3. **Note:** *The finite difference approximation of nonlinear parabolic PDEs such as the one in this question is not covered in the lecture notes, but the final problem sheet for the course does contain such an exercise. Therefore the words "Extension of bookwork to unseen nonlinear problem." below should be understood in this sense.*

(a) The explicit Euler finite difference approximation of the initial-boundary-value problem (4) is

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} - (\cos x_j)\,\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} &= \frac{2}{\pi}\arctan U_j^m, && \text{for } j = 1, \dots, N-1 \text{ and } m = 0, \dots, M-1, \\ U_0^{m+1} = 0, \quad U_N^{m+1} &= 0 && \text{for all } m = 0, \dots, M-1, \\ U_j^0 &= u_0(x_j) && \text{for all } j = 0, 1, \dots, N. \end{aligned} \tag{7}$$

[S/N: Extension of bookwork to unseen nonlinear problem.] **[6 marks]**

(b) We let $\mu := \Delta t/(\Delta x)^2$ and rewrite the scheme (7) as

$$U_j^{m+1} = \mu(\cos x_j)(U_{j+1}^m + U_{j-1}^m) + (1 - 2\mu(\cos x_j))U_j^m + \frac{2}{\pi}\Delta t \arctan U_j^m$$

for $j = 1, \dots, N-1$, $m = 0, \dots, M-1$, and the same boundary and initial conditions as in (7) above. Suppose that $\mu \leqslant A := \frac{1}{2}$. Then, $2\mu(\cos x_j) \in [0, 1]$ and $1 - 2\mu(\cos x_j) \in [0, 1]$ for all $j = 0, \dots, N$. Hence, by taking the maximum over all $j \in \{1, \dots, N-1\}$, we have

$$|U_j^{m+1}| \leqslant 2\mu(\cos x_j) \max_{0 \leqslant j \leqslant N} |U_j^m| + (1 - 2\mu(\cos x_j)) \max_{0 \leqslant j \leqslant N} |U_j^m| + \Delta t \qquad \text{for } m = 0, \dots, M-1,$$

because $|\arctan U_j^m| \leqslant \frac{\pi}{2}$. Thus, and because $U_0^{m+1} = 0$ and $U_N^{m+1} = 0$, we have that

$$\max_{0 \leqslant j \leqslant N} |U_j^{m+1}| \leqslant \max_{0 \leqslant j \leqslant N} |U_j^m| + \Delta t \qquad \text{for } m = 0, \dots, M-1.$$

Consequently,

$$\max_{0 \leqslant j \leqslant N} |U_j^m| \leqslant \max_{0 \leqslant j \leqslant N} |U_j^0| + m\,\Delta t \qquad \text{for } m = 1, \dots, M.$$

[S/N: Extension of bookwork to unseen nonlinear problem.] **[6 marks]**

(c) The consistency error of the scheme is defined by

$$\mathcal{T}_j^m := \frac{u_j^{m+1} - u_j^m}{\Delta t} - (\cos x_j)\,\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} - \frac{2}{\pi}\arctan u_j^m,$$

for $j = 1, \dots, N-1$ and $m = 0, \dots, M-1$, where $u_j^m := u(x_j, t_m)$. Letting $e_j^m := u(x_j, t_m) - U_j^m = u_j^m - U_j^m$, subtracting from the definition of the consistency error the definition of the implicit Euler method we arrive at

$$e_j^{m+1} = \mu(\cos x_j)(e_{j+1}^m + e_{j-1}^m) + (1 - 2\mu(\cos x_j))e_j^m + \frac{2}{\pi}\Delta t\left[\arctan u_j^m - \arctan U_j^m\right] + \mathcal{T}_j^m$$

for $j = 1, \dots, N-1$, $m = 0, \dots, M-1$, $e_0^{m+1} = 0$ and $e_N^{m+1} = 0$ for all $m = 0, \dots, M-1$ and $e_j^0 = 0$ for $j = 0, \dots, N$.

Noting that the mapping $u \mapsto \arctan u$ is Lipschitz continuous on the real line with Lipschitz constant 1, we deduce for $\mu = \Delta t/(\Delta x)^2 \leqslant \frac{1}{2}$, as in the previous part of the question, that

$$\begin{aligned}\max_{0\leqslant j\leqslant N} |e_j^{m+1}| &\leqslant \max_{0\leqslant j\leqslant N} |e_j^m| + \frac{2}{\pi}\Delta t \max_{0\leqslant j\leqslant N} |e_j^m| + \Delta t \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m| \\ &\leqslant \left(1 + \frac{2}{\pi}\Delta t\right) \max_{0\leqslant j\leqslant N} |e_j^m| + \Delta t \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m|\end{aligned}$$

for all $m = 0, \dots, M-1$. As $\max_{0\leqslant j\leqslant N} |e_j^0| = 0$, it follows that

$$\max_{0\leqslant j\leqslant N} |e_j^m| \leqslant \left[\left(1 + \frac{2}{\pi}\Delta t\right)^{m-1} + \cdots + \left(1 + \frac{2}{\pi}\Delta t\right) + 1\right] \Delta t \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m|$$

for all $m = 1, \dots, M$. Therefore,

$$\begin{aligned}\max_{0\leqslant j\leqslant N} |e_j^m| &\leqslant \frac{\left(1 + \frac{2}{\pi}\Delta t\right)^m - 1}{\left(1 + \frac{2}{\pi}\Delta t\right) - 1} \Delta t \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m| \\ &= \frac{\pi}{2}\left[\left(1 + \frac{2}{\pi}\Delta t\right)^m - 1\right] \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m| \\ &\leqslant \frac{\pi}{2}\left[\exp\left(\frac{2}{\pi}\Delta t\, m\right) - 1\right] \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m| \\ &\leqslant \frac{\pi}{2}\left[\exp\left(\frac{2}{\pi}T\right) - 1\right] \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m| \quad \text{for all } m \in \{1, \dots, M\}.\end{aligned}$$

Hence,

$$\max_{1\leqslant m\leqslant M} \max_{0\leqslant j\leqslant N} |u(x_j, t_m) - U_j^m| \leqslant C_0 \max_{0\leqslant m\leqslant M-1} \max_{1\leqslant j\leqslant N-1} |\mathcal{T}_j^m|,$$

where $C_0 := \frac{\pi}{2}(\mathrm{e}^{2T/\pi} - 1)$.

[S/N: Extension of bookwork to unseen nonlinear problem.]

Note: Full marks will be given to complete answers that end up with a larger, but otherwise correct, constant C_0.

[6 marks]

(d) By Taylor series expansions of u about the point (x_j, t_m) with remainder terms we have that

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = \frac{\partial u}{\partial t}(x_j, t_m) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \tau_m),$$

where $\tau_m \in (t_m, t_{m+1})$, and

$$\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} = \frac{\partial^2 u}{\partial x^2}(x_j, t_m) + \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_m),$$

where $\xi_j \in (x_{j-1}, x_{j+1})$. Hence, and by recalling the partial differential equation satisfied by u, we have that

$$\begin{aligned}\mathcal{T}_j^m &:= \frac{\partial u}{\partial t}(x_j, t_m) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \tau_m) - (\cos x_j)\left[\frac{\partial^2 u}{\partial x^2}(x_j, t_m) + \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_m)\right] - \frac{2}{\pi}\arctan u_j^m \\ &= \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \tau_m) - (\cos x_j)\left[\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_m)\right]\end{aligned}$$

for $j = 1, \ldots, N-1$ and $m = 0, \ldots, M-1$. Thus,

$$\max_{0 \leqslant m \leqslant M-1} \max_{1 \leqslant j \leqslant N-1} |\mathcal{T}_j^m| \leqslant \frac{\Delta t}{2} M_{2t} + \frac{(\Delta x)^2}{12} M_{4x},$$

where

$$M_{2t} = \max_{(x,t) \in [0,1] \times [0,T]} \left| \frac{\partial^2 u}{\partial t^2}(x,t) \right| \quad \text{and} \quad M_{4x} = \max_{(x,t) \in [0,1] \times [0,T]} \left| \frac{\partial^4 u}{\partial x^4}(x,t) \right|.$$

Inserting this bound on the consistency error into the bound from part (c) we deduce that

$$\max_{1 \leqslant m \leqslant M} \max_{0 \leqslant j \leqslant N} |u(x_j, t_m) - U_j^m| \leqslant C_1 \left(\Delta t + (\Delta x)^2\right),$$

where $C_1 = C_0 \max\left(\frac{1}{2} M_{2t}, \frac{1}{12} M_{4x}\right)$; that is, $C_1 = \frac{\pi}{2}(\mathrm{e}^{2T/\pi} - 1) \max\left(\frac{1}{2} M_{2t}, \frac{1}{12} M_{4x}\right)$.

S: [Refinement of bookwork, requiring truncation of Taylor expansions with explicit remainder terms.]

Note: Full marks will be given to complete answers that end up with a larger, but otherwise correct, constant C_1.

[7 marks]

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