Power law bounds for critical long-range percolation.

T. Hutchcroft 10/11/2020

$J : \mathbb{Z}^d \longrightarrow [0,\infty)$ integrable $\sum J(x) < \infty$

symmetric $J(-x) = J(x)$

Long-range percolation: Random graph with vertex set $\mathbb{Z}^d$, include a possible edge $\{x,y\}$ w/prob

$$1 - e^{-\beta J(y-x)} \approx \beta J(y-x) \text{ when } \|y-x\| \text{ large.}$$

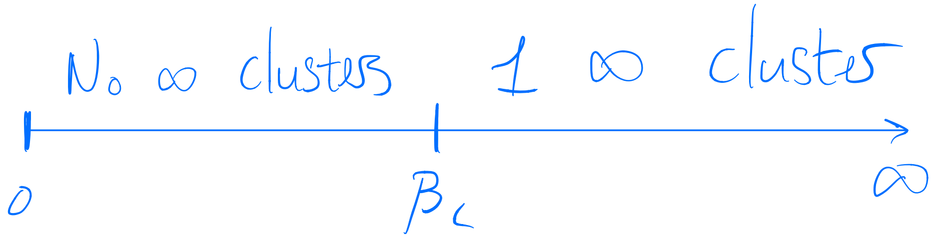
a parameter

Any pair $\{x,y\}$ could be an edge.
No special role for nearest neighbours

"clusters" = connected components.

Mostly interested in the case $J(x) \approx \|x\|^{-d-\alpha}$, $\alpha > 0$

needed for integrability!



$d = 1$: $\beta_c < \infty$ iff $\alpha \leq 1$
Newman Shulman '86

$d > 1$: $\beta_c < \infty$ $\forall$ $\alpha > 0$.

Surprisingly, long-range percolation is better understood than nearest-neighbour!

Thm Aizenman & Newman '86

New proof on the arXiv this morning!

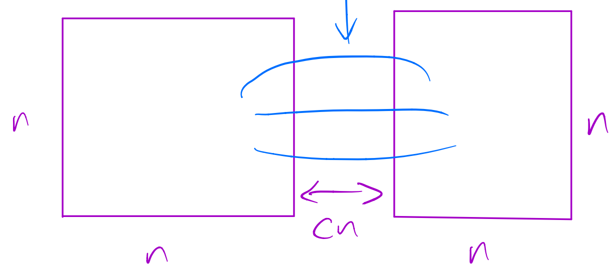
When $d = 1$ and $\alpha = 1$, the phase transition is discontinuous: there exists an ∞ cluster at β_c a.s.

Thm Berger '02

If $\alpha < d$ then the phase transition is continuous: there are no ∞ clusters at β_c a.s.

Why is $\alpha < d$ important?

number of edges between two boxes



$\approx (n^d)^2 n^{-d-\alpha} = n^{d-\alpha}$

large when $\alpha < d$
constant when $\alpha = d$
zero whp when $\alpha > d$.

Berger's proof works by showing that the set

$$\{\beta > 0 : \exists \text{ an } \infty \text{ cluster at } \beta\}$$

is open, and does not yield quantitative control of critical percolation.

constants

Today: cluster of origin

$J(x) \geq C \|x\|^{-d-\alpha}$ $\forall x$ with $\|x\| \geq r_0$

$$\boxed{\begin{array}{l}\text{Thm (H. 2020) If } \alpha < d \text{ then } \exists C \text{ st.} \\ \mathbb{P}_{\beta_c}(|K_0| \geq n) \leq C n^{-(d-\alpha)/(2d+\alpha)} \\ \frac{1}{r^d} \sum_{x \in [-r,r]^d} \mathbb{P}(0 \leftrightarrow x) \leq C r^{-2(d-\alpha)/3d}\end{array}}$$

- Based on a completely different method to Berger.
- Most interesting when $d < 6$, $\alpha > \frac{d}{3}$ where the model is not expected to be mean field, no previous power law upper bounds known
- Same proof gives similar results for groups other than $\mathbb{Z}^d$.

Critical Exponents

Conjecture: For each $d \geq 1$ and $\alpha > 0$ there exist critical exponents δ and η such that

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) \approx n^{-1/\delta}$$

$$\mathbb{P}_{\beta_c}(x \longleftrightarrow y) \approx \|x-y\|^{-d+2-\eta}$$

Intentionally vague!

$d > 6$ or $\alpha < d/3 \Rightarrow \delta = 2$, $\eta = 0$

"Mean field behaviour"

Hara & Slade 1990

Chen & Sakai 2015

Aizenman & Barsky 1987

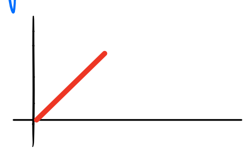
Always have $\delta \geq 2$, $2-\eta \leq d$

Our results: If δ, η are well-defined then

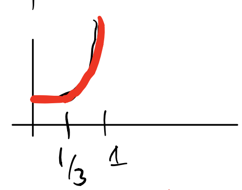
$$\boxed{\delta \leq \frac{2d+\alpha}{d-\alpha} \qquad 2-\eta \leq \frac{d}{3} + \frac{2\alpha}{3}}$$

There is a surprisingly simple prediction for the true value of these exponents:

$d=1$: $2-\eta = \alpha$



$$\delta = \frac{1+\alpha}{1-\alpha} \vee 2$$



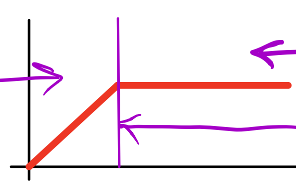
$d \geq 2$: $2-\eta = \alpha \wedge (2-\eta_{SR})$

nearest neighbour values of η, δ

$$\delta = \frac{d+\alpha}{d-\alpha} \vee 2 \wedge \delta_{SR}$$

See Chen & Sakai 2019

long-range dominant



short-range dominant

Interesting things can happen here (log corrections)

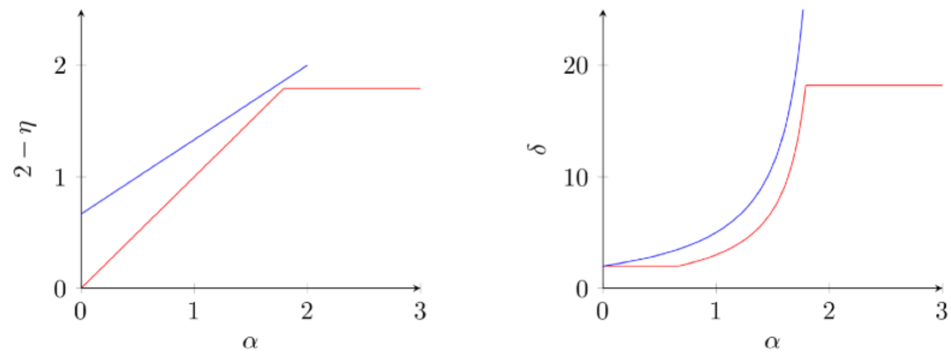


Figure 1: Our upper bounds (blue) vs. the conjectured true values (red) of $2-\eta$ and δ when $d=2$.

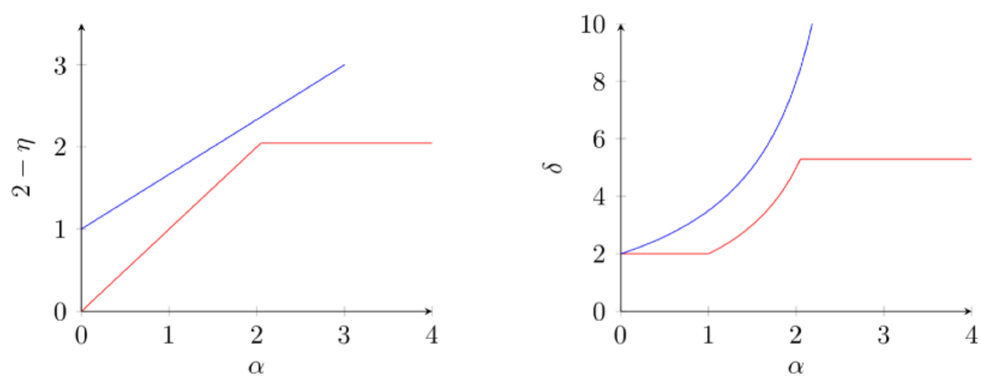


Figure 2: Our upper bounds (blue) vs. the conjectured true values (red) of $2-\eta$ and δ when $d=3$. Here we use the numerical values $\alpha^*(3) = 2 - \eta_{\mathrm{SR}}(3) \approx 2.0457$ and $\delta_{\mathrm{SR}}(3) \approx 5.2886$ obtained by applying the scaling and hyperscaling relations to the numerical estimates on the exponents ν and β/ν obtained by Wang et al. in [71]. When $\alpha = 2.0457 \approx \alpha^*(3)$ our upper bound on δ is about 8.43.

Our bounds are of reasonable order inside the conjectured "long-range dominant" regime.

Proof overview:

Two basic strategies for proving no percolation at p_c / β_c

"Supercritical strategy" Prove that if infinite clusters exist, ~~then~~ they must be 'large' in some way that guarantees they have $p_c < 1$.

This shows that $\{p : \infty \text{ clusters exist}\}$ is open, does not typically yield quantitative control of critical percolation.

E.g. Harris $\mathbb{Z}^2$ (1960), Benjamini, Lyons, Peres, Schramm nonamenable groups 1999

Half-spaces in $\mathbb{Z}^d$ Barsky, Grimmett, Newman 1991

Slabs $\mathbb{Z}^2 \times [0,n]^{d-2}$ Duminil-Copin, Sidoravicius, Tassion 2016

Benjamini's long range proof 2002

"Subcritical strategy": Try to prove that

$\{p : \infty \text{ clusters do not exist}\}$ is closed by proving that some non-trivial upper bound on the distribution of K_0 holds uniformly on $(0, p_c)$.

Often uses a bootstrapping argument

Prove that some non-trivial bound implies a strictly stronger version of itself

E.g., Hara-Slade lace expansion method roughly works by showing that in high-dimensions

$$\Big(\mathbb{P}_p(x \leftrightarrow y) \leq 3\, G(x,y) \quad \forall\, x,y \in \mathbb{Z}^d\Big)$$

$$(*) \Rightarrow \Big(\mathbb{P}_p(x \leftrightarrow y) \leq 2\, G(x,y) \quad \forall\, x,y \in \mathbb{Z}^d\Big)$$

Where $G(x,y) \asymp \|y-x\|^{-d+2}$ is the Greens function

If $(*)$ is established, a continuity argument yields that the strong form of the bound holds uniformly on $(0, p_c)$ and hence at p_c also.

Our proof is also based on a bootstrapping argument.

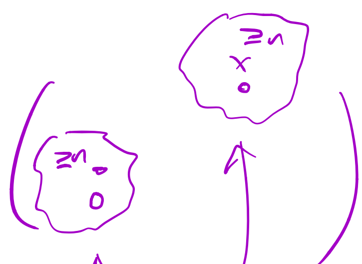
- Builds on ideas originally used to analyze percolation on some 'big' groups, some joint with Jonathan Hermon.

- One key ingredient is the <u>two-ghost</u> inequality

<u>Thm</u> Normalize so that $\sum J(x) = 1$.

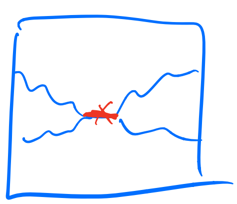
universal constant

$$\sum_x \underbrace{\sqrt{J(x)\left(e^{\beta J(x)}-1\right)}}_{\lesssim\ \sqrt{\beta}J(x)} \ \mathbb{P}_\beta\left(\cdots \right) \le \frac{C}{\sqrt{n}}$$



distinct each with at least n verts, at least 1 finite.

$$\mathbb{P}\left(\cdots \right) \le C \frac{\log n}{\sqrt{n}}$$



Uses ideas from Aizenman-Kesten-Newman '86.

We'll see stronger versions later.

Let's now prove that if $\alpha < d/4$ then

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) \leq C n^{-\frac{d-4\alpha}{4d}} \qquad = \theta$$

Suffices to prove that $\exists\, C$ s.t. if $\frac{\beta_c}{2} \leq \beta < \beta_c$ then $\forall A < \infty$

$$\left(\mathbb{P}_\beta(|K_0| \geq n) \leq A n^{-\theta} \ \forall n \geq 1\right)$$

$$(*) \quad \Rightarrow \left(\mathbb{P}_\beta(|K_0| \geq n) \leq C\sqrt{A+1}\, n^{-\theta} \ \forall n \geq 1\right)$$

Indeed, if we define $A(\beta)$ to be minimal st $\mathbb{P}_\beta(|K_0| \geq n) \leq A(\beta) n^{-\theta}$ $\forall n \geq 1$ then

Sharpness of the phase transition $\Rightarrow A(\beta) < \infty$ $\forall \beta < \beta_c$

$$(*) \Rightarrow A(\beta) \leq C\sqrt{A(\beta)+1}$$

$$A(\beta) \leq 4C^2 \qquad \forall \frac{\beta_c}{2} \leq \beta < \beta_c$$

Let's now prove $(*)$. Let $\Lambda_r = [-r,r]^d$.

Fix $\frac{\beta_c}{2} \le \beta < \beta_c$ and let A be such that $\mathbb{P}_\beta(|K_0| \ge n) \le A n^{-\theta}$.

Let $S_{x,n} = \{$ 

$\}$ then

two-ghost inequality $\Rightarrow$

$$\sum_{x \in \Lambda_r} \mathbb{P}(S_{x,n}) \le C r^{d+\alpha} \sum_{x \in \mathbb{Z}^d} \sqrt{J(x)(e^{\beta J(x)}-1)\, \mathbb{P}(S_{x,n})}$$

$$\le C r^{d+\alpha} / n^{1/2}.$$

But we also have

$$\mathbb{P}(S_{x,n}) \ge \mathbb{P}(|K_0|, |K_x| \ge n) - \mathbb{P}(0 \leftrightarrow x)$$

$$\ge \mathbb{P}(|K_0| \ge n)^2 - \mathbb{P}(0 \leftrightarrow x)$$

FKG.

And $\sum_{x \in \Lambda_r} \mathbb{P}_\beta(0 \leftrightarrow x) = \mathbb{E}_\beta |K_0 \cap \Lambda_r|$

$$\leq \mathbb{E}_\beta |K_0| \wedge (2r+1)^d$$

$$= \sum_{1}^{(2r+1)^d} \mathbb{P}_\beta(|K_0| \geq n)$$

$$\leq C A r^{d(1-\theta)}$$

Summing over x and rearranging gives

$$\mathbb{P}(|K_0| \geq n)^2 \leq \frac{1}{|\Lambda_r|} \sum_{x \in \Lambda_r} \mathbb{P}(0 \leftrightarrow x)$$

$$+ \frac{1}{|\Lambda_r|} \sum_{x \in \Lambda_r} \mathbb{P}(S_{x,n})$$

$$\leq \frac{CAr^{d(1-\theta)}}{r^d} + \frac{C r^{d+\alpha}}{r^d n^{1/2}}$$

Optimize over r by taking $r = n^{\frac{1-4\theta}{2\alpha}}$

$$\leq C(A+1) r^{-2\theta}$$

θ was chosen to make this step work. $\square$

Two ingredients to improve this proof:

- Find a better way to convert volume-tail bounds into two-point function bounds

$\mathbb{P}_\beta(x \leftrightarrow y)$

- Improved two ghost inequality.

Thm $\sum J(x) = 1$. Assume that
$\mathbb{P}(|K_0| \geq n) \leq A n^{-\theta}$ $\forall n \geq 1$, some $A < \infty$ and $0 \leq \theta < 1/2$.
Then

$$\sum_x (e^{\beta J(x)} - 1) \mathbb{P}(S_{x,n})^2 \leq \frac{CA^2}{(1-2\theta)^2 n^{1+2\theta}}$$

- Gets something out of the bootstrapping hypothesis
- Handles large x better.

Universal tightness of the maximum cluster size.

$G = (V, E, J)$ weighted graph

$J : E \to [0, \infty)$

percolation defined as before.

Given $\Lambda \subseteq V$ finite, define

$$|K_{max}(\Lambda)| = \max \{ |K_v \cap \Lambda| : v \in V \}.$$

$$M_\beta(\Lambda) = \min \{ n : \mathbb{P}_\beta(|K_{max}(\Lambda)| \geq n) \leq \tfrac{1}{e} \}$$

"Typical value", essentially the median.

makes eqns come out more nicely

We will prove that $|K_{max}(\Lambda)|$ is always of order $M_\beta(\Lambda)$, with universal upper and lower tail bounds.

Theorem 2.2 (Universal tightness of the maximum cluster size)**.** *Let $G = (V, E, J)$ be a countable weighted graph and let $\Lambda \subseteq V$ be finite and non-empty. Then the inequalities*

$$\mathbf{P}_\beta\Big(|K_{\max}(\Lambda)| \geq \alpha M_\beta(\Lambda)\Big) \leq \exp\left(-\frac{1}{9}\alpha\right) \tag{2.5}$$

$$\text{and} \qquad \mathbf{P}_\beta\Big(|K_{\max}(\Lambda)| < \varepsilon M_\beta(\Lambda)\Big) \leq 27\varepsilon \tag{2.6}$$

hold for every $\beta \geq 0$, $\alpha \geq 1$, and $0 < \varepsilon \leq 1$. Moreover, the inequality

$$\mathbf{P}_\beta\Big(|K_u \cap \Lambda| \geq \alpha M_\beta(\Lambda)\Big) \leq e\mathbf{P}_\beta\Big(|K_u \cap \Lambda| \geq M_\beta(\Lambda)\Big) \exp\left(-\frac{1}{9}\alpha\right) \tag{2.7}$$

holds for every $\beta \geq 0$, $\alpha \geq 1$, and $u \in V$.

We will deduce this theorem as a corollary of the following more general inequality.

Theorem 2.3. *Let $G = (V, E, J)$ be a countable weighted graph and let $\Lambda \subseteq V$ be finite and non-empty. Then the inequalities*

$$\mathbf{P}_\beta\big(|K_{\max}(\Lambda)| \geq 3^k\lambda\big) \leq \mathbf{P}_\beta\big(|K_{\max}(\Lambda)| \geq \lambda\big)^{3^{k-1}+1} \tag{2.8}$$

$$\text{and} \qquad \mathbf{P}_\beta\big(|K_u \cap \Lambda| \geq 3^k\lambda\big) \leq \mathbf{P}_\beta\big(|K_{\max}(\Lambda)| \geq \lambda\big)^{3^{k-1}}\mathbf{P}_\beta\big(|K_u \cap \Lambda| \geq \lambda\big) \tag{2.9}$$

hold for every $\beta \geq 0$, $\lambda \geq 1$, $k \geq 0$, and $u \in V$.

This has the following consequence:

If G is such that $\mathbb{P}(|K_u| \geq n) \leq An^{-\theta}$ $\forall n \geq 1$

then in fact

$$\mathbb{P}(|K_u \cap \Lambda| \geq n) \leq C_\theta A n^{-\theta} e^{-\frac{n}{18M_\beta(\Lambda)}}$$

$\forall n \geq 1.$

(Proof is just calculus)

constant $= e(18)^\theta$

This inequality is extremely useful!

$$\boxed{\begin{array}{l}\textbf{Corollary} \text{ If } A<\infty \text{ and } 0\le\theta<1 \text{ are such that } \mathbb{P}_\beta(|K_u|\ge n)\le A n^{-\theta} \\ \forall n\ge 1 \text{ and } u\in V \text{ then} \\ \qquad M_\beta(\Lambda)\le C_\theta A^{1/1+\theta}|\Lambda|^{1/1+\theta} \\ \text{and} \\ \frac{1}{|\Lambda|}\sum_{v\in\Lambda}\mathbb{P}_\beta(u\leftrightarrow v)\le C_\theta A^{\frac{2}{1+\theta}}|\Lambda|^{-\frac{2\theta}{1+\theta}} \\ \forall \Lambda\subseteq V \text{ finite and } u\in V.\end{array}}$$

- Can be used to give slick new proofs of some classical things. E.g.,

max cluster size of critical Erdős-Rényi $\le n^{2/3}$ whp.

Bollobás & Łuczak 1980s

Why? Stochastic domination by branching process gives that $\mathbb{P}(|K|\ge n)\le A n^{-1/2}$ $\forall n\ge 1$.

Similarly lets us get upper bound for high-dim tori ~~at~~ p_c given $\mathbb{Z}^d$ results of Hara & Slade.

- Yields the "hyperscaling inequality"

$$2-\eta \leq d\,\frac{\delta-1}{\delta+1}$$

$$\mathbb{P}_{\beta_c}(x \leftrightarrow y) \approx \|y-x\|^{-d+2-\eta}$$

$$\mathbb{P}_{\beta_c}(|K_0| \geq n) \approx n^{-1/\delta}$$

Believed to be an equality in low dimensions.

Deduction of the corollary from the theorem:

Write $M = M_3(\Lambda)$. As before,

$$\sum_{v \in \Lambda} \mathbb{P}(u \leftrightarrow v) = \mathbb{E}|K_u \cap \Lambda| = \sum_{n \geq 1} \mathbb{P}(|K_u \cap \Lambda| \geq n)$$

$$\leq C_\theta A \sum n^{-\theta} e^{-\frac{n}{18M}}$$

$$\leq C_\theta A M^{1-\theta} \qquad \text{calculus}$$

On the other hand,

$$\sum_{u,v \in \Lambda} \mathbb{P}(u \leftrightarrow v) = \mathbb{E}\Big[\sum_{u,v \in \Lambda} \mathbb{1}(u \leftrightarrow v)\Big] \geq \mathbb{E}\big[|K_{max}(\Lambda)|^2\big]$$

$$\geq c M^2$$

Comparing these two inequalities and rearranging yields the claim! $\square$

Let's now prove the universal tightness theorem.

Key combinatorial lemma:

Let $G = (V,E)$ be a connected, locally finite graph and let $A \subseteq V$ be finite, $|A| \geq 3$.

Then $\exists\, E_1, E_2 \subseteq E$ disjoint such that

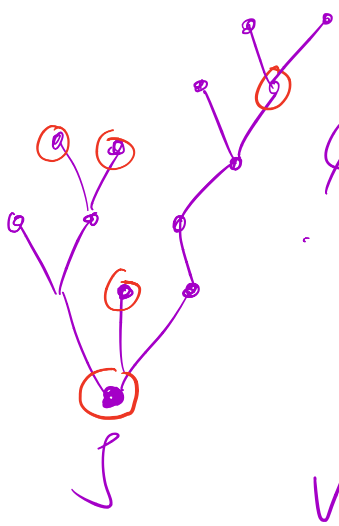
- E_1, E_2 both span connected subgraphs of G
- The sets $V(E_1)$ and $V(E_2)$ of vertices incident to the two edge sets satisfy

$$\frac{1}{3}|A| \leq |V(E_i) \cap A| \leq \frac{2}{3}|A| \quad i=1,2.$$

Proof. Suffices to consider the case G is a tree, taking a spanning tree otherwise.

We will take E_1, E_2 to be a partition of the edge set.

Root T at an element of A.



Recursively construct

$$\emptyset = E^0 \subseteq E^1 \subseteq \dots \subseteq E^N$$

so that E^i and $E \setminus E^i$ span connected subgraphs $1 \leq i \leq N$

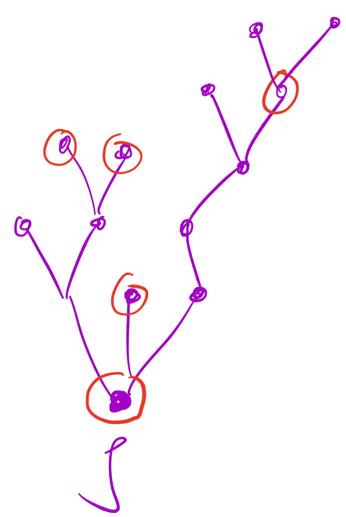
$V^0 = \rho$, V^i = unique vertex incident to E^i and $E \setminus E^i$

- If V^i has exactly one child not incident to E_i
$E^{i+1} = E^i \cup \{$edge connecting V^i to its child$\}$

- Otherwise v^i has at least 2 children not incident to E_i.

Pick the child that has the fewest elements of A descended from it.

$E^{i+1} = E^i \cup \{$ all edges in subtree with this child $\}$



Stop when every vertex of A touches E^i

$V_i = V(E^i)$, $i \geq 1$, $V_0 = \{\rho\}$

$\{0 \leq n \leq N : |V_n| > \frac{|A|}{3}\}$ contains N but not 0

$$m = \min\left\{0 \le n \le N : |V_n| > \frac{|A|}{3}\right\}$$

$$\frac{|A|}{3} < |V_m \cap A| \le |V_{m-1} \cap A| + \max\left\{1, \tfrac{1}{2}|A \setminus V_m|\right\}$$
$$\le \frac{2}{3}|A|$$

as required $\square$

Applying this lemma iteratively we get:

<u>Lemma</u> If $|A| \ge 3^k$ then there exists $m \ge 3^{k-1}+1$ and a collection of disjoint sets $E_1, \dots, E_m$ such that

- E_i spans a connected subgraph of G $\forall\ 1 \le i \le m$,
- $$\frac{|A|}{3^k} \le |A \cap V(E_i)| \le \frac{|A|}{3^{k-1}}$$ $\forall\ 1 \le i \le m$

Given this lemma, the universal tightness theorem follows from the BK inequality!

$$\{|K_{max}(\Delta)| \geq 3^k \lambda\}$$
$$\subseteq \underbrace{\{|K_{max}(\Delta)| \geq \lambda\} \circ \cdots \circ \{|K_{max}(\Delta)| \geq \lambda\}}_{3^{k-1}+1 \text{ times}}$$

disjoint occurrence

$$\{|K_u \cap \Delta)| \geq 3^k \lambda\}$$
$$\subseteq \{|K_u \cap \Delta| \geq \lambda\}$$
$$\circ \underbrace{\{|K_{max}(\Delta)| \geq \lambda\} \circ \cdots \circ \{|K_{max}(\Delta)| \geq \lambda\}}_{3^{k-1} \text{ times}}$$

BK inequality: $\mathbb{P}(A_1 \circ \cdots \circ A_n) \leq \prod \mathbb{P}(A_i)$

A_i increasing.

The Aizenman-Kesten-Newman method.

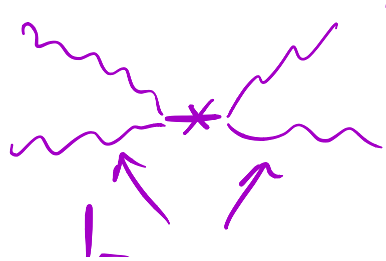
(If we have extra time)

ω_p : Bernoulli p bond percolation

Theorem Aizenman, Kesten, Newman (1987)

On $\mathbb{Z}^d$, ω_p has at most one ∞ cluster a.s.

Moreover

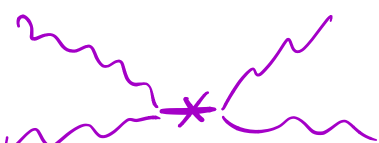
$\mathbb{P}_p\Big($  $\Big) \le C_{p,d}\frac{\log n}{\sqrt{n}}$

diam $\geq n$

Simplified presentation by Gandolfi, Grimmet & Russo

Only 4 pages!

Cerf 2013:

$$\mathbb{P}_{p_c}\Big($$  $$\Big) \leq n^{-\frac{1}{2}-\varepsilon_d}$$

diam $\geq n$

$$\varepsilon_3 = 1/46$$

H. 2018: New version that works for all Cayley graphs, works directly with <u>volumes</u>

$S_{e,n} = \{ e$ closed, endpoints

are in distinct clusters both of which touch at least n edges, at least one finite$\}$

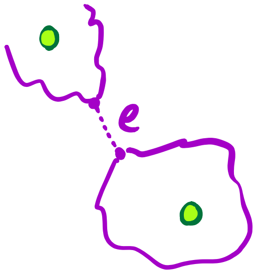
Thm Let G be a transitive unimodular graph of degree d. Then

$$\mathbb{P}_p(S_{e,n}) \leq 42\, d \left[\frac{1-p}{pn}\right]^{1/2}$$

$\forall e \in E,\ n \geq 1,\ p \in [0,1].$

$\mathcal{G}_h \subseteq \{0,1\}^E$ ghost field independent of ω_p which contains each edge w/prob $1-e^{-h}$.

$T_e = \{$ e closed, endpoints of e in distinct clusters



(each of which touches a green edge and at least one of which is finite.

η uniform random edge incident to 0

<u>Thm'</u> $\mathbb{P}_{p,h}(T_\eta) \leq 21 \left[\frac{1-p}{p} h\right]^{1/2}$

Sketch of proof:

F_e: both clusters touching e are finite

$$\mathbb{P}(F_e \cap T_e) = \mathbb{E}\, \#\{\text{finite clusters touching } e \text{ \& the ghost}\}\, \mathbb{1}(e \text{ closed})$$

$- \mathbb{1}(\exists$ finite cluster touching e & the ghost $)$

$\mathbb{1}(e \text{ closed})$.

$\mathbb{P}\big((\exists$ finite cluster touching e & the ghost $) \cap (e \text{ closed}) \big)$

$= \mathbb{P}(T_e \setminus F_e)$

$+ \mathbb{P}\big(\underbrace{F_e \cap \{\text{at least one cluster touching } e \text{ touches ghost}\}} \cap \{e \text{ closed}\} \big)$

independent!

Algebra $\rightsquigarrow$

$\mathbb{P}(T_e) =$

$$\mathbb{E}\left[\left(\mathbb{1}(e \text{ closed}) - \frac{1-p}{p} \mathbb{1}(e \text{ open}) \right) \cdot \#\left\{ \begin{array}{c} \text{finite clusters touching} \\ e \text{ \& ghost} \end{array} \right\} \right]$$

Mass-transport $\rightsquigarrow$

$$\mathbb{P}(T_\eta) = \mathbb{E}\left[\sum \frac{h_p(K)}{|K|} \right]$$

K a finite cluster touching η and ghost.

$\sum \mathbb{1}(e \text{ closed}) - \frac{1-p}{p} \mathbb{1}(e \text{ open})$

$$\mathbb{P}(T_\eta) \leq 2\, \mathbb{E} \frac{|h_p(K_0)|}{|K_0|} \left(1 - e^{-h|K_0|}\right)$$

edge volume.

$$= 2\,\mathbb{E}\left|\frac{Z_T}{T}\right|\left(1-e^{-hT}\right)\mathbb{1}(T<\infty)$$

for some martingale Z and stopping time T.

Finish w/ martingale analysis. $\square$