Notes on modules of finite upper rank

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0.1 Quasi-f.g. modules

A G-module M is *quasi-f.g.* if there exists a finitely generated group that is an extension of M by G. The first lemma is useful for finding a 'good' f.g. submodule inside a quasi-f.g. module, in some circumstances.

Lemma 1 *Suppose that*

$$E = \langle \Gamma \rangle \rhd N = \left\langle X^E \right\rangle \geq Z = Z^E \geq A = A^E \geq \gamma_{c+1}(N)[Z, N] \geq A' = 1.$$

Assume that $X = X^{-1},\ \Gamma = \Gamma^{-1}$ *and set* $Y = Z \cap X$*. Let*

$$B = \langle [x, y]^w,\ [x^\gamma, y]^w \mid x \in X,\ y \in Y,\ \gamma \in \Gamma,\ w \in E \rangle .$$

Write $^- : E \to E/A$*,* $R = \mathbb{Z}\overline{Z}$*. Then the* R*-module* A/B *is* Λ*-torsion if* Λ *is a multiplicatively closed subset of* R *satisfying*

$$(\Lambda^g + 1) \cap \overline{Y} \neq \varnothing \tag{1}$$

for each $g \in E$*.*

Proof. Note that $\gamma_{i+1}(N)$ is generated by the elements $v_i(\mathbf{x}, \mathbf{w})^g$ for $g \in E$ and

$$v_i(\mathbf{x}, \mathbf{w}) = [x_0, x_1^{w_1}, \ldots, x_i^{w_i}],$$

$x_j \in X,\ w_j \in E$. Put

$$A_i = \langle [v_i(\mathbf{x}, \mathbf{w}), z]^g \mid x_j \in X,\ z \in Y,\ g, w_j \in E \rangle ,$$
$$B_i = \langle [x^v, y]^g \mid x \in X,\ y \in Y,\ g, v \in E,\ l(v) \leq i \rangle$$

where $l(v)$ denotes the least n such that $v = \gamma_1 \ldots \gamma_n$ $(\gamma_j \in \Gamma)$. Note that $B_1 = B$.

Claim: A/A_c is Λ-torsion.

To see this, choose $y \in Y$ with $\overline{y} - 1 \in \Lambda$. Then (mixing additive and multiplicative notation)

$$A(\overline{y} - 1)^c = [A,_c y] \subseteq \gamma_{c+1}(N) \subseteq A.$$

Given a generator $v_c(\mathbf{x},\mathbf{w})^g$ of $\gamma_{c+1}(N)$, choose $z \in Y$ such that $\overline{z}^g - 1 \in \Lambda$. Then

$$v_c(\mathbf{x},\mathbf{w})^g(\overline{z}^g - 1) = [v_v(\mathbf{x},\mathbf{w}), z]^g \in A_c.$$

Claim: For $i > 1$, B_i/B_{i-1} is Λ-torsion.

To see this, say $b = [x^{\gamma u}, y]^g$ is a generator of B_i where $l(u) \le i-1$. Choose $z \in Y$ such that $\overline{z}^{ug} - 1 \in \Lambda$. Then

$$\begin{aligned}(b(\overline{z}^{ug} - 1))^{-g^{-1}y} &= [x^{\gamma u}, y, z^u]^{-y^{-1}} \\ &= [z^u, x^{-\gamma u}, y^{-1}]^{x^{\gamma u}} + [y^{-1}, z^{-u}, x^{\gamma u}]^{z^u}.\end{aligned}$$

The first term lies in $B_1 \le B_{i-1}$ and the second term lies in B_{i-1}. The claim follows since B_{i-1} is E-invariant.

Claim: Write $B_\infty = \bigcup_j B_j$. Then for $i > 1$, $A_i \subseteq B_\infty + A_{i-1}$.

To see this, let $x = (\mathbf{x}', x)$ and $\mathbf{w} = (1, w_1, \ldots) = (\mathbf{w}', w)$ be $(i+1)$-tuples in X, E respectively, and let $z \in Y$. Then

$$\begin{aligned}[v_i(\mathbf{x},\mathbf{w}), z]^{-x^{-w}} &= [v_{i-1}(\mathbf{x}',\mathbf{w}'), x^w, z]^{-x^{-w}} \\ &= [x^w, z^{-1}, v_{i-1}(\mathbf{x}',\mathbf{w}')]^z + [z, v_{i-1}(\mathbf{x}',\mathbf{w}')^{-1}, x^w]^{v_{i-1}(\mathbf{x}',\mathbf{w}')}.\end{aligned}$$

The first term lies in B_∞ and the second term lies in A_{i-1}. The claim follows since each of these modules is E-invariant.

The three claims together now imply that A/B is Λ-torsion. ∎

The point of the lemma is that if both Γ and X are finite, then B is a finitely generated $G := E/A$-module. By judicious choice of the set Λ one ensures that most of the finite quotients of A are quotients of B, and vice versa.

Definition $\mathcal{K}(G)$ denotes the set of all nilpotent normal subgroups K of G such that G/K is virtually abelian.

This is a non-empty set if G is a minimax group.

Remark Suppose that E is f.g., $A \lhd E$ is abelian, and E/A is minimax. Let $N/A \in \mathcal{K}(E/A)$. Then E/N is f.g. and virtually abelian. Hence $N = \langle X^E \rangle$ where X is finite, and the E-module B in Lemma 1 is finitely generated. Also Z/A is an extension of a f.g. subgroup by a divisible group, so we can choose X so that $\overline{Z}/\langle\overline{Y}\rangle$ is divisible, where $Y = Z \cap X$.

Lemma 2 *Same notation as above. Assume that $\overline{Z}/\langle\overline{Y}\rangle$ is divisible. If L is a maximal ideal of finite index in $\mathbb{Z}\overline{Z}$ not containing $\overline{Z} - 1$ then $\Lambda = \mathbb{Z}\overline{Z} \setminus L$ satisfies (1) for every $g \in E$.*

Proof. Write $K = \overline{Z} \cap (L+1)$. If 1 fails for g then $K^g \supseteq \overline{Y}$ which implies $K^g = \overline{Z}$ since $|\overline{Z} : K|$ is finite. Hence $K = Z$ and so $L \supseteq \overline{Z} - 1$. ∎

Lemma 3 *Let G be a finitely generated minimax group and M a quasi-f.g. G-module. Suppose $V \leq_{\mathbb{Z}G} M$ and M/V has finite rank. Then there exist $K_0 \in \mathcal{K}(G)$, and a f.g. G-submodule B of V, such that if $K_0 \geq K \in \mathcal{K}(G)$, then (i) $M(K-1)^d \leq V$ for some finite d and (ii) V/B is $\Lambda = (\mathbb{Z}Q \smallsetminus L)$-torsion whenever $L \ngeq Q-1$ is a maximal ideal of finite index in $\mathbb{Z}Q$ and $Q = Q^G \leq \mathrm{Z}(K)$.*

Proof. Suppose to begin with that M is f.g., and set $E = M \rtimes G$. Then E/V is f.g. of finite rank, hence minimax. Hence there exists $K_0 \in \mathcal{K}(G)$ such that $M(K_0-1)^d \leq V$ for some finite d.

Now let $K_0 \geq K \in \mathcal{K}(G)$. Taking $A = V$ in the *Remark* we can take $N = MK$. Let $Q = Q^G \leq \mathrm{Z}(K)$.

Put $W = V + M(K-1)$. Now apply Lemma 1 with $A = W$ and $Z = MQ$ to find a f.g. G-submodule B_1 of W such that W/B_1 is Λ-torsion for Λ as described (a maximal ideal L of $\mathbb{Z}Q$ corresponds to the maximal ideal $(\overline{M}-1)\mathbb{Z}\overline{M}Q + L\mathbb{Z}\overline{M}Q$ of $\mathbb{Z}\overline{Z} = \mathbb{Z}\overline{MQ}$).

If $d = 1$ we have $W = V$ and we are done. If $d > 1$, set $M_1 = B_1$ and $V_1 = V \cap M_1$. Then $M_1(K-1)^{d-1} \leq V_1$ and $M_1/V_1 \cong (M_1+V)/V$ has finite rank. Arguing by induction on d we may suppose that V_1 contains a f.g. G-submodule B such that V_1/B is Λ-torsion for Λ as described. As $V/V_1 \cong (V+M_1)/M_1 \leq W/M_1$ is also Λ-torsion we have V/B Λ-torsion as required.

In the general case, we have a f.g. group E with $M \lhd E$ and $E/N = G$. As above, E/V is f.g. of finite rank, hence minimax. Let $N_0/V \in \mathcal{K}(E/V)$, with $N \geq M$, and put $K_0 = N_0/M$. Then $K_0 \in \mathcal{K}(G)$ and $M(K_0-1)^d \leq V$ for some finite d. Now let $K_0 \geq K = N/M \in \mathcal{K}(G)$ and suppose that $Q = Z/M = Q^G \leq \mathrm{Z}(K)$. Then $[Z,N] \leq M$.

Put $A = V + [Z,N]$, and apply Lemma 1 to find a f.g. G-submodule M_0 of A such that A/M_0 is Λ-torsion for Λ as described. Put $V_0 = M_0 \cap V$. Now we apply the special case above, replacing M by M_0; note that $M_0(K_0-1)^d \leq V_0$. This shows that V_0 contains a f.g. G-submodule B such that V_0/B is Λ-torsion for each set Λ as described. Finally, $V/V_0 \cong (V+M_0)/M_0 \leq A/M_0$ is also Λ-torsion, so V/B is Λ-torsion as required. ■

Lemma 4 *Same notation as in Lemma 1. Assume that E/N is polycyclic and $\overline{E}$ is reduced minimax. Then $\mathrm{ur}_p(A_E) = \mathrm{ur}_p(A_N)$ for each prime p.*

Proof. Obviously $\mathrm{ur}_p(A_N) \geq \mathrm{ur}_p(A_E)$. Suppose the inequality is strict (so $\mathrm{ur}_p(A_E) = r$ is finite). Then there exists $B = B^N < A$ such that $B \geq A^p$ and $r < \mathrm{rk}(A/B) < \infty$. There exists $m \in \mathbb{N}$ such that $N_1 := AN^m$ satisfies

$$K := A^p[A, N_1] \leq B,$$
$$N_1/A \text{ is torsion-free.}$$

Since $K \lhd E$ and $\mathrm{rk}(A/K) \geq \mathrm{rk}(A/B) > r$ it follows that A/K is infinite.

Now N_1/K is nilpotent of class at most $c+1$, so $N_2 := AN_1^{p^{c+2}}$ satisfies $N_2 \cap A = K$. It follows that $V := N_2/N_2'N_2^p$ is infinite. But $E^* := E/N_2'N_2^p$ is finitely generated and abelian-by-polycyclic, so residually finite. As the image of V in every finite quotient of E^* has order at most p^r it follows that $|V| \leq p^r$, a contradiction. ■

Lemma 5 *Let $N = \langle S \rangle$ be a nilpotent minimax group where $S \supseteq \langle x \rangle$ for each $x \in S$. Then $N = S^{*k}$ for some finite k.*

Proof. If N is abelian this is Kropholler's lemma. If not, put $Z = \mathrm{Z}(N)$ and suppose inductively that $N = ZS^{*k}$. Write X for the set of commutators in N. Then $X \subseteq S^{*4k}$. There exists f such that $N' = X^{*f}$ (*Words*, Theorem 1.3.2). So $N' \subseteq S^{*4kf}$. Again by the abelian case we have $N = N'S^{*l}$ for some l, and so $N = S^{*4kf+l}$. ■

Application: N is normal in a finitely generated group $G = NH$, and $N = \langle Y^G \rangle$ for a finitely generated subgroup Y of N. Then $N = N' \langle Y^H \rangle$, whence $N = \langle Y^H \rangle$. Taking $S = Y^H$ we deduce that $N = (Y^H)^{*f}$ for some finite f. If H is polycyclic this gives Kropholler's theorem that G is a product of cyclic subgroups.

0.2 Just-infinite rank

Assume that G is a minimax group, with spectrum π. Then G has a nilpotent normal subgroup K such that G/K is virtually abelian; let $\mathcal{K}(G)$ denote the set of all such K, define $c^*(G)$ to be the minimal nilpotency class of any member of $\mathcal{K}(G)$, and $\mathcal{K}_0(G)$ to be the set of $K \in \mathcal{K}(G)$ of class $c^*(G)$. We fix a set of primes σ containing π.

Definition A G-module M is *qrf* if $\mathrm{C}_G(a)$ is profinitely closed in G for every $a \in M$.

Lemma 6 *If M is* (a) *qrf, resp.* (b) *σ-torsion-free, and $Y \subseteq \mathbb{Z}G$, then $M/^*Y$ is* (a) *qrf, resp.* (b) *σ-torsion-free.*

Lemma 7 *Residually qrf modules and locally qrf modules are qrf.*

Lemma 8 *Suppose M is additively a π-minimax group. If M is σ-torsion-free then M is residually finite as a G-module; if M is periodic then M is locally finite as a G-module. In either case, M is qrf.*

Lemma 9 (i) *G has ACC on closed subgroups;* (ii) *every chain (H_i) in G with $H_i \lhd H_{i+1}$ and H_{i+1}/H_i σ-torsion free for all i is finite.*

Lemma 10 *If (U_i) is a chain in M with union V and M/U_i is qrf for each i then M/V is qrf.*

Proof. $\mathrm{C}_G(a \mod V) = \bigcup \mathrm{C}_G(a \mod U_i)$, a chain of closed subgroups. But G has ACC on closed subgroups. ■

Lemma 11 *Assume that G is nilpotent-by-(virtually polycyclic). Let $A \leq B \leq G$. If B is closed in G and A is closed in B then A is closed in G.*

Proof. Suppose first that G is nilpotent. It's enough to prove the claim when $A = B^n$ for some n. If B is normal in G the claim now follows from the fact that the class of residually finite minimax groups is extension closed. In general, note that B closed implies $\mathrm{N}_G(B)$ closed, and argue by induction on the subnormal defect of B in G.

In general, let $N \lhd G$ be nilpotent with G/N virtually polycyclic. By the next lemma, B is closed in G if and only if $B \cap N$ is closed in N, and the result now follows from the previous case. ■

Lemma 12 *Suppose that $K \lhd G$ and G/K is virtually polycyclic. If $C \leq G$ and $C \cap K$ is closed in K then C is closed in G.*

Proof. For each n the group G/K^n is virtually polycyclic, so K^nC is closed in G. So

$$\overline{C} = \bigcap_n K^n C.$$

Therefore

$$\overline{C} \cap K = \bigcap_n K^n (C \cap K) = C \cap K$$

since $C \cap K$ is closed in K. It follows that $\overline{C} = KC \cap \overline{C} = C(K \cap \overline{C}) = C$. ■

Corollary 13 *With K as above, M is qrf for G if and only if M is qrf for K.*

Lemma 14 *Assume that G is nilpotent, let N be a G-module and $\delta : G \to N$ a derivation. If N is σ-torsion-free and qrf then $\ker \delta$ is closed in G.*

Proof. We argue by induction on the nilpotency class c of G, starting with $c = 0$ when $G = 1$. Now suppose that $c \geq 1$.

Put $N_0 = 0$ and for $i \geq 1$ set $N_i/N_{i-1} = \mathrm{C}_{N/N_{i-1}}(G)$, $H_i = \delta^{-1}(N_i)$. Then δ induces a homomorphism $H_i \to N_i/N_{i-1}$ with kernel H_{i-1}, so H_i/H_{i-1} is a σ-torsion-free abelian minimax group. It follows that (i) for some k we have $H_{k+1} = H_k$, and (ii) H_{i-1} is closed in H_i for each i. As $\ker \delta = H_0$, it will suffice to prove that H_k is closed in G. Now $H_k = \ker \delta_k$ where $\delta_k : G \to N/N_k$ is induced by δ, and N/N_k is σ-torsion-free and qrf. So replacing N by N/N_k and δ by δ_k we reduce to the case where

$$K = \ker \delta = \delta^{-1}(\mathrm{C}_N(G)).$$

Put $B = \mathrm{C}_G(N)$. For $x \in K \cap B$ and $g \in G$ we have

$$\delta(x^g) = \delta(g)(1 - x^g) + \delta(x)g = 0$$
$$\delta(xg) = \delta(x)g + \delta(g) = \delta(g),$$

so $K \cap B \lhd G$ and δ induces a derivation $G/(K \cap B) \to N$, with kernel $K/(K \cap B)$. So replacing G by $G/(K \cap B)$ we may assume that $K \cap B = 1$.

Now put $Z = \mathrm{Z}(G)$. If $z \in Z \cap B$ and $g \in G$ then $\delta(z)(g-1) = \delta(g)(z-1) = 0$; thus $\delta(z) \in \mathrm{C}_N(G)$ and so $z \in K \cap B = 1$. Thus $Z \cap B = 1$ and so $B = 1$ as G is nilpotent. This implies in particular that G is residually finite, since now $1 = B = \mathrm{C}_G(N)$ is closed because N is qrf. It follows that Z is closed in G.

Let $T = \mathrm{C}_N(Z)$. Then for $g \in G$ we have $\delta(g) \in T$ iff $\delta(z)(g-1) = \delta(g)(z-1) = 0$ for all $z \in Z$, so $\delta^{-1}(T) = \mathrm{C}_G(\delta(Z)) = H$, say, a closed subgroup of G since N is qrf.

If $h \in H$ and $z \in Z \cap H$ then $\delta(z)(h-1) = \delta(h)(z-1) = 0$, so $\delta(Z \cap H) \subseteq \mathrm{C}_T(H)$. Therefore δ induces a derivation $\overline{\delta} : H/(Z \cap H) \to T/\mathrm{C}_T(H)$. Now $T/\mathrm{C}_T(H)$ is σ-torsion-free and qrf for $H/(Z \cap H)$, and $H/(Z \cap H)$ has class at most $c-1$, so by inductive hypothesis $H_1/(Z \cap H) := \ker \overline{\delta}$ is closed in $H/(Z \cap H)$, whence H_1 is closed in G.

Finally, $\delta_{|H_1}$ is a homomorphism $H_1 \to \mathrm{C}_T(H)$ with kernel K, so H_1/K is a σ-torsion-free abelian π-minimax group, hence residually finite. Thus K is closed in H_1, hence closed in G. ■

Lemma 15 *Assume that G is nilpotent. Let $N \leq M$. Suppose that N is σ-torsion-free. If N and M/N are qrf then M is qrf.*

Proof. Let $a \in M$ and put $C = \mathrm{C}_G(a)$, $G_1 = \mathrm{C}_G(a \mod N)$. Then $\delta(g) = a(g-1)$ defines a derivation $\delta : G_1 \to N$ with $\ker \delta = C$. Thus C is closed in G_1 by the preceding lemma, and G_1 is closed in G because M/N is qrf. The result follows. ■

Theorem 16 *Assume that G is f.g. Let M be a f.g. G-module that is σ-torsion-free, qrf and of infinite rank. Then M has a submodule V that is maximal subject to M/V having these three properties.*

Proof. Fix $K \in \mathcal{K}_0(G)$. We will use the fact that a G-module is qrf iff it is qrf for K.

Suppose $(U_i)_{i \in \mathbb{N}}$ is an ascending chain in M such that M/U_i has the three properties for each i. Let $V = \bigcup U_i$. We will prove that M/V has the properties - the result then follows by ZL (note that M is countable).

We argue by induction on $c^*(G)$. The induction starts with $c^*(G) = 0$, when G is virtually abelian. In that case, M is Noetherian and the result is clear.

Assume now that $c = c^*(G) \geq 1$, and that our claim is false. Clearly M/V is σ-torsion-free, and M/V is qrf by Lemma 10. So M/V has finite rank. By Lemma 3, there exist $K_1 \in \mathcal{K}(G)$, with $K_1 \leq K$, and a f.g. G-submodule B of V such that (i) $M(K_1 - 1)^d \leq V$ for some d, and (ii) V/B is Λ-torsion whenever $\Lambda = \mathbb{Z}Q \setminus L$ and $L \lhd_f \mathbb{Z}Q$ is a maximal ideal with $Q - 1 \subsetneq L$, where $Q = \mathrm{Z}(K_1)$. Note that K_1 has nilpotency class c.

Since B is f.g. we have $B \leq U_i$ for some i. Replacing M by M/U_i we may suppose that V is Λ-torsion for all Λ as above.

Put $R = \mathbb{Z}Q$. Suppose $0 \neq a \in M$ and $P = \mathrm{ann}_R(a)$ is a prime ideal of R. Then $R/P \cong aR$ is σ-torsion-free and qrf for Q, so $\mathrm{char}(R/P) \notin \pi$ and $Q/P^\dagger = Q/\mathrm{C}_Q(a)$ is residually finite. It follows that P is an intersection of maximal ideals of finite index ([S] Theorem 4.2). Suppose that $P \not\supseteq \Delta = (Q-1)\mathbb{Z}Q$, and let $x \in \Delta \smallsetminus P$. Suppose $L \supseteq P$ is a maximal ideal of finite index in R; then $L \not\supseteq \Delta$, so $ax^d\lambda = 0$ for some $\lambda \in \mathbb{Z}Q \smallsetminus L$, whence $x^d\lambda \in P$, a contradiction. It follows that $P \supseteq \Delta$.

Let F be a finite generating set for the G-module M. [S], Lemma 6.5 shows that $F\Delta^n = 0$ for some finite n, and as $\Delta = \Delta^G$ this implies that $M\Delta^n = 0$. If $n = 1$ then M is a (G/Q)-module, and $c^*(G/Q) < c$, whence M/V has infinite rank by inductive hypothesis, contradiction (note that $K_1/Q \in \mathcal{K}(G/Q)$).

Suppose $n > 1$. Now Q has a finitely generated subgroup S such that Q/S is a π-group. Put $\Xi = (S-1)R$ and set $M_1 = F\Xi\mathbb{Z}G$. Since Ξ is finitely generated as an ideal of R, M_1 is a finitely generated G-module. Also $\Delta/(\Xi + \Delta^n)$ is a π-group (cf. [S], Prop. 1.8), so $M\Delta/M_1$ is a π-torsion group.

Since $\Xi \leq \Delta$ we have $M_1\Delta^{n-1} = 0$. Now $M_1/(M_1 \cap V)$ has finite rank, so arguing by induction on n we may suppose that $M_1/(M_1 \cap U_l)$ has finite rank for some l. This implies that $M_1 \cap V = M_1 \cap U_j$ for some j (since $(M_1 \cap V)/(M_1 \cap U_l)$ cannot contain an infinite ascending chain with σ-torsion-free factors). As $M\Delta/M_1$ is a π-torsion group and V/U_j is σ-torsion-free it follows that $V \cap M\Delta \leq U_j$.

Let $W_i/(U_i + M\Delta)$ denote the σ-torsion subgroup of $M/(U_i + M\Delta)$, and set $W = \bigcup U_i$. Then for $i \geq j$ we have

$$V \cap (U_i + M\Delta) = U_i + (V \cap M\Delta) = U_i$$

so $(V \cap W_i)/U_i$ is a σ-torsion group and so $V \cap W_i = U_i$. Thus $V/U_i \cong (V + W_i)/W_i \leq W/W_i$. It follows that M/W_i is both σ-torsion-free and of infinite rank. As $M\Delta \leq W_j$ we may apply the previous case ($n = 1$) to the module M/M_j and infer that M/W_i cannot be qrf, for infinitely many values of $i \geq j$.

Fix $i \geq j$ and let $Y_i/(V+W_i)$ be the σ-torsion subgroup of $M/(V+W_i)$. Then Y_i/W_i is an extension of the σ-torsion-free qrf module $(V + W_i)/W_i \cong V/U_i$ by $Y_i/(V + W_i)$, which is periodic minimax. Hence Y_i/W_i is qrf by Lemma 15. Also Y_i/W_i is σ-torsion-free, so applying Lemmas 15 and 8 again we see that M/W_i is qrf, the final contradiction. ■

0.3 Finite upper p-ranks

A is an abelian minimax group.

Lemma 17 *Let $H \leq G$ and let $M \leq_G U \uparrow_H^G$ be subdirect, where U is an H-module. Suppose that $V < U$ and put $N = M \cap V \uparrow_H^G$. If M/N has finite rank and U/V is torsion-free then $|G : H| < \infty$ and U/V has finite rank.*

Proof. Projecting M onto U shows that U/V is an image of M/N, hence U/V has finite rank. Now we have $N \leq E \leq M$ with M/E periodic and E/N f.g. as $\mathbb{Z}$-module. Then $E \leq N + UL$ where $H \leq_f L \leq G$. Suppose $y \in G \smallsetminus L$ and write $\pi : M \to Uy$ for the projection. Then $\pi(UL) = 0$ and $\pi(N) \leq Vy$ so $\pi(M)/Vy$ is periodic. But $\pi(M) = Uy$ and Uy/Vy is torsion-free, whence $Uy = Vy$, contradiction. It follows that $G = L$ so $|G : H|$ is finite. ■

Lemma 18 *Let U be a π-torsion-free R-module where $R = \mathbb{Z}A$, and let P be a regular prime ideal of R. If $UP^m = 0$ and $U/^*P$ has finite rank then UP has finite rank.*

Proof. By [S] Prop. 1.8 there is a f.g. ideal $P^\natural \leq P$ such that $P/(P^\natural + P^m)$ is a π-group. Then $UP/UP^\natural$ is a π-group, so it suffice to show that $UP^\natural$ has finite rank. But $UP^\natural$ is an image of $(U/^*P)^{(k)}$ if $P^\natural$ can be generated by k elements. ∎

Proposition 19 *Assume that G is f.g. and let M be a finitely generated G-module with $r_p := \mathrm{ur}_p(M) < \infty$ for all primes p. Suppose that M has infinite upper rank. Assume that $\sigma \supseteq \pi$ and that σ is finite. Then M has a torsion-free residually finite quotient M_1 of infinite upper rank such that: if M_1/N is a proper, σ-torsion-free quotient of M_1 and M_1/N is qrf then M_1/N has finite rank.*

Proof. Let's call a module 'just-bad' if it has the property ascribed to M_1 after the colon, and has infinite rank. Denote by $M(p)$ the smallest submodule of M such that $M/M(p)$ is a finite $\mathbb{F}_pG$-module, and for each set of primes μ put $M(\mu) = \bigcap_{p\in\mu} M(p)$. Then $M/M(\sigma')$ has infinite rank since $\sup_{p\in\sigma'} r_p = \infty$, so replacing M by $M/M(\sigma')$ we may suppose that $M(\sigma') = 0$.

Now M is both residually finite and σ-torsion-free. Hence by Theorem 16 it has a just-bad σ-torsion-free quotient M_1. In particular, M_1 has infinite upper rank, by [MS], Theorem A.

Put $N = M_1(\sigma')$. Suppose $N \neq 0$. Then M_1/N is again σ-torsion-free and residually finite, so M_1/N has finite rank r say. But $M_1(p) \geq N$ for every prime $p \notin \sigma$, so the upper rank of M_1 is at most $\max\{r,\ r_p\ (p \in \sigma)\}$, contradiction. Hence $M_1(\sigma') = 0$.

Now for each prime p we have $M_1(p) \cap M_1(p') \subseteq M_1(\sigma') = 0$. It follows that $M_1(p')$ is precisely the p-torsion subgroup of M_1. Suppose $M_1(p') \neq 0$ for some p. Then $M_1/M_1(p')$ is a proper residually finite σ-torsion-free quotient of M_1, hence has finite rank, and as before we infer that M_1 has finite upper rank, contradiction. Thus M_1 is torsion free. ∎

Henceforth, let G and M be as in Proposition 19, and assume that M has the properties ascribed to M_1. Assume also that M is faithful for G. (We replace M by M_1 and G by $G/\mathrm{C}_G(M_1)$. Since M is residually finite as a G-module it follows that G is residually finite, hence virtually torsion-free; we fix $K \in \mathcal{K}_0(G)$ and assume in addition that K is torsion-free. By [PS] theorem ?, *G is not abelian-by-polycyclic*, so K is not virtually abelian and not finitely generated. It follows by the next lemma that $\mathrm{Z}(K)$ is not f.g..

Lemma 20 *If K is a torsion-free nilpotent minimax group and $\mathrm{Z}(K)$ is f.g. then K is f.g..*

Proof. arguing by induction on nilpotency class, it will suffice to show that Z_2/Z_1 is f.g., where $Z_i = \zeta_i(K)$. Now Z_2/Z_1 embeds in $\mathrm{Hom}(K^{\mathrm{ab}}, Z_1)$ which is f.g. since K^{ab} is an etension of a f.g. abelian group by a divisible group. ∎

We also make the inductive assumption:

$\mathcal{H}$: *if G^* is a f.g. minimax group with $h(G^*) < h(G)$ and M^* is a quasi-f.g. G^*-module with $\mathrm{ur}_p(M^*) < \infty$ for all primes p then $\mathrm{ur}(M^*)$ is finite.*

Lemma 21 *If N is a G-submodule of M and N has finite rank then $N = 0$.*

Proof. Suppose $N \neq 0$ and $\mathrm{rk}(N) = s < \infty$. Put

$$V = \bigcap_{p\in\sigma'} (N + M(p)).$$

Then M/V is residually finite and σ-torsion-free, so M/V has finite rank r say. For $p \in \sigma'$ we have

$$r_p = \mathrm{rk}(M/M(p)) \leq r + s$$

whence

$$\mathrm{ur}(M) \leq \max\{r + s,\ r_p \mid p \in \sigma\} < \infty,$$

contradiction. ■

Applying Lemma ? we observe:

Lemma 22 *If $K \leq H \leq G$ then for each prime p the upper p-rank of M as an H-module is equal to r_p.*

Fix $A = A^G \leq \mathrm{Z}(K)$, and consider M as a module for $R = \mathbb{Z}A$. We use the notation of [S].

Lemma 23 *Let $\mathcal{Y}$ be a set of prime ideals of R. Then $M/M(\mathcal{Y})$ is torsion-free and qrf for K.*

Proof. See [S], Lemma 6.5(i). (The proof gives this result though the statement only refers to qrf for $\dot{A}$.) ■

Lemma 24 *$\mathcal{P}(M) = \mathcal{M}(M) = P^G$ for some prime ideal P of R.*

Proof. Suppose $\mathcal{P}(M) \neq \mathcal{M}(M)$. Then $\mathcal{Y} := \mathcal{P}(M) \setminus \mathcal{Q}(M)$ is non-empty. According to [S] Proposition 6.7 we have $M > M_1 > 0$ where $M_1 = M(\mathcal{Y})$, and $\mathcal{P}(M/M_1) = \mathcal{Q}(M)$. Now M/M_1 is torsion-free and qrf for K, so M/M_1 has finite rank. Let $P \in \mathcal{M}(M)$ and suppose that $P \geq Q \in \mathcal{Q}(M)$. Then $R/Q \cong aR \leq M/M_1$ for some $a \in M/M_1$, so R/Q has finite rank and hence Krull dimension 1 since $Q \cap \mathbb{Z} = 0$. Then R/P also has finite rank, and hence Krull dimension 1 for the same reason; consequently $Q = P$.

It follows that $\mathcal{M}(M) = \mathcal{P}(M) = \mathcal{Q}(M)$. If $P, Q \in \mathcal{M}(M)$ then $M/M(P^G)$ and $M/M(Q^G)$ each have finite rank for the same reason as above; if $P^G \neq Q^G$ then $P^G \cap Q^G = \varnothing$ and then $M(P^G) \cap M(Q^G) = 0$ by [S], Lemma 7.1. This now implies that M has finite rank, contradiction. Therefore $P^G = Q^G$, and the result follows. ■

Now fix P as in the preceding lemma and put $H = \mathrm{N}_G(P)$, $\mathcal{X} = \mathcal{P}(M) \smallsetminus \{P\}$, $N = M(\mathcal{X})$, $U = M/N$. Let T be a transversal to the right cosets of H in G, with $1 \in T$. Now [S], Proposition 7.3 gives:

Lemma 25 (i) *U is torsion-free, qrf for K and finitely generated for $\mathbb{Z}H$;*

(ii) *$\mathcal{P}(U) = \{P\}$ and $UP^e = 0$ for some $e \in \mathbb{N}$;*

(iii) *the natural mapping $M \to \prod_{t\in T} M/Nt$ is injective and maps M into $\bigoplus_{t\in T} M/Nt \cong U\uparrow_H^G$.*

We take e as in (ii) to be minimal. Write

$$W = U\uparrow_H^G = \bigoplus_{t\in T} Ut$$

and identify M with its image as a subdirect sum in W.

Proposition 26 $e = 1$.

Proof. Suppose that $e > 1$. Put $V = {}^*P < U$ and set $M_1 = M \cap \bigoplus_{t\in T} Vt$. Then M/M_1 is torsion-free and qrf for K. I claim that $M_1 \neq 0$. To see this, let $0 \neq v \in V$. There exist distinct elements $t_1, \ldots, t_k \in T \smallsetminus \{1\}$ and elements $u_j \in U$ such that

$$a := v + u_1 t_1 + \cdots + u_k t_k \in M.$$

Put $D = \prod_{j=1}^{k} (P^{e-1})^{t_j}$. Then $u_i t_i D \subseteq V t_i$ for each i, so $aD \subseteq M_1$. On the other hand, $vD \neq 0$ since $v^* = P \not\supseteq D$, and so $aD \neq 0$.

It follows that M/M_1 has finite rank. Now Lemma 17 shows that $|G : H|$ is finite and that U/V has finite rank. Hence UP has finite rank by Lemma 18. Now put $J = P^{t_1} \ldots P^{t_m}$ where $T = \{t_1 = 1, \ldots, t_m\}$. Then $MJ \leq \bigoplus_{i=1}^{m} UPt_i$ so MJ is a G-submodule of M of finite rank, whence $MJ = 0$ by Lemma 21. Now let $w \in U \smallsetminus V$. Then $wP \neq 0$ so $(wP)^* \leq P$; but

$$0 = wJ = (wP)P^{t_2} \ldots P^{t_m}$$

implies $P^{t_2} \ldots P^{t_m} \leq P$, contradiction. ∎

Lemma 27 *If $|G : H|$ is infinite then U has no nontrivial finite quotient as a K_1-module for any $K_1 \leq_f K$.*

Proof. Suppose V is a K_1-submodule of finite index in U, and assume wlog that $pU \leq V$ for some prime p. Put $W_1 = V\mathbb{Z}G$. Then $C = K^f \leq \mathrm{C}_{K_1}(U/V)$ for some finite f, and W/W_1 is a finitely generated (G/C)-module, hence residually finite as G/C is f.g. and virtually abelian. Hence so is $M/(M \cap W_1)$. As $Mp \leq M \cap Wp \leq M \cap W_1$ it follows that $M \cap W_1 \geq M(p)$ and hence that $(M + W_1)/W_1 \cong M/(M \cap W_1)$ is finite. Therefore

$$M \leq W_1 + \sum_{t\in X} Ut$$

for some finite subset X of T. Then for $t \in T \smallsetminus X$ the projection of M into Ut has image contained in Vt; so $Vt = Ut$ and so $V = U$. ∎

Proposition 28 *$|G : H|$ is finite.*

Proof. Suppose not. Say $T = \{t_i \mid i \in \mathbb{N}\}$ (with $t_1 = 1$) and put $P_i = P^{t_i}$, $U_i = Ut_i$,

$$M_m = M \cap \bigoplus_{i=1}^{m} U_i, \ M'_m = M \cap \bigoplus_{i=2}^{m} U_i$$

Since U is f.g. as an H-module, there exists m such that M_m projects onto $U_1 = U$ with kernel M'_m. Now let D be an H-submodule of finite index in M_m. Then $M_m = M'_m + D$ by Lemma 27, so

$$M_m P_2 \ldots P_m \leq D.$$

As $M_m \leq M$ is residually finite as an H-module it follows that $M_m P_2 \ldots P_m = 0$. Therefore $M_1 P_2 \ldots P_m = 0$. But $M_1 = M \cap U_1 = {}^*P_1$ so $M_1^* = P_1$; consequently $P_2 \ldots P_m \leq P_1$, contradiction. ■

Lemma 29 *$\mathrm{C}_H(U)$ is finite and $P^\dagger = 1$.*

Proof. Recall that $P^\dagger = \mathrm{C}_A(U)$. As $A \leq H$ and A is torsion-free, the second claim will follow from the first. Suppose now that $\mathrm{C}_H(U)$ is infinite. Then U is a f.g. module for $H/\mathrm{C}_H(U)$ and $\mathrm{ur}_p(U) \leq r_p < \infty$ for each prime p; so U has finite upper rank by Hypothesis $\mathcal{H}$. Let V be the finite residual of U as H-module; then U/V has finite rank by [MS] Theorem A.

Note that $|G : H| > 1$ since if $H = G$ then $M = U$ and $\mathrm{C}_H(U) = \mathrm{C}_G(M) = 1$. Now we use the notation of the preceding proof, taking $T = \{t_1, \ldots, t_m\}$ where $m = |G : H|$, and write $M' = M'_m = \ker \pi$ where $\pi : M \to U_1 = U$ is the projection. Set $K = \pi^{-1}(V)$. Then $K/M' \cong V$ has no nontrivial finite quotient as an H-module, so as above we see that $KP_2 \ldots P_m \leq D$ for each H-submodule D of finite index in M, and hence that $KP_2 \ldots P_m = 0$. If $K \cap U_1 \neq 0$ then $(K \cap U_1)^* = P_1$ whence $P_1 \geq P_2 \ldots P_m$, contradiction. Therefore $K \cap U_1 = 0$, and so $U = U_1 \hookrightarrow M/K \cong U/V$. Thus U has finite rank and then M has finite rank, contradiction. ■

Corollary 30 *If $0 \neq v \in M$ then $\mathrm{C}_A(v) = 1$.*

Corollary 31 *$R/P = (R_0/P_0) \uparrow_{A_0}^{A}$ where A_0 is a f.g. subgroup of A. Hence if $A = \mathrm{Z}(K)$ then R/P has infinite rank.*

Proof. First claim is [S]. ???. Second claim follows since $\mathrm{Z}(K)$ is not f.g.. ■
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Put $A_0 = \Delta_G(A) = \{a \in A \mid |G : \mathrm{C}_H(U)| < \infty\}$. Since $P^\dagger = 1$ and $|G : \mathrm{N}_H(P)| < \infty$, [B] Theorem A shows that $P = (P \cap \mathbb{Z}A_0)\mathbb{Z}A$. As $P \cap \mathbb{Z} = 0$ it follows that *either* $P = 0$ *or* $A_0 \neq 1$.

If A happens to be rationally irreducible for G, then $A_0 \neq 1$ implies $\mathrm{C}_G(A) = \mathrm{C}_G(A_0) \leq_f G$. So we have:

Theorem 32 *Suppose that A is rationally irreducible for G. Then either M is torsion-free as a $\mathbb{Z}A$-module or $|G : \mathrm{C}_G(A)|$ is finite.*

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Alternative line

We have established that $|G : H|$ is finite, so H is f.g.. Keep the notation $M \leq U_1 \oplus \cdots \oplus U_m$.

Suppose that U has finite upper rank (for $\mathbb{Z}H$). The second paragraph in the proof of Lemma 29 shows that then M has finite rank, contradiction. Consequently U has infinite upper rank (for $\mathbb{Z}H$). Then U has a torsion-free residually finite quotient U/V that satisfies the conditions stipulated for M with H in place of G. Put $W_0 = V_1 \oplus \cdots \oplus V_m$. If $M \cap W_0 \neq 0$ then $M/(M \cap W_0)$ has finite rank; but $M/(M \cap W_0)$ projects onto $U_1/V_1 \cong U/V$ which has infinite upper rank, contradiction. Therefore $M \cap W_0 = 0$.

Let $Q \in \mathcal{P}(U/V)$, $H_1 = \mathrm{N}_H(Q)$. As above, we deduce that $|H : H_1|$ is finite. Put $J = \prod_{h \in H} Q^h$, and set $Y/V = {}^*J$. Then U/Y has finite rank, so $M \cap (Y_1 \oplus \cdots \oplus Y_m) \neq 0$. Let $0 \neq a \in M \cap (Y_1 \oplus \cdots \oplus Y_m)$. Then

$$aJ \subseteq M \cap (Y_1 \oplus \cdots \oplus Y_m) = M \cap W_0 = 0,$$

so $J \subseteq P^t$ for some $t \in T$. As $UP = 0$ it follows that $P \subseteq Q \subseteq P^g$ for some g and hence that $Q = P$.

Replacing G by H, M by U/V we reduce to the case where $\mathcal{P}(M) = \{P\}$, $MP = 0$ and $P = P^G$, $P^\dagger = 1$.

If $A_0 = 1$ we deduce from Brookes that $P = 0$, so M is torsion-free for $\mathbb{Z}A$.

If $A_0 > 1$ and M is not torsion-free for $\mathbb{Z}A$, Brookes shows that $P = (P \cap \mathbb{Z}A_0)\mathbb{Z}A$, and replacing A by A_0 and G by $\mathrm{C}_G(A_0)$ we reduce to the case where $A \leq \mathrm{Z}(G)$ and M is torsion-free as a module for $\mathbb{Z}A/P$.

0.4 Further reductions

Note that $\bigcap_{p \in \tau} M(p) = 0$ for every set of primes τ such that $\{r_p \mid p \in \tau\}$ is unbounded, since if $N = \bigcap_{p \in \tau \smallsetminus \sigma} M(p) \neq 0$ then for each $p \in \tau \smallsetminus \sigma$ we have $r_p \leq \mathrm{rk}(M/N) < \infty$. In particular, M is residually finite as a G-module, so also G is residually finite.

We call a set τ as above *large.*

For each prime p we choose a maximal G-submodule $T_p/M(p)$ of $M/M(p)$ with $\mathrm{rk}(M/T_p)$ as large as possible.

Lemma 33 *Suppose that $H \lhd \Gamma$ is nilpotent and let V be a finite $\mathbb{F}_p\Gamma$ module. If H acts unipotently on every simple Γ-quotient of V then H acts unipotently on V.*

Proof. Assume wlog that Γ acts faithfully on V, and let Q be the p'-component of H. Then $V = \mathrm{C}_V(Q) \oplus [V,Q]$. If $Q \neq 1$ then $\mathrm{C}_V(Q) \leq N < V$ for some maximal Γ-submodule N of V, and then $[V,Q] \leq N$ whence $N = V$, contradiction. Therefore H is a p-group and the result follows. ∎

Lemma 34 *Let τ be a large set of primes. Then $\{\mathrm{rk}(M/T_p) \mid p \in \tau\}$ is unbounded, and $\bigcap_{p\in\tau} T_p = 0$.*

Proof. Suppose the first claim is false. Then there exists $r < \infty$ such that $\mathrm{rk}(M/T) \leq r$ whenever $M(p) \leq T < M$ with M/T a simple $\mathbb{F}_pG$-module and $p \in \tau$.

Now there exists $G_1 \lhd_f G$ such that $M(G_1' - 1) \leq T$ for every such T, where G_1 depends only on r (Mal'cev's Theorem). Put $L = K \cap G_1'$. Then L acts nilpotently on $M/M(p)$ for each $p \in \tau$, by Lemma 33.

Thus $(M/M(p)) \rtimes L$ is nilpotent and residually finite for each $p \in \tau$. Since $\bigcap_{p\in\tau} M(p) = 0$ it follows that $M \rtimes L$ is residually finite-nilpotent. Now [PS] Proposition 4.2 shows that $M \rtimes G$ is a minimax group, contradiction since M has infinite upper rank.

This establishes the first claim. The second claim follows since if $D := \bigcap_{p\in\tau} T_p \neq 0$ then M/D has finite rank, which bounds $\mathrm{rk}(M/T_p)$ for all $p \in \tau$. ∎

Let H be a subgroup of K containing A. Then H is subnormal in G so M/T_p is a direct sum of simple $\mathbb{Z}H$-modules. Denote by $s_p(H)$ the maximal rank of these as modules over R.

Corollary 35 *If H is not virtually abelian and τ is large then the set $\{s_p(H) \mid p \in \tau\}$ is unbounded.*

Proof. If U is an H-composition factor of M/T_p then $R/U^* = k$ is a field, U is a simple kH-module and $\dim_k(U) \leq s_p(H)$. Now suppose that $s_p(H) \leq r < \infty$ for all $p \in \tau$. Then there exists $H_1 \lhd_f H$ such that $U(H_1' - 1) = 0$ for every such U with $p \in \tau$, and it follows that $M(H_1' - 1) \leq \bigcap_{p\in\tau} T_p = 0$. Thus $H_1' = 1$, so H is virtually abelian. ∎

Corollary 36 *Suppose that $H \leq K$ and set $E = \mathbb{Z}(A \cap H)$. If N is an H-submodule of M and $H/\mathrm{C}_H(N)$ is not virtually abelian then N has infinite upper rank as an E-module. More precisely: let $\lambda \in E$ be a non zero-divisor on N. Then for each r there exists $S < N$ with $N\lambda \nsubseteq S$ such that N/S is a simple H-module of rank at least r as an E-module.*

Proof. Suppose that every finite simple H-quotient U of N with $U\lambda \neq 0$ has rank at most $r - 1$ as an E module. Then there exists $H_1 \lhd_f H$ such that $U(H_1' - 1) = 0$ for every such U. Therefore every finite simple H-quotient U of M satisfies $U(H_1' - 1)\lambda = 0$. Now each H-quotient $N/(N \cap T_p)$ is a direct sum of H-modules like U, so $N(H_1' - 1)\lambda \leq \bigcap_p N \cap T_p = 0$. As λ is not a zero divisor on N it follows that $N(H_1' - 1) = 0$, so $H/\mathrm{C}_H(N)$ is virtually abelian. ∎

Lemma 37 *Let H and N be as above. If $H/\mathrm{C}_H(N)$ is virtually abelian then $H' \cap A = 1$.*

Proof. Put $C = \mathrm{C}_H(N)$. Then $C \cap A = 1$. Now $H' \leq T$ where T/C is the torsion subgroup of H/C, since H/T is torsion-free nilpotent of finite rank; hence $H' \cap A = 1$ as A is torsion-free. ■