Motivation

[S], Section 1 considers a prime module M over the group algebra kG where k is a finitely generated commutative ring and G is an abelian minimax group; the main result shows (under a non-singularity hypothesis) that M is induced from a finitely generated subgroup of G. Key steps in the proof are the following lemmas, which hold modulo some assumptions on the prime p.

Lemma 1 *Let $F = F(y_0) \subseteq \ldots \subseteq F(y_1) \subseteq F(y_n)$ be a chain of fields, where $y_i^p = y_{i-1}$ for each i. If $F(y_n) = F(y_{n-1})$ then $F(y_n) = F$.*

Lemma 2 *Let $R = k[A]$ be a commutative domain, where $A \leq R^*$. Let B be a subgroup of A with $A/B \cong C_{p^\infty}$. Then A/B has a finite subgroup C/B such that* either

$$R = k[C] \uparrow_C^A$$

or *R is contained in the field of fractions of $k[C]$.*

The aim is to generalize as much as possible of the theory to the case where G is not abelian, but is nilpotent.

Skewfields

p a prime, F a skewfield, σ an auto of F, $R = F[t;\sigma]$ the skew polynomial ring with

$$at = ta^\sigma \quad (\forall a \in F). \tag{1}$$

We will usually write $F[t]$ for $F[t;\sigma]$. Assume that F contains a central σ-invariant primitive p^2th root of unity $\zeta = \zeta_{p^2}$. Then

$$g^\phi(t) := g(\zeta t) \quad (g(t) \in F[t])$$

defines an automorphism ϕ of R, of order p^2. Put $\phi_i = \phi^{p^i}$ $(\in \{0, 1\})$.

We use the facts: (1) R is a principal right ideal ring; (2) A ring A that is simple as a right A-module is a skewfield; (3) a finite-dimensional F-algebra contained in a skewfield is a skewfield.

When $F \subseteq E$ is a pair of skewfields, we say that F is *invariant* in E if $x^{-1}Fx = F$ for all $x \in E$.

Lemma 3 *If $g = g^{\phi_i}$ then $g = h(t^{p^{2-i}})$ for some h. If $gR = g^{\phi_i}R$ then $g = h(t^{p^{2-i}})t^j$ for some h and some $j \geq 0$.*

Proof. We may suppose that $g \neq 0$. Suppose that $gR = g^{\phi_i}R$. Then for some $\lambda \in F^*$ we have

$$g = \sum_0^k a_l t^l = \lambda \sum_0^k a_l \zeta^{p^i l} t^l.$$

Suppose first that $a_0 \neq 0$. Then $\lambda a_0 = a_0$ implies $\lambda = 1$, and then $(1-\zeta^{p^i l})a_l = 0$ for each l; consequently $a_l = 0$ unless $l \equiv 0 \pmod{p^{2-i}}$. Thus $g = h(t^{p^{2-i}})$ for a suitable h. In general, $g = t^j g_1$ where g_1 has non-zero constant term. Clearly $g_1 R = g_1^{\phi_i} R$ and we apply the previous case.

If we assume that $g = g^{\phi_i}$ then $\lambda = 1$ and the result follows as above. ■

For $\lambda \in F$ and $m \geq 0$ define

$$\lambda^{m_\sigma} = \prod_{i=0}^{m-1} \lambda^{\sigma^i}.$$

Lemma 4 *('Remainder theorem') For $m \geq 1$ and $\lambda \in F$,*

$$t^m = (t-\lambda)q(t) + \lambda^{m_\sigma}.$$

Proof. Clear if $m = 1$. Suppose $n > 1$ and induct.

$$t^{m-1} = (t-\lambda)q_1(t) + \lambda^{(m-1)_\sigma}$$
$$\Longrightarrow$$
$$t^m = (t-\lambda)q_1(t)t + (t-\lambda)(\lambda^{(m-1)_\sigma})^\sigma + \lambda(\lambda^{(m-1)_\sigma})^\sigma$$

giving the claim since $\lambda(\lambda^{(m-1)_\sigma})^\sigma = \lambda^{m_\sigma}$. ■

Corollary 5 *Suppose that $\mu = \lambda^{p_\sigma}$, $\lambda \in F$. Then $(t^p - \mu)R < (t-\lambda)R$.*

Proof. Modulo $(t-\lambda)R$ we have

$$t^p \cong \lambda^{p_\sigma}$$

so $(t^p - \mu)R \leq (t-\lambda)R$. Of course $t - \lambda \notin (t^p - \mu)R$. ■

Lemma 6 *Let $\mu = \mu^\sigma \in F$. If $a\mu = \mu a^{\sigma^m}$ $(\forall a \in F)$ then $(t^m - \mu)R$ is a 2-sided ideal of R.*

Proof. $(t^m - \mu)$ commutes with t, and if $a \in F$ then $a(t^m - \mu) = (t^m - \mu)a^{\sigma^m}$. ■

Proposition 7 *Let $\mu = \mu^\sigma \in F \setminus \{0\}$. At least one of the following holds:*

(a) There exists $\lambda \in F$ such that

$$\mu = \lambda^{p_\sigma};$$

(b) $(t^p - \mu)R = I$ is a maximal right ideal of R. If in addition $a\mu = \mu a^{\sigma^p}$ $(\forall a \in F)$ then I is a 2-sided ideal and R/I is a skewfield.

Proof. Suppose that $I = (t^p - \mu)R \leq L \underset{\max_r}{\lhd} R$. Put $L_i = L^{\phi_1^{i-1}}$ for $1 \leq i < p$. Since $I = I^{\phi_1}$ we have $L_i \geq I$ for each i. Now $\widetilde{L} := L_1 \cap L_2 \cap \ldots \cap L_p = gR$ for some g, and then $g^{\phi_1} R = gR$, so $g = h(t^p)t^j$.

Now $(t^p - \mu) = gr$ for some r. As $\mu \neq 0$ it follows that $j = 0$. Set $d = \deg g = \dim(R/\widetilde{L})$ (dimension as a right F-module). Then $p \geq d = p \deg h \geq 1$, so in fact $d = p$ and $\widetilde{L} = I$.

It follows that the composition factors of the R-module R/I are all $\langle \phi_1 \rangle$-conjugate to R/L, hence have dimension $d^* \in \{1, p\}$.

(a) Suppose that $d^* = 1$. Then $L = (t - \lambda)R$ for some $\lambda \in F$. In this case we have

$$\lambda^{p_\sigma} - \mu = (\lambda^{p_\sigma} - t^p) + (t^p - \mu) \in (t - \lambda)R \cap F = 0.$$

(b) Suppose that $d^* = p$. Then $I = L$ is a maximal right ideal.

The final claim follows by Lemma 6.

(This result does not require the presence of $\zeta_{p^2} \in F$; it suffices to have $\zeta_p \in F$ (central and σ-invariant).) ■

Remark Case (b) holds when $t^p - \mu$ is irreducible. In Case (a), the composition factors of the R-module R/I are all 1-dimensional: this is equivalent to $t^p - \mu$ being a product of p linear factors.

Corollary 8 (of proof) *Suppose that $F \subseteq E = F(x)$ where $0 \neq x^p \in F$ and F is invariant in E (the notation means: E is the skewfield generated by F and x). Then $\dim_F E \in \{1,\ p\}$.*

Proof. Conjugation by x effects an automorphism σ on F. We have an F-algebra homorphism $\pi : R = F[t; \sigma] \to E$ with $t\pi = x$ and $L := \ker \pi \supseteq I = (t^p - \mu)R$ where $\mu = x^p$. Then $R\pi$ is a finite-dimensional F-subalgebra of E, so it is a skewfield; so in fact $R\pi = E$. Hence L is maximal as a right ideal of R, hence has codimension p or 1. ■

Proposition 9 *Let $\mu = \mu^\sigma \in F \setminus \{0\}$. Assume that $a\mu = \mu a^{\sigma^{p^2}}$ $(\forall a \in F)$. Suppose that*

$$(t^{p^2} - \mu)R = I \leq L \lhd R$$

and that R/L is a skewfield. Then one of the following holds:

(a) There exists $\lambda \in F$ such that

$$\mu = \lambda^{p_{\sigma(1)}}$$

where $\sigma(1) = \sigma^p$.

(b) $L = I$.

Remark We have to assume here that L is a *two-sided ideal.* It is *not* necessarily the case that I is a maximal right ideal if (a) does not hold. In other words, $(t^{p^2} - \mu)$ can be reducible while $s^p - \mu$ is irreducible in $F[s; \sigma^p]$. This cannot happen in the commutative situation (provided $\zeta_p \in F$), by Lemma 1. A counterexample is given below.

Proof. Put $L_i = L^{\phi^{i-1}}$ for $1 \le i < p^2$. Since $I = I^\phi$ we have $L_i \ge I$ for each i. Now $\widetilde{L} := L_1 \cap L_2 \cap \ldots \cap L_{p^2} = gR$ for some g, and then $g^\phi R = gR$, so $g = h(t^{p^2})t^j$.

Now $(t^{p^2} - \mu) = gr$ for some r. As $\mu \ne 0$ it follows that $j = 0$. Set $d = \deg g = \dim(R/\widetilde{L})$. Then $p^2 \ge d \ge 1$, so in fact $d = p^2$ and $\widetilde{L} = I$.

It follows that the composition factors of the R-module R/I are all $\langle\phi\rangle$-conjugate to R/L, hence have dimension $d^* = \dim_F(R/L) \in \{1, p, p^2\}$.

Since L_i is the annihilator of the right R-module R/L_i, we have $R/L_i \cong R/L_j$ as right R-modules if and only if $L_i = L_j$.

(a) Suppose that $d^* = 1$. Then $L = (t - \pi)R$ for some $\pi \in F$. In this case we have

$$\pi^{p_\sigma^2} - \mu = (\pi^{p_\sigma^2} - t^{p^2}) + (t^{p^2} - \mu) \in (t - \pi)R \cap F = 0.$$

Setting $\lambda = \pi^{p_\sigma}$ we obtain

$$\mu = \pi^{p_\sigma^2} = \lambda^{p_{\sigma(1)}}.$$

(a') Suppose that $d^* = p$. Then R/I has composition length p, so at most p of the L_i can be distinct. Hence $L_i = L_j$ for some $i \ne j$ and it follows that $L^{\phi^p} = L$. Thus L is ϕ_1-invariant.

Now $L = wR$ where $\deg w = p$. As above we infer that $w = v(t^p)$, where v has degree 1; say $w = t^p - \lambda$ where $\lambda \in F$. Put $s = t^p$. Then $(s^p - \mu)F[s] = I \cap F[s] < L \cap F[s] = (s - \lambda)F[s]$. Now Proposition 7 (with s in place of t) shows that $\mu = \overline{\lambda}^{p_{\sigma(1)}}$ for some $\overline{\lambda} \in F$ (actually the proof gives $\overline{\lambda} = \lambda$).

(b) Suppose that $d^* = p^2$. Then $L = I$. ■

We can now generalize Lemma 1:

Theorem 10 *Let $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ be skewfields, where $n \ge 2$ and $\zeta_{p^2} \in F_0$. Suppose that $F_i = F_{i-1}(x_i)$ where $x_i^p = x_{i-1}$ for each i and $0 \ne x_0 \in F_0$. Assume that F_i is invariant in F_{i+2} for $0 \le i \le n-2$. If $\dim_{F_{n-1}} F_n < p$ then $F_n = F_0$.*

Proof. Corollary 8 shows that $\dim_{F_{n-1}} F_n < p$ is equivalent to $F_n = F_{n-1}$. Let m be minimal such that $F_n = F_{m-1}$. If $m = 1$ we conclude that $F_n = F_0$. Supposing that $m \ge 2$ we will derive a contradiction.

Now $F_{m-2} < F_{m-1} = F_m$, and so $\dim_{F_{m-2}} F_m = p$. Set $F = F_{m-2}$ and $E = F_m$. Then $E = F(y) = F(x) > F$ where $x^p := \mu \in F$ and $y^p = x \ne 0$. Let σ be the automorphism of F effected by conjugation by y.

We have an F-algebra homorphism $\pi : R = F[t;\sigma] \to E$ with $t\pi = y$ and $L := \ker \pi \supseteq I = (t^{p^2} - \mu)R$. Then $R\pi$ is a finite-dimensional F-subalgebra of E, so it is a skewfield; so in fact $R\pi = E$. Hence L is maximal as a right ideal of R and has codimension p. It follows by Proposition 9 that $\mu = \lambda^{p_{\sigma(1)}}$ for some $\lambda \in F$.

Now restrict π to $F[s]$ where $s = t^p$ (so $F[s] = F[s;\sigma(1)]$). As before π maps $F[s]$ onto E, so $K := \ker \pi \cap F[s]$ is maximal as a right ideal of $F[s]$. But

$$K \supseteq (s^p - \mu)F[s]$$

and comparing codimensions we conclude that

$$\begin{aligned} K &= (s^p - \mu)F[s] \\ &< (s - \lambda)F[s] \end{aligned}$$

by Corollary 5. This is the desired contradiction. ■

The counterexample

We exhibit a field F with an automorphism σ of order $p = 3$ and an element $\mu = \mu^\sigma \in F \smallsetminus F^3$ such that $I = (t^{p^2} - \mu)R$ is not a maximal right ideal of $R = F[t;\sigma]$.

Note that $\sigma(1) = \sigma^3$ is the identity here, so $\mu = \lambda^{p_{\sigma(1)}}$ for some $\lambda \in F$ is equivalent to $\mu \in F^3$.

In this example we have : $(F[t^p] + I)/I$ is a skewfield of dimension p over F, but R/I has a maximal right ideal of codimension p. So a simple right module for $F[t]/I$ is induced from F as a module for $F[t^p]$ but not as a module for $F[t]$.

Take $p = 3$. Let k_0 be a field containing ζ a primitive cube root of unity. $k_0(q)$ the field of rational functions, q an indeterminate. Set

$$z = (3q(q^2 + 1))^{-1/2}$$

and let $k = k_0(q)(z)$. Set $F = k(y)$ where $y^3 = q$. Then $\mathrm{Gal}(F/k) = \langle \sigma \rangle$ where σ sends y to ζy.

Put

$$\mu = q^{-1}(q^2 - 1)z^3.$$

Thus $0 \neq \mu = \mu^\sigma \in k \subset F$.

1. We have $\sigma(1) = \sigma^3 = 1$. If $\mu = \lambda^{3_{\sigma(1)}}$ for some $\lambda \in F$ then $\mu \in F^3$ and then

$$q^{-1}(q^2 - 1) \in F^3 \cap k.$$

This implies that

$$q^r(q^2 - 1) \in k^3$$

where $r \in \{0, \pm 1\}$. As $(k : k_0(q)) = 2$ it follows that $q^r(q^2 - 1) \in k_0(q)^3$, which is plainly false.

(2). Let s be an indeterminate and set $I_0 = (s^3 - \mu)F[s]$,

$$D = F[s]/I_0 = F(x)$$

where $x = s + I_0$, so $x^3 = \mu$. Since $\mu \notin F^3$ we have $(D : F) = 3$. Extend σ to an automorphism of D that fixes x. (All the field extensions are Galois since $\zeta \in k$.)

Set

$$\begin{aligned} a &= x - qz \in k(x) \\ b &= qx + z \in k(x) \\ \lambda &= ay + b \in D. \end{aligned}$$

Then

$$\begin{aligned} \lambda\lambda^{\sigma}\lambda^{\sigma^2} &= qa^3 + b^3 \\ &= A + Bx + Cx^2 \end{aligned}$$

where

$$\begin{aligned} A &= q(\mu - q^3z^3) + \mu q^3 + z^3 \\ &= (1-q^4)z^3 + \mu q(q^2+1) \\ &= (q^2+1)((1-q^2)z^3 + \mu q) = 0 \end{aligned}$$

since $\mu q = (q^2-1)z^3$;

$$C = 3(-q^2z + q^2z) = 0;$$

and

$$\begin{aligned} B &= 3(q^3z^2 + qz^2) \\ &= 3q(q^2+1)z^2 = 1. \end{aligned}$$

Thus $\lambda\lambda^{\sigma}\lambda^{\sigma^2} = x$.

Proof. Now consider the ring $D[t;\sigma]$. By Lemma 4 we have $t^3 - x \in (t - \lambda)D[t;\sigma] := L$. Note that $D[t;\sigma]/L \cong D \cong F^3$ as F-vector space.

Recall that $I = (t^9 - \mu)R$ where $R = F[t;\sigma]$. Put $s = t^3$; recall that $F[s] \cap I = I_0 = (s^3 - \mu)F[s]$ and $D = F[s]/I_0$. Then

$$\frac{R}{I} \cong \frac{D[t;\sigma]}{(t^3-x)D[t;\sigma]} > \frac{L}{(t^3-x)D[t;\sigma]}.$$

Thus R/I has a right ideal $\widetilde{L}$of F-codimension 3, so I is not a maximal right ideal of R.

Explicitly, $\widetilde{L} = (t - v)R$ where $v \in F + Ft^3$ is the pre-image of λ under the F-algebra epimorphism $F[t^3] \to F[s]/I_0 = D$ that sends t^3 to s.

(*Remark*: we can probably replace q by an element of k_0 by specialization, if k_0 is a Hilbertian field. So for example k could be an algebraic number field of degree 6 over $\mathbb{Q}(\zeta)$.) ■

References

[S] On the group rings of abelian minimax groups, *J. Algebra* **237** (2001), 64 - 94.