Notes on pseudo-finite groups

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1 Embedding

$G = \langle g_1, \ldots, g_d \rangle$ is a pseudo-finite group. $\mathcal{T}$ is the set of first-order formulas ϕ with at most d free variables such that $G \models \phi(g_1, \ldots, g_d)$, and $\mathcal{T}_0$ (the theory of G) is the subset of $\mathcal{T}$ consisting of sentences.

We enumerate $\mathcal{T} = \{\phi'_n \mid n \in \mathbb{N}\}$ and set

$$\phi_n = \bigwedge_{j=1}^{n} \phi'_j$$
$$\psi_n = \exists \mathbf{x}.\phi_n(\mathbf{x}).$$

Since G is pseudo-finite, for each n there is a finite group M_n with $M_n \models \psi_n$. Thus there exist $h_{in} \in M_n$ such that $M_n \models \phi_n(h_{1n}, \ldots, h_{dn})$. Set $H_n = \langle h_{1n}, \ldots, h_{dn} \rangle$.

Put

$$\mathbf{h}_i = (h_{in})_n \in \prod_{n=1}^{\infty} H_n = L \leq \prod_{n=1}^{\infty} M_n = M,$$
$$H = \langle \mathbf{h}_1, \ldots, \mathbf{h}_d \rangle \leq L.$$

For $\mathbf{a} = (a_n) \in M$ put $\sup(\mathbf{a}) = \{n \mid a_n \neq 1\}$ and $\mathrm{cosup}(\mathbf{a}) = \{n \mid a_n = 1\}$.

Let $F = \langle f_1, \ldots, f_d \rangle$ be free of rank d and define

$$\alpha : F \twoheadrightarrow G; \ f_i \mapsto g_i$$
$$\theta : F \twoheadrightarrow H; \ f_i \mapsto \mathbf{h}_i.$$

Lemma 1 *Let $w \in F$. The following are equivalent:*

- $w\alpha = 1$
- $\sup(w\theta)$ is finite
- $\mathrm{cosup}(w\theta)$ is infinite.

Proof.

$$\begin{aligned} w\alpha = 1 &\Longrightarrow [w(\mathbf{x}) = 1] = \phi'_j \text{ (some } j) \\ &\Longrightarrow w(h_{1n}, \ldots, h_{dn}) = 1\ \forall n \geq j \Longrightarrow \sup(w\theta) \subseteq [1, j-1]. \end{aligned}$$

$$\begin{aligned} w\alpha \neq 1 &\Longrightarrow [w(\mathbf{x}) \neq 1] = \phi'_j \text{ (some } j) \\ &\Longrightarrow w(h_{1n}, \ldots, h_{dn}) \neq 1\ \forall n \geq j \Longrightarrow \operatorname{cosup}(w\theta) \supseteq [j, \infty). \end{aligned}$$

■

Write $D < M$ for the restricted direct product of the M_n, let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and set

$$\begin{aligned} N &= \{\mathbf{a} \in M \mid \operatorname{cosup}(\mathbf{a}) \in \mathcal{U}\} \\ &= \ker\left(M \to \left(\prod_{n=1}^{\infty} M_n\right)/\mathcal{U} := \widetilde{M}\right). \end{aligned}$$

Then $D \leq N$ and $\widetilde{M} = M/N$ is a non-principal ultraproduct of the M_n. Lemma 1 shows that $\ker\theta \leq \ker\alpha$, so α induces an epimorphism

$$\begin{aligned} \pi : H \cong F/\ker\theta &\twoheadrightarrow F/\ker\alpha \cong G; \\ \mathbf{h}_i\pi &= g_i \ (i = 1, \ldots, d) \end{aligned}$$

Moreover,

$$\begin{aligned} h = w\theta \in \ker\pi &\Longleftrightarrow w \in \ker\alpha \\ &\Longleftrightarrow h \in D \\ &\Longleftrightarrow h \in N, \end{aligned}$$

since $h \in N$ implies that $\operatorname{cosup}(\mathbf{a})$ is infinite, while if $\sup(h)$ is finite then $\operatorname{cosup}(\mathbf{a})$ is cofinite hence belongs to $\mathcal{U}$, whence $h \in N$. Thus

$$\ker\pi = H \cap D = H \cap N,$$

and we have an induced embedding

$$\begin{aligned} \beta : G \cong HN/N &\rightarrowtail \widetilde{M}; \\ g_i\beta &= \mathbf{h}_i N \ (i = 1, \ldots, d). \end{aligned}$$

I'll write '$\forall_{\mathcal{U}} n$' to mean '$\forall n \in U$, for some $U \in \mathcal{U}$', and '$\forall n >> 0$' to mean '$\forall n \geq n_0$, for some $n_0 \in \mathbb{N}$'.

Proposition 2 *Let ϕ be a formula with at most d free variables. (i) The following are equivalent:*

(a) $G \models \phi(g_1, \ldots, g_d)$, i.e. $\phi \in \mathcal{T}$;

(b) $M_n \models \phi(h_{1n}, \ldots, h_{dn})$ *for infinitely many* n;
(c) $M_n \models \phi(h_{1n}, \ldots, h_{dn})$ $\forall_{\mathcal{U}} n$.
(d) $M_n \models \phi(h_{1n}, \ldots, h_{dn})$ $\forall n >> 0$;

(ii) Let

$$C(\mathbf{y}, \mathbf{x}) = \bigwedge_{j=1}^{m} w_j(y_1, \ldots, y_k, x_1, \ldots, x_d)\#1,$$

$$D(\mathbf{y}, \mathbf{x}) = \bigvee_{j=1}^{m} w_j(y_1, \ldots, y_k, x_1, \ldots, x_d)\#1$$

where $\#$ *is either* $=$ *or* $\neq$ *(not necessarily the same each time), and each* w_j *is a group word in* $k+d$ *variables. Then*

$$\widetilde{M} \models \exists y_1 \ldots \exists y_k . C(\mathbf{y}, g_1\beta, \ldots, g_d\beta) \Longleftrightarrow G \models \exists y_1 \ldots \exists y_k . C(\mathbf{y}, g_1, \ldots, g_d) \quad (1)$$

and

$$\widetilde{M} \models \exists y_1 \ldots \exists y_k . D(\mathbf{y}, g_1\beta, \ldots, g_d\beta) \Longleftrightarrow G \models \exists y_1 \ldots \exists y_k . D(\mathbf{y}, g_1, \ldots, g_d). \quad (2)$$

(iii) If ϕ *is a sentence, then* $\phi \in \mathcal{T}_0$ *iff* $\widetilde{M} \models \phi$.

Proof. (i) is similar to the proof of Lemma 1: observe that (a)⇒(d)⇒(c)⇒(b) and (a')⇒(d') where (a') is like (a) with $\neg\phi$ in place of ϕ, etc.

(iii) follows by Lós's Theorem. For (ii), suppose there exist $\mathbf{a}_1, \ldots, \mathbf{a}_k \in M$ such that

$$w_j(\mathbf{a}_1 N, \ldots, \mathbf{a}_k N, g_1\beta, \ldots, g_d\beta)\#1 \quad (3)$$

for every j (case (1)), resp. some j (case (2)). Put $\mathbf{b}_j = w_j(\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{h}_1, \ldots, \mathbf{h}_d)$ and set

$$T_j = \begin{cases} \operatorname{cosup}(\mathbf{b}_j) & \text{when } \# \text{ is } = \\ \operatorname{sup}(\mathbf{b}_j) & \text{when } \# \text{ is } \neq \end{cases} .$$

Then $T_j \in \mathcal{U}$ whenever (3) holds.

Suppose (3) holds for some j. Then

$$\{n \mid M_n \models \exists y_1 \ldots \exists y_k . w_j(y_1, \ldots, y_k, h_{1n}, \ldots, h_{dn})\#1\} \supseteq T_j \in \mathcal{U}.$$

Hence by (i) we have $G \models \exists y_1 \ldots \exists y_k . w_j(y_1, \ldots, y_k, g_1, \ldots, g_d)\#1$. This implies $G \models \exists y_1 \ldots \exists y_k . D(\mathbf{y}, g_1, \ldots, g_d)$, giving the implication $\Longrightarrow$ in (2).

Now suppose that (3) holds for every $j = 1, \ldots, m$. Then the set

$$S = \{n \mid M_n \models \exists y_1 \ldots \exists y_k . \bigwedge_{j=1}^{m} w_j(y_1, \ldots, y_k, h_{1n}, \ldots, h_{dn})\#1\}$$

contains $T_1 \cap \ldots \cap T_m$, so $S \in \mathcal{U}$, and as before we may infer $G \models \exists y_1 \ldots \exists y_k . C(\mathbf{y}, g_1, \ldots, g_d)$. This establishes the implication $\Longrightarrow$ in (1).

The reverse implications are clear, since $\beta : G \to \widetilde{M}$ is a homomorphism. ■

Corollary 3 *(i) G is elementarily equivalent to $\widetilde{M}$.*
(ii) A finite set of equations and inequations with parameters in $G\beta$ is solvable in $G\beta$ iff it is solvable in $\widetilde{M}$.

Proof. (i) is just (iii) of the Proposition. For (ii), first rewrite each equation or inequation $v(y_1, \ldots, y_k, c_1, \ldots, c_r)\#1$ with $c_1, \ldots, c_r \in G\beta$ in the form

$$w_j(\mathbf{y}, \mathbf{g}\beta) := v(y_1, \ldots, y_k, u_1(\mathbf{g}\beta), \ldots, u_r(\mathbf{g}\beta))\#1$$

where u_j is a word such that $u_j(g_1, \ldots, g_d)\beta = c_j$. The claim then follows from (1) and the fact that $\beta : G \to G\beta$ is an isomorphism. ■

1.1 The finitely presented case

Suppose in addition that

$$G = \langle g_1, \ldots, g_d; R \rangle$$

is finitely presented, so R is a finite set of words on $g_1, \ldots, g_d$. In this case, we choose ϕ_1' to be the formula

$$\bigwedge_{r \in R} r(x_1, \ldots, x_d) = 1.$$

Then $r(h_{1n}, \ldots, h_{dn}) = 1$ for each $r \in R$ and every n, so $r(\mathbf{h}_1, \ldots, \mathbf{h}_d) = 1$ for each $r \in R$ and we obtain an epimorphism $\gamma : G \to H$ with $g_i\gamma = \mathbf{h}_i$ ($i = 1, \ldots, d$). Thus γ and $\pi : H \to G$ are mutually inverse, so γ is an isomorphism and

$$H \cap N = H \cap D = \ker \pi = 1.$$

As $H \leq \prod_{n=1}^{\infty} H_n$ we deduce:

Proposition 4 *If G is finitely presented then G is residually finite.*

2 Definable subgroups

Let κ be a formula with one free variable. Write

$$G_\kappa = \{a \in G \mid G \models \kappa(a)\}.$$

The following is evident (here G denotes an arbitrary group):

Lemma 5 *Let $K = G_\kappa$, let $e \in \mathbb{N}$ and let $A = \{a_1, \ldots, a_e\}$ be a finite group of order e.*

(i) K is a subgroup of G iff $G \models s(\kappa)$ where

$$s(\kappa) = \forall x \forall y. (\kappa(x) \wedge \kappa(y)) \to \left(\kappa(x^{-1}) \wedge \kappa(xy)\right).$$

(ii) K is a normal subset of G iff $G \models n(\kappa)$ where

$$n(\kappa) = \forall x \forall y. \kappa(x) \to \kappa(y^{-1}xy).$$

(iii) $|G:K| \leq e$ *iff* $G| = \exists y_1 \ldots \exists y_e \forall x. \bigvee_{i=1}^{e} \kappa(xy_i)$.

(iv) $|G:K| = e$ *iff* $|G:K| \leq e$ and (not $|G:K| \leq e-1$).

(v) If K is a normal subgroup of G, then $G/K \cong A$ iff $|G:K| = e$ and

$$G| = \exists y_1 \ldots \exists y_e. \left(\bigwedge_{i,j=1}^{e} \kappa(y_i y_j y_{m(i,j)}^{-1}) \wedge \forall x. \bigvee_{i=1}^{e} \kappa(xy_i) \right)$$

where $a_i a_j = a_{m(i,j)}$.

(vi) Let w be a group word in k variables. Put

$$\kappa(x) = \exists z_{11} \ldots \exists z_{kf}. \bigvee_{\eta \in \{\pm 1\}^f} x = \prod_{j=1}^{f} w(z_{1j}, \ldots, z_{kj})^{\eta_j}.$$

Then $G_w^{*f} = G_\kappa$*, and* $w(G) = G_w^{*f}$ *iff* $G| = s(\kappa)$.

Lemma 6 *If G is pseudo-finite and G_κ is a subgroup of G then G_κ is pseudo-finite. If also $G_\kappa \lhd G$ then G/G_κ is pseudo-finite.*

Proof. To a formula ϕ associate the formula $\phi_\natural(\kappa)$ obtained as follows: replace each occurrence of $\forall x.(\ldots$ by $\forall x.(\kappa(x) \to \ldots$ and each occurrence of $\exists x.(\ldots$ by $\exists x.(\kappa(x) \wedge \ldots$. Then

$$G_\kappa| = \phi \Longleftrightarrow G| = \phi_\natural(\kappa). \tag{4}$$

Suppose $G_\kappa| = \phi$ for a sentence ϕ. Then there exists a finite group M such that $M| = s(\kappa) \wedge \phi_\natural(\kappa)$, and then M_κ is a finite group satisfying $M_\kappa| = \phi$.

Now suppose that $G_\kappa \lhd G$. Let $\phi_\flat(\kappa)$ be the formula obtained from ϕ by replacing each occurrence of an atomic subformula $w(x_1, \ldots, x_k) = 1$ with $\kappa(w(x_1, \ldots, x_k))$. Then

$$G/G_\kappa| = \phi \Longleftrightarrow G| = \phi_\flat(\kappa). \tag{5}$$

Suppose $G/G_\kappa| = \phi$ for a sentence ϕ. Then there exists a finite group M such that $M| = s(\kappa) \wedge n(\kappa) \wedge \phi_\flat(\kappa)$, and then M/M_κ is a finite group satisfying $M/M_\kappa| = \phi$. ■

Henceforth G is as in section 1. I will call a word w *good* if there exists $f = f(w) \in \mathbb{N}$ such that $w(H) = H_w^{*f}$ for every d-generator finite group H (f may depend on d).

Proposition 7 *If w is good then* $w(G) = G_w^{*f}$ *where* $f = f(w)$.

Proof. We will establish the stronger statement

$$w(H)(H \cap D) = H \cap w(M)N = H \cap M_w^{*f} N = H_w^{*f}(H \cap D); \tag{6}$$

the claim follows on applying $\pi : H \to G$.

Let $h = v(\mathbf{h}_1, \ldots, \mathbf{h}_d) \in H \cap w(M)N$. Then

$$h = \prod_{j=1}^{s} w(\mathbf{z}_j)^{\varepsilon_j} \cdot y$$

for some $\mathbf{z}_1, \ldots, \mathbf{z}_s \in M^{(k)}$, $\varepsilon_j = \pm 1$ and $y \in N$. Hence

$$\forall_{\mathcal{U}} n : M_n \models \exists z_{11} \ldots \exists z_{sk}. v(h_{1n}, \ldots, h_{dn}) = \prod_{j=1}^{s} w(z_{j1}, \ldots, z_{jk})^{\varepsilon_j}. \tag{7}$$

Therefore

$$G \models \exists z_{11} \ldots \exists z_{sk}. v(g_1, \ldots, g_d) = \prod_{j=1}^{s} w(z_{j1}, \ldots, z_{jk})^{\varepsilon_j},$$

so there exist $\mathbf{z}_1, \ldots, \mathbf{z}_s \in H^{(k)}$ and $b \in D \cap H$ such that

$$h = v(\mathbf{h}_1, \ldots, \mathbf{h}_d) = \prod_{j=1}^{s} w(\mathbf{z}_j)^{\varepsilon_j} \cdot b. \tag{8}$$

Since $w(H_n) = H_{n,w}^{*f}$ for each n, by considering the H_n-components separately we find $\mathbf{t}_1, \ldots, \mathbf{t}_f \in H^{(k)}$ such that $\prod_{j=1}^{s} w(\mathbf{z}_j)^{\varepsilon_j} = \prod_{j=1}^{f} w(\mathbf{t}_j)^{\eta_j}$ (for some $\eta_j = \pm 1$). Thus (7) holds with $s = f$ and η_j for ε_j, and therefore so does (8).

Thus $H \cap w(M)N \subseteq H_w^{*f}(H \cap D)$. The other inclusions in (6) are immediate. ■

Proposition 8 [OP] *If G is abelian or has finite exponent then G is finite.*

Theorem 9 [OP] (i) *$G/G^{(l)}$ is finite for each $l \in \mathbb{N}$;*
(ii) *G/G^q is finite for each $q \in \mathbb{N}$.*

Proof. The words $[x, y]$ and x^q are good by [NS]. Therefore G' and G^q are definable subgroups of G, by Proposition 7 and Lemma 5. Hence G/G' and G/G^q are again pseudo-finite, so they are finite. It also follows that G' is pseudo-finite and f.g.; supposing inductively that $G'/G'^{(l-1)}$ is finite we infer that $G/G^{(l)}$ is finite. ■

2.1 Profinite completion

Write $X = H \cap D$; this is the kernel of $\pi : H \twoheadrightarrow G$, and we will sometimes identify G with H/X.

Lemma 10 *For each $q \in \mathbb{N}$ we have*

$$M/M^q N \cong G/G^q$$
$$H \cap M^q N = H^q X$$
$$H M^q N = M.$$

Proof. Recall that M/N is elementarliy equivalent to G. It follows using Lemma 5, Proposition 7 and Theorem 9 that $M/M^qN \cong G/G^q$, a finite group. The second claim follows from (6) on taking $w = x^q$. This now implies the third claim since $H/H^qX \cong G/G^q$. ■

Lemma 11 *H/H^q is finite for each $q \in \mathbb{N}$.*

Proof. $H/H^qX \cong G/G^q$ is finite, by Theorem 9. Therefore H^qX is finitely generated. But $X \leq D$ is locally finite, so H^qX/H^q is finite. ■

Write
$$M(n) = 1 \times \prod_{j>n} M_j \leq M.$$
For $q \in \mathbb{N}$ we write $H_{(q)} = \{a^q \mid a \in H\} = H_{v_q}$ where $v_q(x) = x^q$.

Lemma 12 *Let $q \in \mathbb{N}$. Then there exists $n = n(q)$ such that*
$$H \cap M(n) \leq H^qX.$$

Proof. SInce H/H^q is finite, there exists r such that $H^qX = H^qD_r$, where $D_r = H \cap M_1 \times \cdots \times M_r$. Now suppose that
$$h \in \bigcap_{n\in\mathbb{N}} (H \cap M(n))H^qD_r.$$
Then for each $n > r$ we have $h_n \in H_n^q = H_{n,(q)}^{*f} \subseteq M_{n,(q)}^{*f}$ where $f = f(v_q)$, so
$$h \in H \cap M_{(q)}^{*f}N = H^qX = H^qD_r$$
by (6). Thus the descending chain $(H \cap M(n))H^qD_r$ $(n \in \mathbb{N})$ intersects in H^qD_r. As H^qD_r has finite index in H, this chain stabilizes at some point $n = n(q)$, and the result follows. ■

Now $M = \prod M_n$ is a profinite group, with the product topology. The closure of a subset S in M is
$$\overline{S} = \bigcap_{n\in\mathbb{N}} SM(n).$$
If S is a subgroup, the inclusion $S \to M$ induces an epimorphism $\lambda_S : \widehat{S} \to \overline{S}$.

Proposition 13 *$\overline{X} \lhd \overline{H}$, and the map λ_H induces an isomorphism*
$$\frac{\widehat{H}}{X} \to \frac{\overline{H}}{\overline{X}}.$$

Proof. Since H^q has finite index in H for each q, Lemma 12 implies that the system

$$\frac{(H \cap M(n))X}{X} \quad (n \in \mathbb{N})$$

is cofinal with the system of all normal subgroups of finite index in $H/X = G$. Our claim is a formal consequence of this.

To spell this out, choose integers q_n so that $H_n^{q_n} = 1$ and q_n is a multiple of both q_{n-1} and n. Then the projection $H \to H_n$ induces an epimorphism $\rho_n : H/H^{q_n} \to H_n$, and together these induce an epimorphism

$$\rho : P = \prod H/H^{q_n} \to \prod H_n = L.$$

We may identify $\widehat{H}$ with the closure of the image of H in P, and $\overline{H}$ is the closure of H in L. Let $\widetilde{X}$ denote the closure of the image of X in P, so $\widetilde{X}$ is the kernel of the natural epimorphism from $\widehat{H}$ to $\widehat{G}$. As a morphism of profinite groups, ρ is both continuous and a closed map; it follows that $\widehat{H}\rho = \overline{H}$ and $\widetilde{X}\rho = \overline{X}$. Thus to establish the claim it remains to show that $\rho^{-1}(\overline{X}) \subseteq \widetilde{X}$.

An element x of P lies in $\widehat{H}$, respectively $\widetilde{X}$, if and only if

$$x = (h(n)H^{q_n}) \tag{9}$$

where the $h(n) \in H$, respectively X, satisfy $h(j)H^{q_n} = h(n)H^{q_n}$ whenever $j \geq n$. Suppose now that $x \in \widehat{H}$ is of the form (9) and $x\rho \in \overline{X}$. Then $x\rho \in XM(j)$ for every j, so for each j there exists $b(j) \in X$ such that

$$h(n)_n = (x\rho)_n = b(j)_n \quad \forall n \leq j.$$

Now given $m \in \mathbb{N}$, choose $j \geq \max\{m,\ n(q_m)\}$. Then, since $H_n^{q_n} = 1$, we have

$$h(j)_n = h(n)_n = b(j)_n \quad \forall n \leq j,$$

so

$$b(j)^{-1}h(j) \in H \cap M(j) \leq H^{q_m}X.$$

Thus

$$h(m)H^{q_m} = h(j)H^{q_m} = c(m)H^{q_m}$$

with $c(m) \in X$. As this holds for each m we see that $x \in \widetilde{X}$, as required. ∎

Let $\widetilde{N}$ denote the profinite closure of N in M, that is,

$$\widetilde{N} = \bigcap \{K \mid N \leq K \lhd_f M\}.$$

Lemma 14 $\overline{H} \cap \widetilde{N} = \overline{X}$ *and* $\overline{H}.\widetilde{N} = M$.

Proof. Let $q \in \mathbb{N}$. Then $\widehat{G}^q$ is open in $\widehat{G}$ and $\overline{H}^q$ is open in $\overline{H}$, by [NS]. It follows that $\overline{H}^q \overline{X} = \overline{H}^q X$ and hence by Lemma 10 that

$$\begin{aligned} \frac{\overline{H}}{\overline{H}^q X} \cong \frac{\widehat{G}}{\widehat{G}^q} \cong \frac{G}{G^q} \\ \cong \frac{M}{M^q N} = \frac{\overline{H} M^q N}{M^q N} \cong \frac{\overline{H}}{\overline{H} \cap M^q N}. \end{aligned}$$

As $\overline{H}^q X \leq \overline{H} \cap M^q N$ this implies $\overline{H}^q X = \overline{H} \cap M^q N$.

As $M/M^q N$ is finite for each q, we have $\widetilde{N} = \bigcap_{q \in \mathbb{N}} M^q N$. Thus

$$\overline{H} \cap \widetilde{N} = \bigcap_{q \in \mathbb{N}} \overline{H}^q X = \overline{X}.$$

Now let $a \in M$. For each q put $Y_q = \overline{H} \cap a M^q N$. As $M = \overline{H} M^q N$ and $\overline{H} \cap M^q N = \overline{H}^q \overline{X}$, we see that Y_q is precisely one coset of $\overline{H}^q \overline{X}$ in $\overline{H}$. Obviously $Y_q \supseteq Y_{q'}$ whenever q' is a multiple of q. It follows by compactness that $\bigcap_{q \in \mathbb{N}} Y_q$ is nonempty. If b lies in this intersection then $a^{-1} b \in \bigcap_{q \in \mathbb{N}} M^q N = \widetilde{N}$. Thus $a \in b\widetilde{N} \subseteq \overline{H}.\widetilde{N}$. ■

Corollary 15 *The embedding $\beta : G \to \widetilde{M}$ induces an isomorphism*

$$\widehat{G} \to \frac{\overline{H}}{\overline{X}} \cong \frac{\overline{H} N}{\overline{X} N} \cong \frac{M}{\widetilde{N}}.$$

Corollary 16 *Suppose that G is finitely presented.*

(i) *The embedding $\beta : G \to \widetilde{M}$ extends to an embedding of $\widehat{G}$ into $\widetilde{M}$.*

(ii) *A finite set of equations and inequations with parameters in G is solvable in G iff it is solvable in $\widehat{G}$. In particular, G is conjugacy separable.*

Proof. We have seen in Subsection 1.1 that $X = H \cap D = H \cap N = 1$, so the above isomorphism maps $\widehat{G} \to \overline{H} \cong \overline{H} N/N \leq M/N = \widetilde{M}$. Claim (ii) now follows from Corollary 3. ■

3 Soluble radical

3.1 Finite case

We define the formula σ with one free variable by

$$\sigma(x) =: \exists \mathbf{u}, \mathbf{v} \; . \; x = \prod_{i=1}^{56} [x^{u_i}, x^{v_i}].$$

The following is clear:

Lemma 17 *Let G be any group and $h \in G$. If $G \models \sigma(h)$ then $\langle h^G \rangle = \langle h^G \rangle'$, i.e. $\langle h^G \rangle$ is perfect.*

For the rest of this subsection, G is a finite group. $R(G)$ denotes the soluble radical of G.

Theorem 18 (Wilson [W1]) *G is soluble iff $G| = \sigma^*$ where*

$$\sigma^* =: \forall x.(\sigma(x) \to x = 1).$$

Theorem 19 (Grunewald et al, [GG], [G]) *An element g of G belongs to $R(G)$ iff $\langle g^{x_1}, g^{x_2}, g^{x_3}, g^{x_4} \rangle$ is soluble for all $x_1, \ldots, x_4 \in G$.*

(Flavell [F] originally proved this with 10 in place of 4 but without using CFSG.)

Theorem 20 (Nikolov-Segal [NS]) If $G = \langle y_1, \ldots, y_r \rangle$ *and $H \lhd G$ then*

$$[H, G] = \left\{ \prod_{j=1}^{f} \prod_{i=1}^{r} [x_{ij}, y_i][z_{ij}, y_i^{-1}] \mid x_{ij}, z_{ij} \in H \right\},$$

where $f = f(r)$ depends only on r (in fact $f(r) = O(r^3)$).

Define the formula ρ with two free variables by

$$\rho(y, z) =: \exists \mathbf{x}, \mathbf{s}, \mathbf{t} \ . \ y = \prod_{j=1}^{f(4)} \prod_{i=1}^{4} [s_{ij}, z^{x_i}][t_{ij}, z^{-x_i}].$$

Theorem 21 (Wilson [W2]) *An element g of G belongs to $R(G)$ iff*

$$G| = \forall y.(\rho(y, g) \to (\neg\sigma(y) \vee y = 1)).$$

Proof. Suppose $g \in R(G)$. If $h \in G$ and $G| = \rho(h, g)$ then $h \in R(G)$, so $\langle h^G \rangle$ is soluble. Hence if $h \neq 1$ then $\langle h^G \rangle > \langle h^G \rangle'$, which implies $\neg\sigma(h)$.

Suppose $g \notin R(G)$. Then there exist $x_1, \ldots, x_4$ such that $Q = \langle g^{x_1}, g^{x_2}, g^{x_3}, g^{x_4} \rangle$ is not soluble. Then Q' is not soluble. Hence there exists $h \in Q'$ satisfying both $h \neq 1$ and $\sigma(h)$. Taking $G = H = Q$ in Theorem 20 we see that $\rho(h, g)$ holds in G. ∎

Remark. Wilson's proof in 21 was considerably longer, because [NS] was not yet available and he had to rely on slightly weaker results from [NS1].

3.2 Pseudo-finite case

A group G is *hypo-abelian if* $1 \neq N \lhd G$ implies $N' < N$. This holds in particular if G is residually soluble. From Lemma 17 we deduce

Lemma 22 *Suppose that $K \lhd G$ and K is hypo-abelian. Then $1 \neq h \in K$ implies $G| = \neg\sigma(h)$. If G is hypo-abelian then $G| = \sigma^*$.*

Write

$$\mu(z) =: \forall y.(\rho(y,z) \to (\urcorner\sigma(y) \vee y = 1)).$$

Lemma 23 *If $K \lhd G$ and K is hypo-abelian then $K \leq G_\mu$.*

Proof. If $h \in K$, $1 \neq g \in G$ and $G |= \rho(g,h)$ then $g \in K$, so $G |= \urcorner\sigma(g)$. ■

The Fitting subgroup of G is denoted $\mathrm{Fit}(G)$. This is equal to 1 iff G has no non-trivial abelian normal subgroups, a first-order property:

Lemma 24 $\mathrm{Fit}(G) = 1$ *iff* $G |= \zeta$ *where*

$$\zeta =: \forall y.(\forall x.\,([y, y^x] = 1) \to y = 1).$$

Proposition 25 *Let G be a pseudo-finite group, and put $S = G_\mu$.*

(i) *S is a definable characteristic subgroup of G.*

(ii) *S is pseudo-(finite soluble).*

(iii) *G/S is pseudo-finite and* $\mathrm{Fit}(G/S) = 1$.

Proof. According to Theorem 21, $M_\mu = R(M)$ for every finite group M. So if M is finite, we have

$$M |= s(\mu)$$
$$M_\mu |= \sigma^*$$
$$M/M_\mu |= \zeta.$$

In the notation of (4) and (5), the last two statements are equivalent to

$$M |= \sigma^*_\natural(\mu)$$
$$M |= \zeta_\flat(\mu).$$

All of these therefore hold with G in place of M. The first shows that $S = G_\mu$ is a subgroup of G, and it is clearly characteristic.

Claims (ii) and (iii) then follow from Theorem 18 and Lemma 24 by the argument of Lemma 6. ■

Henceforth, let G be a finitely generated pseudo-finite group, and keep the notation of Sections 1 and 2. Let $S = G_\mu$ as above. Put $S_n = R(M_n)$ for each n and set $\widetilde{S} = \prod_{n\in\mathbb{N}} S_n$.

Proposition 26 (i) *SG^q/G^q is soluble for each $q \in \mathbb{N}$.*

(ii) *Suppose that G is residually finite. Then is S is residually finite-soluble, and is the unique maximal hypo-abelian normal subgroup of G.*

Proof. Put $T = \pi^{-1}(S) \lhd H$. I claim that $T = H \cap \widetilde{S}N$. To see this, observe that

$$\begin{aligned} w(\mathbf{h}_1, \ldots, \mathbf{h}_d) \in T &\iff w(g_1, \ldots, g_d) \in S \\ &\iff G |= \mu(w(g_1, \ldots, g_d)) \\ &\iff M_n |= \mu(w(h_{n1}, \ldots, h_{nd})) \ \ \forall_{\mathcal{U}} n \\ &\iff w(h_{n1}, \ldots, h_{nd}) \in S_n \ \ \forall_{\mathcal{U}} n \\ &\iff w(\mathbf{h}_1, \ldots, \mathbf{h}_d) \in \widetilde{S}N, \end{aligned}$$

by Proposition 2.

Now let $q \in \mathbb{N}$. Then $H/H^q X \cong G/G^q$ is finite by Theorem 9. Lemma 10 shows that $M^q N \cap H = H^q X$. It follows that

$$\begin{aligned} \frac{T}{H^q X \cap T} = \frac{H \cap \widetilde{S}N}{H \cap \widetilde{S}N \cap M^q N} &\cong \frac{(H \cap \widetilde{S}N)M^q N}{M^q N} \\ &\leq \frac{\widetilde{S}M^q N}{M^q N} \cong \frac{\widetilde{S}}{\widetilde{S} \cap M^q N}. \end{aligned}$$

As $\widetilde{S}$ is a prosoluble group, every finite quotient of $\widetilde{S}$ is soluble (see [S], 'Deduction of Theorem 1', page 30). Therefore $T/(H^q X \cap T)$ is soluble. This gives (i) since $SG^q/G^q \cong TH^q/XH^q \cong T/(H^q X \cap T)$.

(ii) follows, in view of Lemma 23. ■

4 Pseudo-nilpotent

We assume in this section that G as in Section 1 is pseudo-(finite nilpotent). Then we may choose each M_n to be nilpotent. By Theorem 9,

$$q := |G/G'| < \infty.$$

By Lemma 5 and Proposition 7, this is a first-order property of G, so we may include it in ϕ_1. Then $|M_n/M_n'| = q$ for all n. This now implies that each M_n is a π-group where π is the set of prime factors of q.

Let W denote the set of all words of length at most q in d variables. Then

$$G = W(g_1, \ldots, g_d)G' = W(g_1, \ldots, g_d)G_\gamma^{*f}$$

where f is the width of $\gamma = [x, y]$ in G, which is finite by Proposition 7. This is expressible by a first-order formula $\phi(g_1, \ldots, g_d)$, and then by Proposition 2 we have $M_n |= \phi(h_{1n}, \ldots, h_{dn})$ for all large n. Re-labelling the M_n we may suppose that this holds for every n. Then $M_n = W(h_{1n}, \ldots, h_{dn})M_{n,\gamma}^{*f} \subseteq H_n M_n'$, and as M_n is nilpotent it follows that $M_n = H_n$ for each n. With Corollary 3 this gives

Proposition 27 *G is pseudo (finite d-generator nilpotent π-groups).*

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