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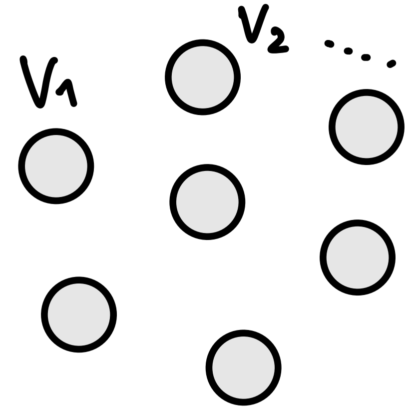

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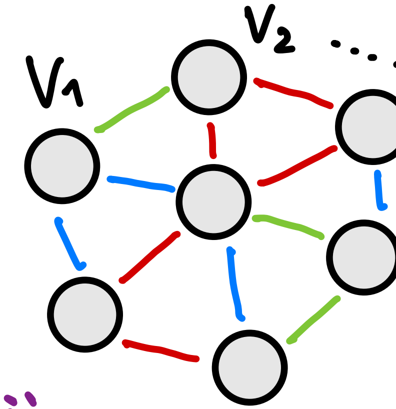

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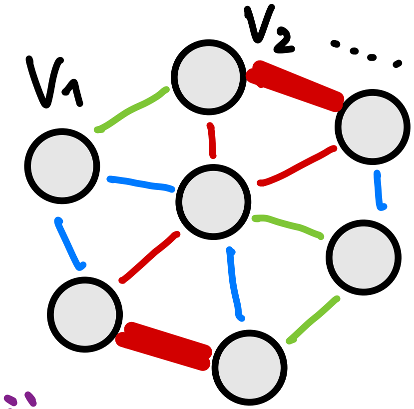
* Form auxiliary graph H: vs $[k]$, edges ij s.t. $G[V_i, V_j]$ is random-like in all colours, colour ij by majority colour. So H is almost complete.



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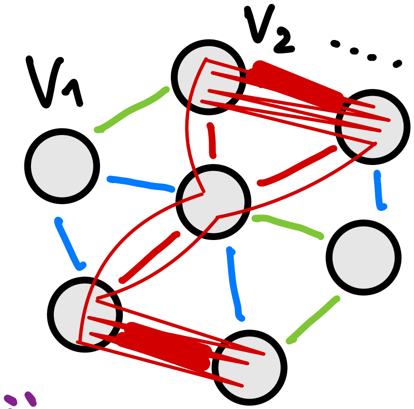

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* Form auxiliary graph H: vs $[k]$, edges ij s.t. $G[V_i, V_j]$ is random-like in all colours, colour ij by majority colour. So H is almost complete.
* Can lift a mono $CM(k')$ in H to a mono $P_{n'}$ in G with $\frac{n'}{n} \approx \frac{k'}{k}$. □

Figaj - Łuczak '07: $\forall \varepsilon > 0$, large n:
$\forall$ almost-complete G on $(\alpha+\varepsilon)n$ vs: $G \xrightarrow{s} CM(n)$ $\Longrightarrow$ $r_s(P_n) \leq (1+o(1))\alpha n$.

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Applications

* Knierim-Su '19: $r_s(P_n) \leq (s - \frac{1}{2})n + o(n)$

  (we saw: $(s-1)n \lesssim r_s(P_n) \lesssim sn$.)

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Drawback: need to consider almost-complete graphs.
Annoying! And cannot use induction ☹

New result

Thm (L. '20+)

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$K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n)$ $\forall \varepsilon > 0$, large n $\Longrightarrow$ $r_s(P_n) \le (\alpha + o(1))n$.

We also prove a similar result for

* asymmetric Ramsey numbers

(where different path lengths are required for different colours)

Thm (L. '20+)

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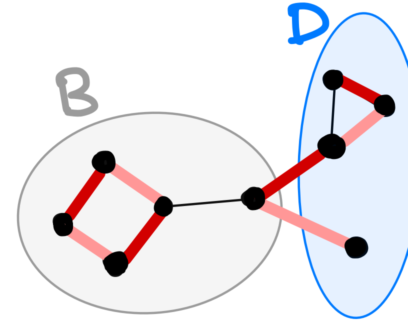
With Bucić & Sudakov '19 we proved a version of above for $K_{n,n}$.

Given graph G, let

$B = \{$vs in every maximum matching$\}$,

$D = V(G) \setminus B$



= examples of maximum matchings

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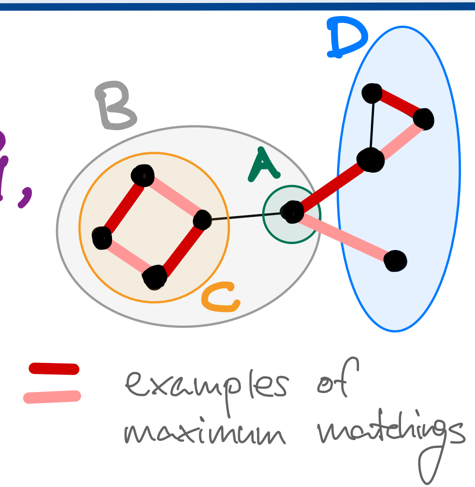

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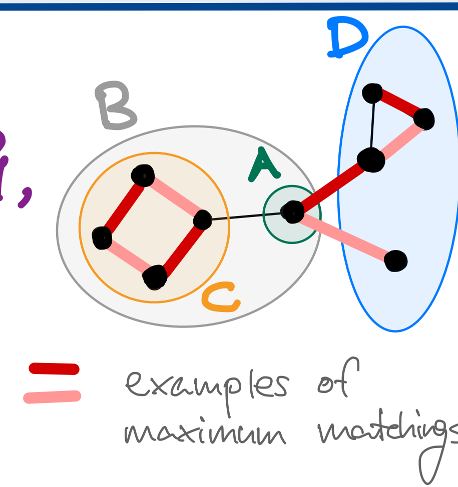

Thm (Edmonds-Gallai). For every maximum matching M in G:

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Gallai-Edmonds decomposition

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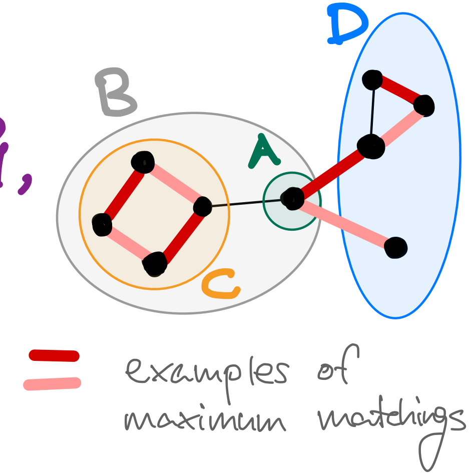

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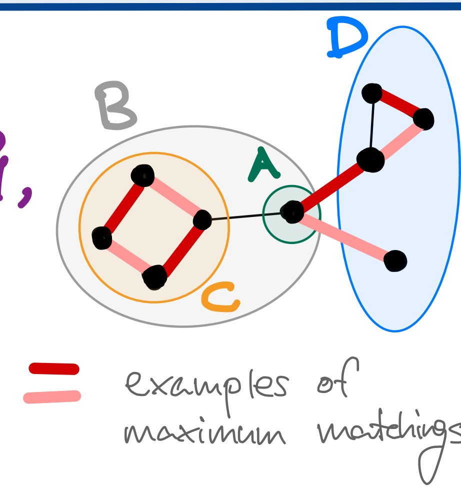

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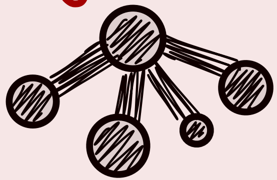

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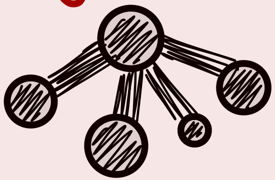

Proof sketch: Let A, C, D be as above.

A useful lemma

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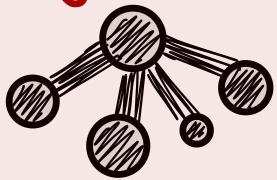

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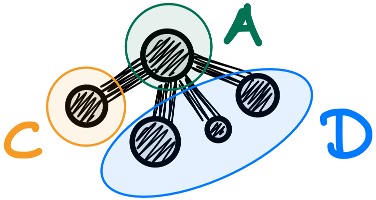

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Thm (L.) $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n)$ $\forall \varepsilon>0$, large n $\Longrightarrow$ $r_s(P_n) \leq (\alpha + o(1))n$.

Proving the theorem

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Figaj-Łuczak '07: $\forall \varepsilon>0$, large n:
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To prove the theorem, enough to show:

$K_N \xrightarrow{s} CM(n)$ $\Longrightarrow$ $\forall \varepsilon>0\ \exists \delta>0$: if $|G| = N + \varepsilon n$ and every vx in G is in $\leq \delta n$ non-edges then $G \xrightarrow{s} CM(n)$.

Suppose: $K_N \xrightarrow{s} CM(n)$.

Fix $\varepsilon > 0$, let $\delta \ll \varepsilon$.

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Let $G_1 \supseteq G$ be a maximal s-multicoloured graph on $V(G)$ with no mono $CM(n)$. (so edges can have several colours)

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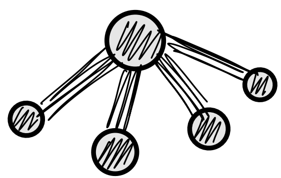

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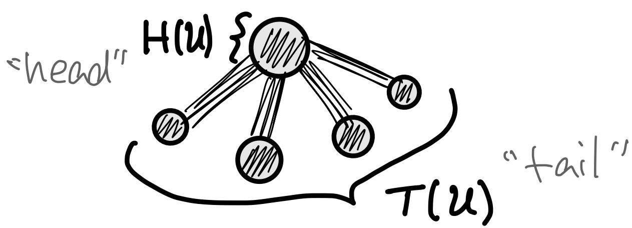

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Contradiction to $K_N \xrightarrow{s} CM(n)$! $\square$

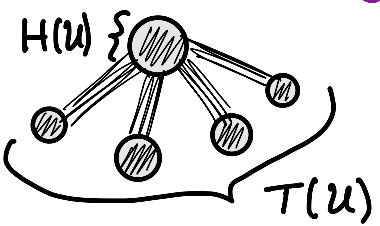
Given vx u, assign a type $\tau(u)$ to it:

$$\tau(u) = (\mathcal{U}_1, \ldots, \mathcal{U}_s, \alpha_1, \ldots, \alpha_s)$$

where:

$\mathcal{U}_i$ is the colour-i compt containing u

$$\alpha_i = \begin{cases} H & \text{if } u \in H(\mathcal{U}_i) \\ T & \phantom{\text{if }} u \in T(\mathcal{U}_i). \end{cases}$$



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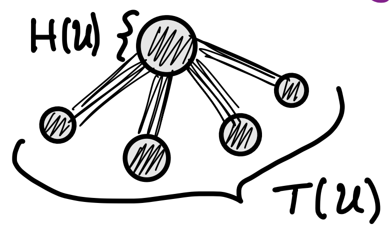

The number of types is $\le \underbrace{\left(\frac{N}{2n}+1\right)}_{\text{upper bound on \# i-coloured compts}}^{s} \cdot 2^s$.

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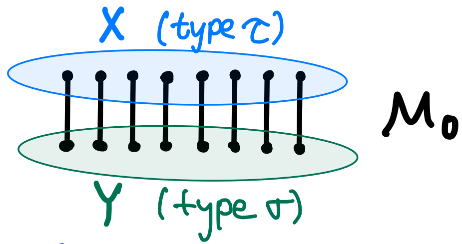

$\exists$ types τ, σ and $M_0 \subseteq M$:

* Edges in M_0 have ends of types τ & σ.
* $|M_0| > 4^s \delta n$. as # types is small and $\delta \ll \varepsilon$.

Proof of claim 1/3

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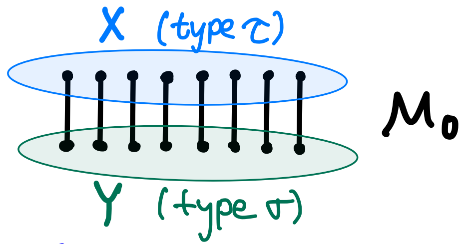

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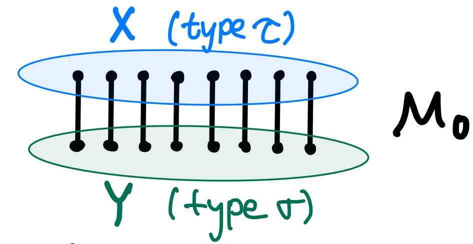

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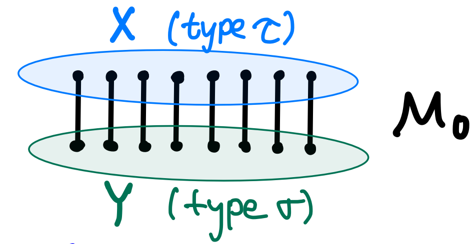

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$\tau = (U_1, \_\_, U_s, \alpha_1, \_\_, \alpha_s)$, $\sigma = (W_1, \_\_, W_s, \beta_1, \_\_, \beta_s)$.

Given M_{i-1}, want $M_i \subseteq M_{i-1}$: * $|M_i| \geq \frac{|M_{i-1}|}{4}$

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$\Rightarrow$ take $M_i = M_{i-1}$.

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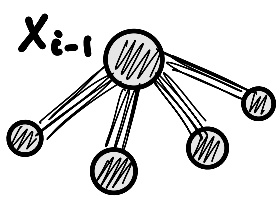


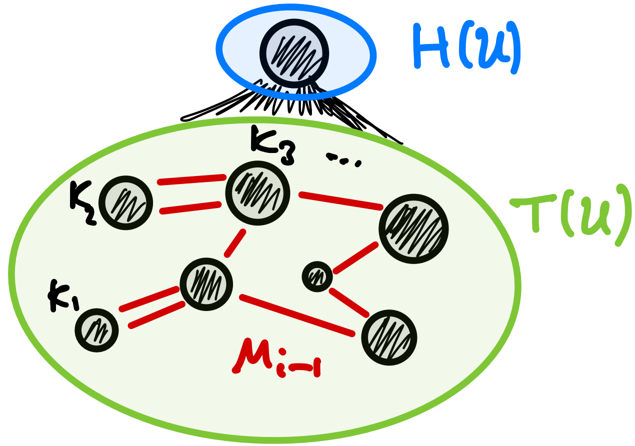
otherwise, if say $\alpha_i = H$, then X_{i-1} is joined to all of $U_i \supseteq Y_{i-1}$

$\Rightarrow$ no non-edges between X_{i-1} & Y_{i-1}

$\Rightarrow$ contradiction to $M_{i-1} \subseteq \overline{G_1}$.

2 X_{i-1}, Y_{i-1} in same i-colour compt $\mathcal{U} := \mathcal{U}_i = W_i$, $X_{i-1}, Y_{i-1} \subseteq T(\mathcal{U})$.

$K_1, \_\_, K_\ell$ i-colour cliques in $T(\mathcal{U})$.



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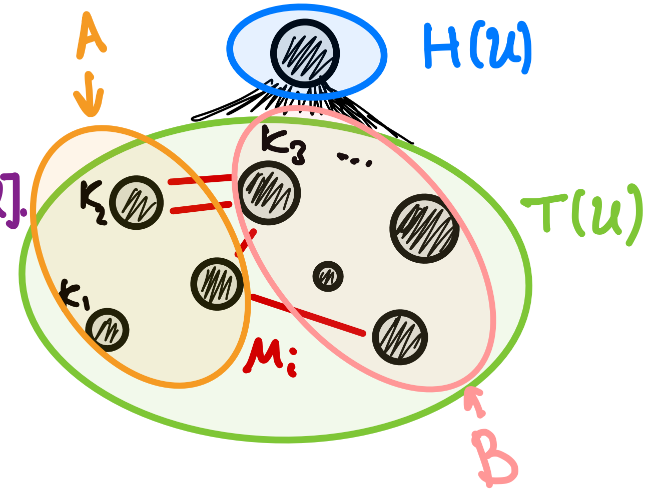

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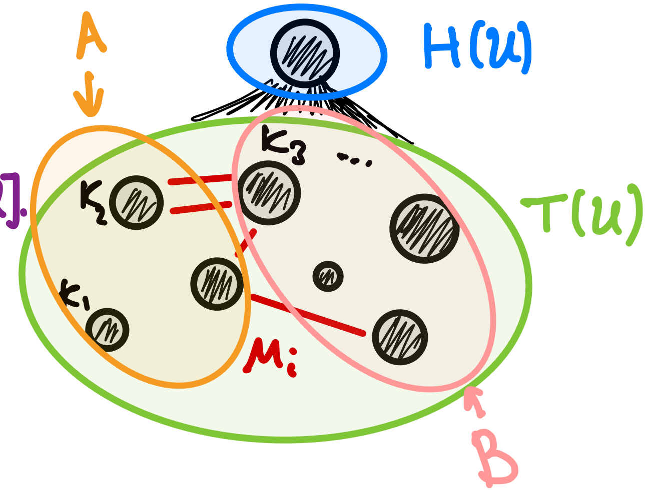

$\forall e \in M_{i-1}$: $\mathbb{P}(e \in M_i) = \frac{1}{4}$ $\implies$ $\mathbb{E}[|M_i|] = \frac{|M_{i-1}|}{4}$ $\implies$ appropriate M_i exists. $\square$

as e has ends in distinct K_a's

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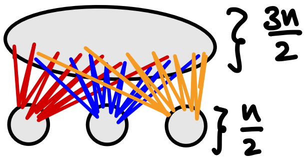

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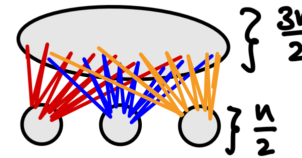

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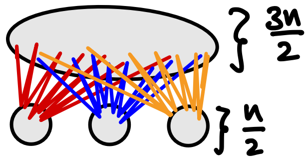

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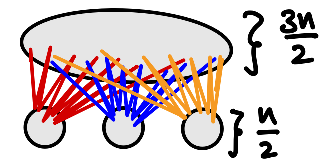

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Summary 2/2

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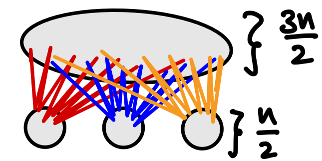

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Thank you for listening!