3. (a) [4 marks] For $\theta \in [0,1]$, the θ-scheme has the form

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = a\theta \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)a\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + F_j^{m+\theta},$$

where

$$F_j^{m+\theta} := \theta f(x_j, (m+1)\Delta t) + (1-\theta) f(x_j, m\Delta t),$$

for $1 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant M-1$, where $\Delta x = 1/N$, $N \geqslant 2$, and $\Delta t = T/M$, $M \geqslant 1$, and

$$U_j^0 = u_0(j\Delta t), \qquad 0 \leqslant j \leqslant N,$$
$$U_0^{m+1} = U_N^{m+1} = 0.$$

[Note: Unseen extension of bookwork, with $f \not\equiv 0$.] **[4 marks]**

(b) [7 marks]

$$\begin{aligned}(1+2\theta\mu)U_j^{m+1} &= \theta\mu(U_{j+1}^{m+1} + U_{j-1}^{m+1}) + (1-\theta)\mu(U_{j+1}^m + U_{j-1}^m)\\ &\quad + (1-2(1-\theta)\mu)U_j^m + \Delta t F_j^{m+\theta}.\end{aligned}$$

Hence, as $1 - 2(1-\theta)\mu \geqslant 0$, and $\theta \in [0,1]$, we have that

$$\begin{aligned}(1+2\theta\mu)|U_j^{m+1}| &\leqslant 2\theta\mu \max_{0\leqslant j\leqslant N} |U_j^{m+1}| + 2(1-\theta)\mu \max_{0\leqslant j\leqslant N} |U_j^m|\\ &\quad + (1-2(1-\theta)\mu) \max_{0\leqslant j\leqslant N} |U_j^m| + \Delta t \max_{1\leqslant j\leqslant N-1} |F_j^{m+\theta}|\end{aligned}$$

for $1 \leqslant j \leqslant N-1$ (and, trivially, also for $j=0$ and $j=N$, since $U_0^{m+1} = U_N^{m+1} = 0$). Taking the maximum over $j \in \{0,\dots,N\}$, we deduce that

$$\begin{aligned}(1+2\theta\mu)\max_{0\leqslant j\leqslant N}|U_j^{m+1}| &\leqslant 2\theta\mu \max_{0\leqslant j\leqslant N} |U_j^{m+1}| + 2(1-\theta)\mu \max_{0\leqslant j\leqslant N} |U_j^m|\\ &\quad + (1-2(1-\theta)\mu) \max_{0\leqslant j\leqslant N} |U_j^m| + \Delta t \max_{1\leqslant j\leqslant N-1} |F_j^{m+\theta}|.\end{aligned}$$

Equivalently,

$$\max_{0\leqslant j\leqslant N}|U_j^{m+1}| \leqslant \max_{0\leqslant j\leqslant N}|U_j^m| + \Delta t \max_{1\leqslant j\leqslant N-1}|F_j^{m+\theta}|,$$

as required.

[Note: Unseen extension of bookwork, with $f \not\equiv 0$.] **[7 marks]**

(c) [7 marks] Let us write $u_j^m = u(j\Delta x, m\Delta t)$. The consistency error of the method is defined by

$$T_j^{m+\theta} := \frac{u_j^{m+1} - u_j^m}{\Delta t} - \left[a\theta\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)a\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} + F_j^{m+\theta}\right]$$

for $1 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant M-1$. Hence, by rearranging the above expression,

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = a\theta\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)a\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} + F_j^{m+\theta} + T_j^{m+\theta}$$

for $1 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant M-1$. By subtracting from this the definition of the θ-scheme from part (a), we have that

$$\frac{e_j^{m+1} - e_j^m}{\Delta t} = a\theta\frac{e_{j+1}^{m+1} - 2e_j^{m+1} + e_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)a\frac{e_{j+1}^m - 2e_j^m + e_{j-1}^m}{(\Delta x)^2} + T_j^{m+\theta}$$

for $1 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant M-1$, with

$$e_j^0 = 0, \qquad 0 \leqslant j \leqslant N,$$

$$e_0^{m+1} = e_N^{m+1} = 0.$$

Clearly, e_j^{m+1} and U_j^{m+1} satisfy the same recursions with respective right-hand sides $T_j^{m+\theta}$ and $F_j^{m+\theta}$. Hence, from part (b) of the question, after replacing U by e and F by T, respectively, we have that

$$\max_{0\leqslant j\leqslant N} |e_j^{m+1}| \leqslant \max_{0\leqslant j\leqslant N} |e_j^m| + \Delta t \max_{1\leqslant j\leqslant N-1} |T_j^{m+\theta}|, \qquad 0 \leqslant m \leqslant M-1,$$

where $T_j^{m+\theta}$, $1 \leqslant j \leqslant N-1$, is the consistency error of the θ-scheme, and $\max_{0\leqslant j\leqslant N} |e_j^0| = 0$. Summing this over $m = 0, \ldots, p-1$ we deduce that

$$\max_{0\leqslant j\leqslant N} |e_j^p| \leqslant \Delta t \sum_{k=0}^{p-1} \max_{1\leqslant j\leqslant N-1} |T_j^{k+\theta}|, \qquad 1 \leqslant p \leqslant M.$$

It remains to rename p into m to obtain the required result.

[Note: Unseen extension of bookwork, with $f \not\equiv 0$.] **[7 marks]**

(d) [7 marks] For $\theta = 0$, after Taylor series expansion,

$$\begin{aligned} T_j^m &= \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{\Delta t} - a\frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{(\Delta x)^2} \\ &= \left(\frac{\Delta t}{2} u_{tt}(x_j, t_m) - a\frac{(\Delta x)^2}{12} u_{xxxx}(x_j, t_m)\right) + \mathcal{O}((\Delta x)^4 + (\Delta t)^2). \end{aligned}$$

where $x_j = j\Delta x$ and $t_m = m\Delta t$.

As $u_t = au_{xx}$, it follows that $u_{tt} = (au_{xx})_t = a(u_t)_{xx} = a^2 u_{xxxx}$. Therefore,

$$\frac{\Delta t}{2} u_{tt}(x_j, t_m) - a\frac{(\Delta x)^2}{12} u_{xxxx}(x_j, t_m) = \left(\frac{a^2 \Delta t}{2} - a\frac{(\Delta x)^2}{12}\right) u_{xxxx}(x_j, t_m) = 0$$

since $\mu = \frac{a\Delta t}{(\Delta x)^2} = \frac{1}{6}$.

We thus find that $T_j^m = \mathcal{O}((\Delta x)^4 + (\Delta t)^2)$. Then, from the previous part of the question we deduce that $\max_{0\leqslant j\leqslant N} |e_j^m| \leqslant \max_{1\leqslant k\leqslant m} \max_{1\leqslant j\leqslant N-1} |T_j^k| = \mathcal{O}((\Delta x)^4 + (\Delta t)^2)$ for all m, $1 \leqslant m \leqslant M$.

[Note: Unseen.] **[7 marks]**