Corotational Hookean models of dilute polymeric fluids: existence of global weak solutions, weak-strong uniqueness, equilibration and macroscopic closure

Tomasz Dębiec[*] Endre Süli[†]

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Abstract

We prove the existence of global weak solutions to the corotational Hookean dumbbell model, a system of PDEs arising in the kinetic theory of dilute polymers, involving the unsteady incompressible Navier–Stokes equations in a bounded domain coupled to a Fokker–Planck type parabolic equation including a centre-of-mass diffusion term, satisfied by the probability density function, modelling the evolution of the configuration of noninteracting polymer molecules in a viscous incompressible solvent. The micro-macro interaction is manifested by the presence of a corotational drag term in the Fokker–Planck equation and the divergence of a polymeric extra-stress tensor on the right-hand side of the Navier–Stokes momentum equation. We also analyse certain properties of weak solutions to this system of PDEs: we use the relative energy method to deduce a weak-strong uniqueness type result, and derive the macroscopic closure of the kinetic model: a corotational Oldroyd-B model with stress-diffusion. Finally, we discuss the existence and uniqueness of global weak solutions to this class of corotational Oldroyd-B models with stress-diffusion.

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1 Introduction

In this paper we study weak solutions to a simple coupled microscopic-macroscopic (micro-macro) model for polymeric fluids. We discuss the existence, conditional uniqueness, and large time behaviour of solutions in d space dimension, $d \in \{2, 3\}$. The presence of polymer molecules in a fluid has substantial macroscopic consequences: in particular, polymer molecules induce elastic properties; thus, the resulting fluid is usually referred to as *viscoelastic*. As such mixtures

[*]Sorbonne Université, Inria, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions UMR7598, F-75005 Paris, France.
[†]Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK.
Email addresses: tomasz.debiec@sorbonne-universite.fr, endre.suli@maths.ox.ac.uk

are ubiquitous in many branches of biology and chemistry, studying coupled models of polymers suspended in a solvent is of great interest. Additionally, these models pose numerous challenges from the point of view of mathematical analysis.

Polymer molecules, being extremely long chains of monomers, are particularly difficult to model. Thus, the starting point for any mathematically tractable mechanical description is to devise a simplified representation of each polymer molecule. We consider a *dumbbell model* wherein each polymer molecule is assumed to consist of two small massless beads connected by an elastic spring. This simple description has its merits, because it accounts both for stretching and rotations of the molecules. It is assumed that the solution is dilute, meaning that there is no interaction between individual polymer chains. An important modelling choice at this stage is the form of the elastic spring-force:

$$F : D \to \mathbb{R}^d, \qquad F\left(\frac{|q|^2}{2}\right) = U'\left(\frac{|q|^2}{2}\right) q,$$

where U is the spring potential. We shall choose $D = \mathbb{R}^d$ and assume that the springs obey Hooke's law, i.e., $U(s) = s$, with the spring constant scaled to 1. Although this choice allows for the polymer molecules to extend indefinitely, which is clearly nonphysical, its simplicity is attractive from the practical point of view. Furthermore, the Hookean potential is the only one for which such a micro-macro model has, at least formally, a macroscopic closure in the form of the well-known Oldroyd-B model [32].

In this paper we are primarily concerned with the following corotational micro-macro system of equations:

$$\begin{aligned} \partial_t u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p &= \operatorname{div}_x \tau, \\ \operatorname{div}_x u &= 0, \\ \partial_t\psi + u\cdot\nabla_x\psi + \operatorname{div}_q(\omega(u)q\psi) - \mu\Delta_x\psi &= \operatorname{div}_q\left(M\nabla_q\left(\frac{\psi}{M}\right)\right), \end{aligned} \tag{1}$$

posed on $(0,T]\times\Omega\times D$, where $\Omega\subset\mathbb{R}^d$ is a bounded open domain and $D = \mathbb{R}^d$ is the conformation domain. We shall mainly focus on the more difficult case $d = 3$, but our arguments work also in the case $d = 2$. By $\omega(u) := \frac{1}{2}(\nabla_x u - \nabla_x^{\mathrm{T}} u)$ we denote the skew-symmetric part of the velocity gradient. The coupling between the Navier–Stokes and Fokker–Planck equations is given by the presence of the *extra stress tensor* $\tau = \tau(\psi)$. It is related to the probability density function, ψ, of dumbbell configurations by the *Kramers expression*

$$\tau(t,x) := \int_D \psi(t,x,q)\, q\otimes q \,\mathrm{d}q - \left(\int_D \psi(t,x,q)\,\mathrm{d}q\right)\mathrm{Id}\,.$$

Thus, τ is a symmetric tensor. The factor M, present in the last term of the Fokker–Planck equation $(1)_3$, is the normalised Maxwellian defined by

$$M(q) := \frac{1}{\mathcal{Z}}\mathrm{e}^{-\frac{|q|^2}{2}}, \quad \mathcal{Z} := \int_D \mathrm{e}^{-\frac{|q|^2}{2}}\,\mathrm{d}q.$$

We will repeatedly use the following crucial properties of the Maxwellian:

$$\nabla_q M = -Mq, \quad \int_D M(q)|q|^r\,\mathrm{d}q < \infty, \quad \sup_{q\in D} M^\alpha |q|^r < \infty, \;\text{ for any } \alpha > 0,\, r \geq 0.$$

Because M serves as a natural weight-function for the solution of the Fokker–Planck equation, we will systematically consider the Maxwellian-normalised probability density $\hat\psi$ and its initial value $\hat\psi_0$, defined by

$$\hat\psi := \frac{\psi}{M}, \quad \hat\psi_0 := \frac{\psi_0}{M}.$$

Consequently, it is useful to rewrite the Fokker–Planck equation as an equation for $\hat{\psi}$:

$$M\partial_t\hat{\psi} + Mu\cdot\nabla_x\hat{\psi} + \mathrm{div}_q\Big(\omega(u)qM\hat{\psi}\Big) - \mu M\Delta_x\hat{\psi} = \mathrm{div}_q\Big(M\nabla_q\hat{\psi}\Big). \tag{2}$$

We supplement the above system of equations with the following initial and boundary/decay conditions:

$$\begin{aligned} u &= 0 && \text{on } (0,T]\times\partial\Omega,\\ u(0,x) &= u_0(x) && \forall\, x\in\Omega \end{aligned}$$

and

$$\begin{aligned} &\left|M\left[\nabla_q\left(\frac{\psi}{M}\right) - \omega(u)q\frac{\psi}{M}\right]\right| \to 0 && \text{as } |q|\to\infty \ \text{ on } (0,T]\times\partial\Omega,\\ &\nabla_x\psi\cdot\hat{n} = 0 && \text{on } (0,T]\times\partial\Omega\times D,\\ &\psi(0,x,q) = \psi_0(x,q) && \forall\,(x,q)\in\Omega\times D. \end{aligned}$$

Here $\hat{n}$ denotes the outward unit normal vector to $\partial\Omega$. If ψ_0 is such that

$$\psi_0 \geq 0, \qquad \int_D \psi_0(x,q)\,\mathrm{d}q = 1 \ \text{ for a.e. } x\in\Omega,$$

then the above boundary/decay conditions guarantee that these properties are propagated to all future time.

It is virtually impossible to give a self-contained review of mathematical results concerning different models of polymeric fluids. The corotational model (1) considered herein is a simplified version of the general noncorotational Hookean model, where, instead of $\omega(u)$, the full velocity gradient $\nabla_x u$ is present in the Fokker–Planck equation. The latter, general noncorotational model, is notoriously difficult to analyse primarily because of the lack of a sufficiently strong *a priori* bound on the extra stress tensor τ. Global existence of weak solutions is known only in two spatial dimensions: it was first shown by Barrett and Süli (under suitable regularity assumptions on the initial data) by exploiting the connection of the model to the macroscopic Oldroyd-B model, see [9]; the result was recently extended to a larger class of data by La [25], who introduced the concept of moment solutions to the Fokker–Planck equation. More complete existence theory is available for models with different spring potentials: for instance, Hookean-type models with light-tailed Maxwellians (i.e., with super-Gaussian Maxwellians that exhibit more rapid decay to 0 as $|q|\to\infty$ than the Gaussian Maxwellian featuring in the classical Hookean model considered herein) and finitely-extensible nonlinear elastic (FENE) models, see [4–7, 24, 27, 29, 30] and references therein, where the Maxwellian is compactly supported in $\mathbb{R}^d$. The latter model corresponds to the situation where the configuration domain, D, is bounded, and the elastic spring-potential modelling the extension of polymer molecules is given by

$$U(s) = -\frac{b}{2}\ln\left(1-\frac{2s}{b}\right), \quad s\in\left[0,\frac{b}{2}\right), \quad b>2.$$

Thus, the function $q\in D\mapsto U(|q|^2/2)\in\mathbb{R}_{\geq 0}$ blows up at the boundary ∂D of $D=B(0,\sqrt{b})$, which then ensures that, unlike in the case of Hookean and Hookean-type models, polymer molecules in FENE models do not stretch indefinitely. Existence of large-data global weak solutions to a general class of FENE-type models with centre-of-mass diffusion, $\mu>0$, and the equilibration of weak solutions as $t\to+\infty$ was proved by Barrett & Süli in [4]. Global existence of weak solutions for the nondiffusive (i.e., when $\mu = 0$) FENE model was subsequently proved in the important paper of Masmoudi [30].

We shall also consider some macroscopic models for polymeric flows (in the corotational case), which have also been studied extensively in the past. In the two-dimensional case with stress-diffusion, existence of weak solutions was shown by Barrett and Boyaval in [1]. Global regularity was then shown by Constantin and Kliegl in [14]. For the nondiffusive case, local well-posedness was shown in critical Besov spaces by Chemin and Masmoudi [13] under a Beale–Kato–Majda type assumption. Hu and Lin proved global existence of weak solutions under the assumption that the initial velocity is small and its gradient is close to an identity matrix [23]; see also [15, 17, 26].

While not completely satisfactory from the physical point of view, the corotational variants of both the micro-macro and fully macroscopic models have been widely discussed in the literature. In [28], Lions and Masmoudi showed global existence of weak solutions for the corotational Oldroyd-B model without stress-diffusion. For the nondiffusive mixed-scale models global well-posedness is known in two space dimensions for the FENE model [29] and the Hookean model [31]. In the presence of centre-of-mass diffusion, existence of global weak solutions was shown for a general class of FENE models in [3]. Our current contribution is to prove existence of weak solutions for the diffusive corotational Hookean model without a restriction on the dimension (i.e. for both $d = 2$ and $d = 3$).

Outline of the paper. In Section 2 we define weak solutions of system (1) and prove global existence (Theorem 2.3) and the property of weak sequential compactness (Theorem 2.6). In Section 3 we prove, by means of the relative energy method, a weak-strong uniqueness result for solutions of system (1). In Section 4 we show that any weak solution of (1) defines a weak solution of a corresponding Oldroyd-B type macroscopic model. Then, in Section 5 we use the energy estimates from the preceding sections to deduce certain long-time asymptotic results for the weak solutions constructed in Theorem 2.3. Finally, in Section 6 we discuss existence and weak-strong uniqueness of solutions for the corotational macroscopic models. We close with some concluding remarks on the corotationality assumption in Section 7.

2 Corotational Hookean dumbbell model: existence of weak solutions

In this section we will prove the existence of global weak solutions to the corotational Hookean dumbbell model (aka Navier–Stokes–Fokker–Planck system) (1).

Before stating the definition of a weak solution, let us introduce some notation regarding the function spaces that we will be working with. Let

$$L^2_{\mathrm{div}}(\Omega) := \Big\{ v \in L^2(\Omega;\mathbb{R}^d) \mid \mathrm{div}_x\, v = 0 \Big\}, \quad V := \Big\{ v \in H^1_0(\Omega;\mathbb{R}^d) \mid \mathrm{div}_x\, v = 0 \Big\},$$

and let $L^p_M(\Omega \times D)$, $p \in [1,\infty)$, denote the Maxwellian-weighted L^p space with the norm

$$\|\phi\|_{L^p(\Omega\times D)} := \left(\int_{\Omega\times D} M|\phi|^p \,\mathrm{d}q\,\mathrm{d}x \right)^{1/p}.$$

Similarly, we introduce the weighted Sobolev space $H^1_M(\Omega \times D)$ by defining

$$H^1_M(\Omega\times D) := \Big\{ \phi \in L^1_{\mathrm{loc}}(\Omega\times D) \mid \|\phi\|_{H^1_M(\Omega\times D)} < \infty \Big\},$$

$$\|\phi\|_{H^1_M(\Omega\times D)} := \left\{ \int_{\Omega\times D} M\big[|\phi|^2 + |\nabla_x \phi|^2 + |\nabla_q \phi|^2\big] \,\mathrm{d}q\,\mathrm{d}x \right\}^{1/2}.$$

Of course, similar definitions can be used to define other M-weighted Lebesgue and Sobolev spaces. Let us also point out that, clearly, the Maxwellian-weighted spaces contain their unweighted counterparts (via a continuous embedding). We recall from [5] the following density and embedding results:

$$
\begin{aligned}
&C^\infty(\bar{\Omega}; C_c^\infty(D)) \text{ is dense in } H^1_M(\Omega \times D),\\
&H^1_M(\Omega \times D) \hookrightarrow\hookrightarrow L^2_M(\Omega \times D).
\end{aligned}
$$

Furthermore, we introduce the mixed regularity spaces

$$
\begin{aligned}
&W_M^{(k,l),p}(\Omega \times D)\\
&:= \Big\{\phi \in L^1_{\mathrm{loc}}(\Omega \times D) \mid \partial_x^\alpha \phi \in L^p_M(\Omega \times D),\ 0 \le |\alpha| \le k,\ \ \partial_q^\beta \phi \in L^p_M(\Omega \times D),\ \ 0 \le |\beta| \le l\Big\}
\end{aligned}
$$

and

$$
Y := \Big\{\phi \in L^1_{\mathrm{loc}}(\Omega \times D) \mid \nabla_q \phi \in W_M^{(2,0),4}(\Omega \times D),\ \ \nabla_x \phi \in L^4_M(\Omega \times D)\Big\},
$$

with the homogeneous seminorm

$$
\|\phi\|_Y := \|\nabla_x \phi\|_{L^4_M(\Omega\times D)} + \sum_{1 \le |\alpha| \le 2} \|\partial_x^\alpha (\nabla_q \phi)\|_{L^4_M(\Omega\times D)}.
$$

Given a normed space X, we denote by $M^{-1}X$ the space of those functions ϕ for which $M\phi \in X$. By X' we denote the continuous dual of the space X. Given a time $t > 0$, we define $\Omega_t := (0,t) \times \Omega$.

Definition 2.1 (Weak solution)**.** Let $d \in \{2,3\}$, and let $(u_0, \hat{\psi}_0)$ be initial data such that

$$
\begin{aligned}
&u_0 \in L^2_{\mathrm{div}}(\Omega), \quad \hat{\psi}_0 \in L^p_M(\Omega \times D) \quad \text{for some } p \in (2,\infty),\\
&\hat{\psi}_0 \ge 0, \quad \int_D M\hat{\psi}_0 \,\mathrm{d}q = 1 \quad \text{for a.e. } x \in \Omega.
\end{aligned}
$$

A pair $(u, \hat{\psi})$ is a weak solution of the corotational Navier–Stokes–Fokker–Planck system if

$$
\begin{gathered}
u \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;V) \cap W^{1,\frac{4}{3}}(0,T;V')\\
\hat{\psi} \in L^\infty(0,T; L^p_M(\Omega \times D)) \cap L^2(0,T; H^1_M(\Omega \times D)) \cap H^1(0,T; M^{-1}Y'),\\
\hat{\psi} \ge 0 \text{ a.e. in } [0,T] \times \Omega \times D, \qquad \int_D M(q)\hat{\psi}(t,x,q)\,\mathrm{d}q = 1 \text{ for a.e. } (t,x) \in [0,T] \times \Omega,
\end{gathered}
$$

and the equations are satisfied in the following weak sense for each $t \in [0,T)$:

$$
\begin{aligned}
&\int_{\Omega_t} u \cdot \partial_t \vartheta \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (u \cdot \nabla_x) u \cdot \vartheta \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_{\Omega_t} \nabla_x u : \nabla_x \vartheta \,\mathrm{d}x\,\mathrm{d}t'\\
&\quad = \int_{\Omega_t} \tau : \nabla_x \vartheta \,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega u(t)\cdot\vartheta(t)\,\mathrm{d}x - \int_\Omega u_0 \cdot \vartheta(0)\,\mathrm{d}x\\
&\qquad\qquad \forall\, \vartheta \in \Big\{\vartheta \in L^2(0,T;V) \mid \partial_t \vartheta \in L^1(0,T;L^2(\Omega;\mathbb{R}^d))\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Omega_t}\int_D M\hat{\psi}\partial_t\phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega\times D} M\hat{\psi}(0)\phi(0)\,\mathrm{d}q\,\mathrm{d}x - \int_{\Omega\times D} M\hat{\psi}(t)\phi(t)\,\mathrm{d}q\,\mathrm{d}x\\
&\quad = -\int_{\Omega_t}\int_D M\hat{\psi} u \cdot \nabla_x \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t}\int_D M\hat{\psi}\omega(u) q \cdot \nabla_q \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\\
&\qquad + \mu \int_{\Omega_t}\int_D M\nabla_x\hat{\psi}\cdot\nabla_x\phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t}\int_D M\nabla_q\hat{\psi}\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\\
&\qquad\quad \forall\, \phi \in \big\{\phi \in W^{1,1}(0,T; L^2_M(\Omega\times D)) \mid \exists \epsilon > 0 : \phi \in L^{2+\epsilon}(0,T;Y)\big\}.
\end{aligned}
$$

Remark 2.2. As usual, we tacitly assume that the velocity field has been modified, if necessary, on a null set, so as to make the weak form of the Navier–Stokes equation in the above definition valid for each time. (Furthermore, u is weakly continuous as mapping from $[0,T]$ into $L^2(\Omega)$.) Since $\hat{\psi} \in L^2(0,T;H^1_M(\Omega \times D))$ (which densely embeds into $L^2(0,T;L^2_M(\Omega \times D))$), and $\partial_t\hat{\psi} \in L^2(0,T;[H^s(\Omega \times D)]')$ for $s > d+1$, we have that $\hat{\psi} \in C([0,T];L^2_M(\Omega \times D))$. (Indeed, note that $L^2_M(\Omega \times D) \equiv [L^2_M(\Omega \times D)]'$ is densely embedded into $[H^s(\Omega \times D)]'$.) We note also that since we have used divergence-free test functions in the weak formulation of the Navier–Stokes equation, the pressure has been eliminated from the weak formulation. In can be recovered in a weak sense by solving a Poisson equation, similarly as for the incompressible Navier–Stokes system.

The main objective of this section is to prove the following existence result.

Theorem 2.3. *Let $d \in \{2,3\}$. There exists a weak solution (as defined in Definition 2.1) of the corotational Navier–Stokes–Fokker–Planck system (1). Furthermore, this solution satisfies the following energy inequality:*

$$\begin{aligned}\int_\Omega \frac{1}{2}|u|^2\,\mathrm{d}x + \int_{\Omega\times D} \frac{1}{2}M\hat{\psi}^2\,\mathrm{d}q\,\mathrm{d}x + \int_0^t \int_{\Omega\times D} M\Big(\mu|\nabla_x\hat{\psi}|^2 + |\nabla_q\hat{\psi}|^2\Big)\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ + \nu\int_{\Omega_t} |\nabla_x u|^2\,\mathrm{d}x\,\mathrm{d}t' \leq \int_\Omega \frac{1}{2}|u_0|^2\,\mathrm{d}x + \int_{\Omega\times D}\frac{1}{2}M\hat{\psi}_0\,\mathrm{d}q\,\mathrm{d}x - \int_0^t\int_\Omega \tau : \nabla_x u\,\mathrm{d}x\,\mathrm{d}t'.\end{aligned} \tag{3}$$

We shall refer to weak solutions satisfying (3) as *dissipative* weak solutions. Let us point out that it is not clear, even in two space dimensions, whether all weak solutions are dissipative. This is because the transport coefficients in the Fokker–Planck equation depend on the velocity gradient, which, for an arbitrary weak solution, only belongs to $L^2(\Omega_T)$.

2.1 *A priori* estimates

We begin our analysis of the Navier–Stokes–Fokker–Planck model (1) by deriving the necessary *a priori* estimates which we will then use to prove weak sequential compactness and apply these weak compactness results to Galerkin approximations. We shall assume to be working with a sequence (u_n, ψ_n) of weak solutions and therefore refer to our estimates as uniform (in n); however, the subscript n is omitted for notational convenience.

Energy estimate
The system (1) comes with a natural energy inequality: testing formally the Navier–Stokes equation with u and the Fokker–Planck equation with $\hat{\psi}$, we obtain, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|u|^2\,\mathrm{d}x + \nu\int_\Omega |\nabla_x u|^2\,\mathrm{d}x = -\int_\Omega \tau : \nabla_x u\,\mathrm{d}x \tag{4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times D}\frac{1}{2}M\hat{\psi}^2\,\mathrm{d}q\,\mathrm{d}x + \mu\int_{\Omega\times D} M|\nabla_x\hat{\psi}|^2\,\mathrm{d}q\,\mathrm{d}x + \int_{\Omega\times D} M|\nabla_q\hat{\psi}|^2\,\mathrm{d}q\,\mathrm{d}x = 0, \tag{5}$$

where we have used the following easily verifiable facts:

$$\mathrm{div}_q(\omega(u)q) = 0, \quad \omega(u) : q\otimes q = 0,$$

$$\int_{\Omega\times D} M\hat{\psi}\,\omega(u)\,q\cdot\nabla_q\hat{\psi}\,\mathrm{d}q\,\mathrm{d}x = \frac{1}{2}\int_{\Omega\times D} M\omega(u)\,q\cdot\nabla_q\hat{\psi}^2\,\mathrm{d}q\,\mathrm{d}x = \frac{1}{2}\int_{\Omega\times D} M\hat{\psi}^2\omega(u) : q\otimes q\,\mathrm{d}q\,\mathrm{d}x = 0.$$

It is in the calculations in the last two lines above where replacement of $\nabla_x u$, featuring in the general noncorotational Fokker–Planck equation, with $\omega(u)$ in the corotational model, plays a crucial role. Next, we observe the following estimates:

$$\begin{aligned}\int_\Omega \tau : \nabla_x u \, \mathrm{d}x &\le \delta \|\nabla_x u\|^2_{L^2(\Omega)} + C_\delta \int_\Omega \left| \int_D M \hat{\psi} q \otimes q \, \mathrm{d}q \right|^2 \mathrm{d}x \\ &\le \delta \|\nabla_x u\|^2_{L^2(\Omega)} + C_\delta \int_\Omega \left(\int_D M|q|^4 \, \mathrm{d}q \right) \left(\int_D M\hat{\psi}^2 \, \mathrm{d}q \right) \mathrm{d}x \\ &\le \delta \|\nabla_x u\|^2_{L^2(\Omega)} + C_\delta \int_{\Omega \times D} M\hat{\psi}^2 \, \mathrm{d}q \, \mathrm{d}x.\end{aligned}$$

Therefore, by choosing $\delta > 0$ small enough and adding (4) to (5), we deduce the following bound:

$$\begin{aligned}&\int_\Omega \frac{1}{2}|u|^2 \, \mathrm{d}x + \int_{\Omega \times D} \frac{1}{2} M\hat{\psi}^2 \, \mathrm{d}q \, \mathrm{d}x + \frac{\nu}{2} \int_0^t \int_\Omega |\nabla_x u|^2 \, \mathrm{d}x \, \mathrm{d}t' \\ &\quad + \int_0^t \int_{\Omega \times D} M \Big(\mu |\nabla_x \hat{\psi}|^2 + |\nabla_q \hat{\psi}|^2 \Big) \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \le \frac{1}{2} \int_\Omega |u_0|^2 \, \mathrm{d}x + \frac{C}{2} \int_{\Omega \times D} M\hat{\psi}_0^2 \, \mathrm{d}q \, \mathrm{d}x.\end{aligned}$$

It follows that

$$u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V)$$

and

$$\hat{\psi} \in L^\infty(0,T; L^2_M(\Omega \times D)) \cap L^2(0,T; H^1_M(\Omega \times D)).$$

Positivity of $\hat{\psi}$

Integrating the Fokker–Planck equation we readily observe that the following conservation property holds:

$$\int_{\Omega \times D} M(q)\hat{\psi}(t,x,q) \, \mathrm{d}q \, \mathrm{d}x = \int_{\Omega \times D} M(q)\hat{\psi}_0(x,q) \, \mathrm{d}q \, \mathrm{d}x \quad \forall\, t \in [0,T].$$

Similarly, one can see that the equation propagates nonnegativity of the initial datum $\hat{\psi}_0$. To this end, we test the Fokker–Planck equation by the nonpositive part $\hat{\psi}_- = \min(0, \hat{\psi})$ of $\hat{\psi}$ and integrate, to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega \times D} M(\hat{\psi}_-)^2 \, \mathrm{d}q \, \mathrm{d}x + \int_{\Omega \times D} M \Big(\mu |\nabla_x \hat{\psi}_-|^2 + |\nabla_q \hat{\psi}_-|^2 \Big) \mathrm{d}q \, \mathrm{d}x = 0,$$

which implies that $\hat{\psi}_- = 0$ a.e. in $\Omega \times D$.

Additional *a priori* estimates for $\hat{\psi}$

A key property of the corotational model, used already to derive the L^2_M estimate for $\hat{\psi}$, is that the integral involving the vorticity tensor vanishes when tested against a large family of functions of $\hat{\psi}$. Indeed, for any differentiable function H, we have, thanks to the skew-symmetry of $\omega(u)$ and the symmetry of $q \otimes q$, that

$$\int_{\Omega \times D} M\omega(u)q \cdot \nabla_q H(\hat{\psi}) \, \mathrm{d}q \, \mathrm{d}x = \int_{\Omega \times D} MH(\hat{\psi})\omega(u) : q \otimes q \, \mathrm{d}q \, \mathrm{d}x = 0.$$

This is in stark contrast with the general noncorotational model, for which even L^2_M bounds on $\hat{\psi}$ are unavailable. For the corotational model under consideration here, we make note of the following L^p bound, $p > 2$. Suppose that

$$\int_{\Omega\times D} M\hat{\psi}_0^p \,\mathrm{d}q\,\mathrm{d}x < \infty.$$

Testing the equation (2) with $\hat{\psi}^{p-1}$ we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times D} M\hat{\psi}^p \,\mathrm{d}q\,\mathrm{d}x + (p-1)\int_{\Omega\times D} M\hat{\psi}^{p-2}\Big(\mu|\nabla_x\hat{\psi}|^2 + |\nabla_q\hat{\psi}|^2\Big)\,\mathrm{d}q\,\mathrm{d}x = 0.$$

Therefore $\hat{\psi}^p$ is bounded in $L^\infty(0,T;L^1_M(\Omega\times D))$.

We observe immediately one useful implication of the above p-estimate. Using the logarithmic Sobolev inequality together with the Fenchel–Young inequality we can see that the function $|q|\hat{\psi}$ is bounded in $L^2(0,T;L^2_M(\Omega\times D))$. With the additional integrability of $\hat{\psi}$, we can upgrade this bound to an L^∞ bound in time. Indeed, Hölder's inequality gives

$$\int_{\Omega\times D} M|q|^2\hat{\psi}^2\,\mathrm{d}q\,\mathrm{d}x \le \left(\int_{\Omega\times D} M\hat{\psi}^p\,\mathrm{d}q\,\mathrm{d}x\right)^{\frac{2}{p}}\left(|\Omega|\int_D M|q|^{\frac{2p}{p-2}}\,\mathrm{d}q\right)^{\frac{p-2}{p}}.$$

Estimate for the time derivative $\partial_t\hat{\psi}$

The last ingredient required in order to obtain strong compactness of $\hat{\psi}$ in a Maxwellian-weighted space is the following estimate for the time derivative.

Lemma 2.4. *We have the following estimate*

$$\left|\int_{\Omega_t}\int_D M\partial_t\hat{\psi}\,\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right| \lesssim \|\phi\|_{L^2(0,T;Y)} \qquad \forall\,\phi\in L^2(0,T;Y).$$

Proof. First, we observe the following straightforward bounds:

$$\int_{\Omega_t}\int_D M\nabla_x\hat{\psi}\cdot\nabla_x\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \le \left\|M^{1/2}\nabla_x\phi\right\|_{L^2(0,T;L^2(\Omega\times D))}\left\|M^{1/2}\nabla_x\hat{\psi}\right\|_{L^2(0,T;L^2(\Omega\times D))},$$

$$\int_{\Omega_t}\int_D M\nabla_q\hat{\psi}\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \le \left\|M^{1/2}\nabla_q\phi\right\|_{L^2(0,T;L^2(\Omega\times D))}\left\|M^{1/2}\nabla_q\hat{\psi}\right\|_{L^2(0,T;L^2(\Omega\times D))}$$

and

$$\int_{\Omega_t}\int_D M\hat{\psi}u\cdot\nabla_x\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \le \left\|M^{1/4}\nabla_x\phi\right\|_{L^2(0,T;L^4(\Omega\times D))}\left\|M^{1/4}u\right\|_{L^2(0,T;L^4(\Omega\times D))}\left\|M^{1/2}\hat{\psi}\right\|_{L^\infty(0,T;L^2(\Omega\times D))}.$$

Thus, each of the above terms can be bounded by the norm of ϕ in $L^2(0,T;Y)$ (indeed, clearly $L^4_M(\Omega\times D)\hookrightarrow L^2_M(\Omega\times D)$).

Finally, we consider the, most delicate, q-advection term. Using the Cauchy–Schwarz inequality we deduce the following bound:

$$\begin{aligned}\int_{\Omega_t}\int_D M\hat{\psi}\,\omega(u)\,q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &\le \int_{\Omega_t}|\nabla_x u|\left(\int_D M\hat{\psi}|q||\nabla_q\phi|\,\mathrm{d}q\right)\mathrm{d}x\,\mathrm{d}t'\\ &\le \left(\int_{\Omega_t}|\nabla_x u|^2\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/2}\left(\int_{\Omega_t}\left(\int_D M\hat{\psi}|q||\nabla_q\phi|\,\mathrm{d}q\right)^2\mathrm{d}x\,\mathrm{d}t'\right)^{1/2}\\ &\le C\left(\int_{\Omega_t}\left(\int_D M|\nabla_q\phi|^2\,\mathrm{d}q\right)\left(\int_D M\hat{\psi}^2|q|^2\,\mathrm{d}q\right)\mathrm{d}x\,\mathrm{d}t'\right)^{1/2}.\end{aligned} \tag{6}$$

Let us now investigate the function

$$F(t) : x \longmapsto \int_D M(q)|\nabla_q \phi(t,x,q)|^2 \,\mathrm{d}q.$$

We claim that for each fixed time $t > 0$ this mapping belongs to $H^2(\Omega)$. Clearly, it belongs to $L^2(\Omega)$:

$$\int_\Omega \left| \int_D M|\nabla_q \phi|^2 \,\mathrm{d}q \right|^2 \mathrm{d}x \leq \left(\int_D M \,\mathrm{d}q \right) \left(\int_{\Omega \times D} M|\nabla_q \phi|^4 \,\mathrm{d}q \,\mathrm{d}x \right),$$

so it only remains to estimate the $L^2(\Omega)$ norm of its second spatial derivatives. To this end, we note that

$$\begin{aligned}
&\int_\Omega \left| \frac{\partial^2}{\partial x_i \partial x_j} \int_D M|\nabla_q \phi|^2 \,\mathrm{d}q \right|^2 \mathrm{d}x \\
&\quad = 4 \int_\Omega \left| \int_D M\Big(\partial_{x_i}(\nabla_q \phi) \cdot \partial_{x_j}(\nabla_q \phi) + \nabla_q \phi \cdot \partial^2_{x_i x_j}(\nabla_q \phi) \Big) \,\mathrm{d}q \right|^2 \mathrm{d}x \\
&\quad \lesssim \int_\Omega \left| \int_D M\partial_{x_i}(\nabla_q \phi) \cdot \partial_{x_j}(\nabla_q \phi) \,\mathrm{d}q \right|^2 \mathrm{d}x + \int_\Omega \left| \int_D M\nabla_q \phi \cdot \partial^2_{x_i x_j}(\nabla_q \phi) \,\mathrm{d}q \right|^2 \mathrm{d}x \\
&\qquad + 2 \int_\Omega \left| \left(\int_D M\partial_{x_i}(\nabla_q \phi) \cdot \partial_{x_j}(\nabla_q \phi) \,\mathrm{d}q \right) \left(\int_D M\nabla_q \phi \cdot \partial^2_{x_i x_j}(\nabla_q \phi) \,\mathrm{d}q \right) \right| \mathrm{d}x.
\end{aligned}$$

Applying Young's inequality to the last term gives additional copies of the first two terms, so it suffices to focus on estimating those. This is achieved by two applications of the Cauchy–Schwarz inequality and Young's inequality, together with the fact that $\int_D M \,\mathrm{d}q = 1$. We then obtain

$$\begin{aligned}
\int_\Omega \left| \int_D M\partial_{x_i}(\nabla_q \phi) \cdot \partial_{x_j}(\nabla_q \phi) \,\mathrm{d}q \right|^2 \mathrm{d}x &\leq \int_\Omega \left(\int_D M|\partial_{x_i}(\nabla_q \phi)|^2 \,\mathrm{d}q \right) \left(\int_D M|\partial_{x_j}(\nabla_q \phi)|^2 \,\mathrm{d}q \right) \mathrm{d}x \\
&\leq \int_{\Omega \times D} M|\partial_{x_i}(\nabla_q \phi)|^4 \,\mathrm{d}q \,\mathrm{d}x + \int_{\Omega \times D} M|\partial_{x_j}(\nabla_q \phi)|^4 \,\mathrm{d}q \,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \left| \int_D M\nabla_q \phi \cdot \partial^2_{x_i x_j}(\nabla_q \phi) \,\mathrm{d}q \right|^2 \mathrm{d}x &\leq \int_\Omega \left(\int_D M|\nabla_q \phi|^2 \,\mathrm{d}q \right) \left(\int_D M|\partial^2_{x_i x_j}(\nabla_q \phi)|^2 \,\mathrm{d}q \right) \mathrm{d}x \\
&\leq \int_{\Omega \times D} M|\nabla_q \phi|^4 \,\mathrm{d}q \,\mathrm{d}x + \int_{\Omega \times D} M|\partial^2_{x_i x_j}(\nabla_q \phi)|^4 \,\mathrm{d}q \,\mathrm{d}x.
\end{aligned}$$

We conclude that

$$\|F(t)\|_{H^2(\Omega)} \lesssim \|\phi(t)\|_Y^2,$$

proving the claim. This implies in particular that $F(t)$ belongs to $L^\infty(\Omega)$. We can use this information in (6) to write

$$\begin{aligned}
\int_{\Omega_t} \int_D M\hat{\psi}\, \omega(u)\, q \cdot \nabla_q \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t' &\lesssim \left(\int_0^T \|F(t)\|_{L^\infty(\Omega)} \int_{\Omega \times D} M\hat{\psi}^2 |q|^2 \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t \right)^{1/2} \\
&\lesssim \left(\sup_{t>0} \int_D M\hat{\psi}^2 |q|^2 \,\mathrm{d}q \right)^{1/2} \|\phi\|_{L^2(0,T;Y)}.
\end{aligned}$$

We can therefore deduce the following bound on the time derivative:

$$\left|\int_{\Omega_t}\int_D M\partial_t\hat{\psi}\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right| \lesssim \|\phi\|_{L^2(0,T;Y)},$$

as required. □

Remark 2.5. Let us make the following observation about the unnormalised density ψ. Applying a strategy as above, one can easily deduce that ψ is bounded in the unweighted space $L^2(0,T;[H^s(\Omega\times D)]')\cap L^\infty(0,T;L^2(\Omega\times D))\cap L^2(0,T;H^1(\Omega\times D))$, for $s>d+1$. (In fact, this is implied by the corresponding bounds on $\hat{\psi}$.) However, such a bound does not suffice to deduce strong compactness, as the embedding $H^1(\Omega\times D)\hookrightarrow L^2(\Omega\times D)$ is not compact. On the other hand, having established compactness of $M\hat{\psi}$ in $L^2(0,T;L^2_M(\Omega\times D))$, we have

$$\int_{\Omega_t}\int_D|\psi|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \lesssim \int_{\Omega_t}\int_D M^{-1}|\psi|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' = \int_{\Omega_t}\int_D M|\hat{\psi}|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t',$$

which implies that ψ is compact in $L^2(0,T;L^2(\Omega\times D))$. Furthermore, the continuous embedding $H^1(\Omega\times D)\hookrightarrow L^4(\Omega\times D)$ for $d=2$ and $H^1(\Omega\times D)\hookrightarrow L^3(\Omega\times D)$ for $d=3$ gives additional integrability of ψ regardless of the integrability of the initial data.

2.2 Weak sequential compactness

We are now ready to prove the following stability result.

Theorem 2.6 (Weak sequential compactness)**.** *Let $(u_n,\hat{\psi}_n)$ be a sequence of weak solutions as in Definition 2.1, with initial data $(u_n^0,\hat{\psi}_n^0)$ such that*

$$u_n^0 \rightharpoonup u^0 \;\text{ weakly in } L^2(\Omega), \quad M^{1/2}\hat{\psi}_n^0 \rightharpoonup M^{1/2}\hat{\psi}^0 \;\text{ weakly in } L^2(\Omega\times D).$$

Then, there exists a subsequence (not relabelled) and a pair $(u,\hat{\psi})$ such that

$$\begin{aligned}&u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V)\cap W^{1,\frac{4}{3}}(0,T;V'),\\ &\hat{\psi}\in H^1(0,T;M^{-1}Y')\cap L^\infty(0,T;L^p_M(\Omega\times D))\cap L^2(0,T;H^1_M(\Omega;\times D))\end{aligned}$$

and the following convergence results hold:

$$\begin{aligned}
u_n &\to u && \text{strongly in } L^2(0,T;L^r(\Omega)),\; r\in[2,6),\\
u_n &\rightharpoonup u && \text{weakly in } L^2(0,T;V),\\
\hat{\psi}_n &\to \hat{\psi} && \text{strongly in } L^p(0,T;L^2_M(\Omega\times D)),\; 2\le p<\infty,\\
\hat{\psi}_n &\rightharpoonup \hat{\psi} && \text{weakly in } L^2(0,T;H^1_M(\Omega\times D));
\end{aligned}$$

and $(u,\hat{\psi})$ is a weak solution with initial data $(u^0,\hat{\psi}^0)$.

Proof. The weak convergence results are a direct consequence of the uniform bounds derived in the previous subsection. Strong compactness of the velocity is classical and follows in the same way as for the incompressible Navier–Stokes equations by the Aubin–Lions lemma and interpolation. It is then easy to pass to the limit in the Navier–Stokes equation. (Note that at this point we only require the weak convergence of $M^{1/2}\hat{\psi}_n$ in $L^2(0,T;L^2(\Omega\times D))$.)

As for the Fokker–Planck equation, passing to the limit in the diffusion terms and the term involving the time derivative poses no problem. For the remaining two terms we shall employ strong convergence of $\hat{\psi}_n$. First, recall that the embedding $H^1_M(\Omega \times D) \hookrightarrow\hookrightarrow L^2_M(\Omega \times D)$ is compact. Second, observe the chain of continuous embeddings

$$H^6(\Omega \times D) \hookrightarrow W^{3,4}(\Omega \times D) \hookrightarrow Y.$$

(The second embedding follows directly from the definition of the space Y, while the first one follows from an application of an extension theorem and the embedding $H^6(\mathbb{R}^{2d}) \hookrightarrow W^{3,4}(\mathbb{R}^{2d})$.) Then, from Lemma 2.4 we deduce that the time derivative $\partial_t \hat{\psi}_n$ is uniformly bounded in the space

$$M^{-1}[H^6(\Omega \times D)]' = \{\zeta \mid M\zeta \in [H^6(\Omega \times D)]'\}.$$

We claim that $L^2_M(\Omega \times D)$ embeds continuously into this space. This follows easily from the following estimate

$$\begin{aligned}\|\zeta\|_{M^{-1}[H^6(\Omega\times D)]'} &= \sup_{\chi \in H^6(\Omega\times D)} \frac{|\langle \chi, M\zeta\rangle|}{\|\chi\|_{H^6(\Omega\times D)}} \\ &\leq \sup_{\chi \in H^6(\Omega\times D)} \frac{\|M^{1/2}\|_{L^\infty(\Omega\times D)} \|M^{1/2}\zeta\|_{L^2(\Omega\times D)} \|\chi\|_{L^2(\Omega\times D)}}{\|\chi\|_{H^6(\Omega\times D)}} \lesssim \|\zeta\|_{L^2_M(\Omega\times D)}.\end{aligned}$$

Applying Dubinskiĭ's compactness theorem [8], we deduce that the sequence $\hat{\psi}_n$ is precompact in $L^2(0,T; L^2_M(\Omega \times D))$. In particular, by extracting a subsequence, we can assume that $\hat{\psi}_n$ converges to $\hat{\psi}$ strongly in $L^2(0,T; L^2_M(\Omega \times D))$. Interpolating with the uniform bounds in $L^\infty(0,T; L^p_M(\Omega \times D))$, we deduce strong convergence in $L^r(0,T; L^q_M(\Omega \times D))$ for any $r \in [2,\infty)$, $q \in [2,p)$.

We are now ready to pass to the limit in the remaining convection terms. First, we write

$$\begin{aligned}&\int_{\Omega_t}\int_D M\hat{\psi}_n u_n \cdot \nabla_x \phi \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' - \int_{\Omega_t}\int_D M\hat{\psi} u \cdot \nabla_x \phi \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \\ &\quad = \int_{\Omega_t}\int_D M(\hat{\psi}_n - \hat{\psi}) u_n \cdot \nabla_x \phi \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' + \int_{\Omega_t}\int_D M\hat{\psi}(u_n - u) \cdot \nabla_x \phi \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t',\end{aligned}$$

and then estimate, in turn,

$$\begin{aligned}&\left| \int_{\Omega_t}\int_D M(u_n - u) \cdot \nabla_x \phi \, \hat{\psi} \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \right| \\ &\quad \lesssim \|\hat{\psi}\|_{L^\infty(0,T;L^2_M(\Omega\times D))} \int_0^T \|u_n - u\|_{L^4(\Omega)} \|\nabla_x \phi\|_{L^4_M(\Omega\times D)} \, \mathrm{d}t \\ &\quad \lesssim \|\hat{\psi}\|_{L^\infty(0,T;L^2_M(\Omega\times D))} \|u_n - u\|_{L^2(0,T;L^4(\Omega))} \|\nabla_x \phi\|_{L^2(0,T;L^4_M(\Omega\times D))},\end{aligned}$$

which converges to zero thanks to strong convergence of the sequence u_n in $L^2(0,T; L^4(\Omega))$; and

$$\begin{aligned}&\left| \int_{\Omega_t}\int_D M u_n \cdot \nabla_x \phi \, (\hat{\psi}_n - \hat{\psi}) \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \right| \\ &\quad \lesssim \int_0^T \|u_n\|_{L^4(\Omega)} \|\hat{\psi}_n - \hat{\psi}\|_{L^2_M(\Omega\times D)} \|\nabla_x \phi\|_{L^4_M(\Omega\times D)} \, \mathrm{d}t \\ &\quad \lesssim \|u_n\|_{L^2(0,T;L^4(\Omega))} \|\hat{\psi}_n - \hat{\psi}\|_{L^r(0,T;L^2_M(\Omega\times D))} \|\nabla_x \phi\|_{L^{2+\epsilon}(0,T;L^4_M(\Omega\times D))},\end{aligned}$$

where $r = \frac{4+2\epsilon}{\epsilon} \in (2,\infty)$.

Finally, we consider the term $\operatorname{div}_q(\omega(u)qM\hat{\psi})$:

$$\left|\int_{\Omega_t}\int_D M\hat{\psi}_n\,\omega(u_n)\,q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t}\int_D M\hat{\psi}\,\omega(u)\,q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right| \tag{7}$$
$$\leq \left|\int_{\Omega_t}\int_D M(\hat{\psi}_n-\hat{\psi})\,\omega(u_n)\,q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right| + \left|\int_{\Omega_t}\int_D M\hat{\psi}\,\omega(u_n-u)\,q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right|.$$

For the second term we observe that, for each $l,k=1,\dots,d$, the function

$$(t,x)\longmapsto \int_D M\hat{\psi}\,q_l\,\partial_{q_k}\phi\,\mathrm{d}q$$

is square-integrable, and it is therefore a viable test function for the weak convergence of the velocity gradient. Indeed, using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\int_{\Omega_t}\left(\int_D M\hat{\psi}\,q_l\,\partial_{q_k}\phi\,\mathrm{d}q\right)^2\mathrm{d}x\,\mathrm{d}t' &\leq \int_{\Omega_t}\left(\int_D M|\nabla_q\phi|^2\,\mathrm{d}q\right)\left(\int_D M|q|^2\hat{\psi}\,\mathrm{d}q\right)\mathrm{d}x\,\mathrm{d}t'\\ &\leq \int_0^T\left\|\int_D M|\nabla_q\phi|^2\,\mathrm{d}q\right\|_{L^\infty(\Omega)}\left(\int_{\Omega\times D} M|q|^2\hat{\psi}^2\,\mathrm{d}q\,\mathrm{d}x\right)\mathrm{d}t\\ &\leq \|M^{1/2}|q|\hat{\psi}\|^2_{L^\infty(0,T;L^2(\Omega\times D))}\|\phi\|^2_{L^2(0,T;Y)}<\infty,\end{aligned}$$

where, for the second inequality we recall the argument used in the proof of Lemma 2.4, which established that

$$\begin{aligned}&\int_0^T\left\|\int_D M|\nabla_q\phi|^2\,\mathrm{d}q\right\|_{L^\infty(\Omega)}\mathrm{d}t\\ &\quad\lesssim \int_0^T\left(\|\nabla_q\phi\|^2_{L^4_M(\Omega\times D)}+\|\nabla_x(\nabla_q\phi)\|^2_{L^4_M(\Omega\times D)}+\|\nabla_x^2(\nabla_q\phi)\|^2_{L^4_M(\Omega\times D)}\right)\mathrm{d}t\\ &\quad\lesssim \|\phi\|^2_{L^2(0,T;Y)}.\end{aligned}$$

It follows that the second term on the right-hand side of (7) converges to zero as $n\to\infty$. The remaining term is treated very similarly. For $q\in(2,p)$, we have

$$\begin{aligned}&\left|\int_{\Omega_t}\int_D M(\hat{\psi}_n-\hat{\psi})\omega(u_n)q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right|\\ &\quad\lesssim\left(\int_0^T\left\|\int_D M|\nabla_q\phi|^2\,\mathrm{d}q\right\|_{L^\infty(\Omega)}\left\|M^{1/2}|q|(\hat{\psi}_n-\hat{\psi})\right\|^2_{L^2(\Omega\times D)}\mathrm{d}t\right)^{1/2}\\ &\quad\lesssim\left(\int_0^T\|\phi\|^2_Y\left\|M^{1/q}(\hat{\psi}_n-\hat{\psi})\right\|^2_{L^q(\Omega\times D)}\mathrm{d}t\right)^{1/2}\\ &\quad\lesssim\|\phi\|_{L^{2+\epsilon}(0,T;Y)}\left\|M^{1/q}(\hat{\psi}_n-\hat{\psi})\right\|_{L^r(0,T;L^q(\Omega\times D))},\end{aligned}$$

where $r=\frac{4+2\epsilon}{\epsilon}$, as before.

We thus deduce that the pair $(u,\hat{\psi})$ is a weak solution of the corotational Navier–Stokes–Fokker–Planck system (1). □

2.3 Galerkin approximation: Proof of Theorem 2.3

With the *a priori* estimates derived in the previous section in hand, it is now a fairly simple matter to prove Theorem 2.3. We shall adapt to our case the two step Galerkin approximation used in [12] in the context of implicitly constituted kinetic models.

The Hilbert space $V \cap H^{d+1}(\Omega)$, equipped with the inner product of $H^{d+1}(\Omega)$, is compactly and densely embedded in $L^2_{\mathrm{div}}(\Omega)$. Therefore there exists a countable set $\{w_i\} \in V \cap H^{d+1}(\Omega)$ of eigenfunctions whose linear span is dense in $L^2_{\mathrm{div}}(\Omega)$. Furthermore, these vectors form an orthonormal set in $L^2(\Omega)$, and an orthogonal set in $H^{d+1}(\Omega)$.

Similarly, the Hilbert space $H^1_M(\Omega \times D) \cap H^1(\Omega \times D) = H^1(\Omega \times D)$ (equipped with the standard inner product) is compactly and densely embedded in $L^2_M(\Omega \times D)$. Therefore, there exists a countable set $\{\phi_i\}$ of eigenfunctions in $H^1(\Omega \times D)$, which are orthogonal in the inner product of $H^1(\Omega \times D)$ and orthonormal and dense in $L^2_M(\Omega \times D)$.

We now fix $n, m \in \mathbb{N}$ and pose the following Galerkin problem: find time-dependent coefficients $c_i^{m,n}, d_i^{m,n}$ such that

$$u^{m,n}(t,x) := \sum_{i=1}^{m} c_i^{m,n}(t) w_i(x),$$
$$\hat{\psi}^{m,n}(t,x,q) := \sum_{i=1}^{n} d_i^{m,n}(t) \phi_i(x,q),$$

solve

$$\int_\Omega \partial_t u^{m,n} \cdot w_i \,\mathrm{d}x - \int_\Omega u^{m,n} \otimes u^{m,n} : \nabla_x w_i \,\mathrm{d}x + \int_\Omega \nabla_x u^{m,n} : \nabla_x w_i \,\mathrm{d}x = - \int_\Omega \tau^{m,n} : \nabla_x w_i \,\mathrm{d}x, \tag{8}$$

for all $i = 1, \ldots, m$ and a.e. $t \in (0,T)$; and

$$\int_{\Omega\times D} M \partial_t \hat{\psi}^{m,n} \phi_i \,\mathrm{d}q \,\mathrm{d}x - \int_{\Omega\times D} M u^{m,n} \hat{\psi}^{m,n} \cdot \nabla_x \phi_i \,\mathrm{d}q \,\mathrm{d}x - \int_{\Omega\times D} M \hat{\psi}^{m,n} \omega(u^{m,n})\, q \cdot \nabla_q \phi_i \,\mathrm{d}q \,\mathrm{d}x + \mu \int_{\Omega\times D} M \nabla_x \hat{\psi}^{m,n} \cdot \nabla_x \phi_i \,\mathrm{d}q \,\mathrm{d}x + \int_{\Omega\times D} M \nabla_q \hat{\psi}^{m,n} \cdot \nabla_q \phi_i \,\mathrm{d}q \,\mathrm{d}x = 0, \tag{9}$$

for all $i = 1, \ldots, n$ and a.e. $t \in (0,T)$; with initial data given by

$$u^{m,n}(0,x) = u_0^m(x) := \sum_{i=1}^{m} (u_0, w_i)_{L^2(\Omega)} w_i(x),$$
$$\hat{\psi}^{m,n}(0,x,q) = \hat{\psi}_0^n(x,q) := \sum_{i=1}^{n} (\hat{\psi}_0, \phi_i)_{L^2_M(\Omega\times D)} \phi_i(x,q).$$

The approximate extra-stress tensor in equation (8) is defined by

$$\tau^{m,n} := \int_D M \hat{\psi}^{m,n}\, q \otimes q \,\mathrm{d}q - \left(\int_D M \hat{\psi}^{m,n} \,\mathrm{d}q \right) \mathrm{Id}\,.$$

A standard application of the Carathéodory existence theorem provides local-in-time existence of such $u^{m,n}, \hat{\psi}^{m,n}$ for any fixed m, n. Then, an extension to the whole time interval $(0,T)$ is possible thanks to the uniform bounds from the previous section.

Multiplying the i^{th} equation of (8) by $c_i^{m,n}$ and summing over $i = 1, \ldots, m$, we deduce that $u^{m,n}$ satisfies (4), while multiplying the i^{th} equation of (9) by $d_i^{m,n}$ and summing over $i = 1, \ldots, n$, shows that $\hat{\psi}^{m,n}$ satisfies (5). Therefore, we obtain the uniform, in both m and n, bounds

$$u^{m,n} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V),$$

and

$$\hat{\psi}^{m,n} \in L^\infty(0,T;L^2_M(\Omega\times D)) \cap L^2(0,T;H^1_M(\Omega\times D)). \tag{10}$$

While at this stage we cannot derive the additional L^p bound on $\hat{\psi}^{m,n}$, we can use the regularity of $u^{m,n}$ to pass to the limit with $n\to\infty$. Indeed, since $H^{d+1}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, we have

$$\|\nabla_x u^{m,n}\|_{L^\infty((0,T)\times\Omega)} \le C(m).$$

Using this bound we can write

$$\int_{\Omega\times D} M\hat{\psi}^{m,n}\omega(u^{m,n})\, q\cdot\nabla_q\phi \,\mathrm{d}q\,\mathrm{d}x \le \|\nabla_x u^{m,n}\|_{L^\infty(\Omega)}\||q|\hat{\psi}^{m,n}\|_{L^2_M(\Omega\times D)}\|\nabla_q\phi\|_{L^2(\Omega\times D)}.$$

Consequently, we obtain an n-uniform bound on the time derivative

$$\partial_t\hat{\psi}^{m,n} \in L^2(0,T;M^{-1}[H^1(\Omega\times D)]').$$

Using the above bounds, together with the Aubin–Lions lemma, we can deduce existence of functions $u^m, \hat{\psi}^m$, that satisfy

$$\int_\Omega \partial_t u^m\cdot w_i\,\mathrm{d}x - \int_\Omega u^m\otimes u^m : \nabla_x w_i\,\mathrm{d}x + \int_\Omega \nabla_x u^m : \nabla_x w_i\,\mathrm{d}x = -\int_\Omega \tau^m : \nabla_x w_i\,\mathrm{d}x, \tag{11}$$

for all $i=1,\dots,m$ and a.e. $t\in(0,T)$, where

$$\tau^m := \int_D M\hat{\psi}^m\, q\otimes q\,\mathrm{d}q - \left(\int_D M\hat{\psi}^m\,\mathrm{d}q\right)\mathrm{Id};$$

and

$$\begin{aligned}\int_{\Omega\times D} M\partial_t\hat{\psi}^m\phi\,\mathrm{d}q\,\mathrm{d}x = \int_{\Omega\times D} M\hat{\psi}^m u^m\cdot\nabla_x\phi\,\mathrm{d}q\,\mathrm{d}x + \int_{\Omega\times D} M\hat{\psi}^m\omega(u^m)q\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\\ -\mu\int_{\Omega\times D} M\nabla_x\hat{\psi}^m\cdot\nabla_x\phi\,\mathrm{d}q\,\mathrm{d}x - \int_{\Omega\times D} M\nabla_q\hat{\psi}^m\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\\ \forall\,\phi\in H^1(\Omega\times D),\ \text{and a.e. } t\in(0,T).\end{aligned} \tag{12}$$

Furthermore, by lower semicontinuity, the following energy inequality holds:

$$\begin{aligned}\int_\Omega \frac{1}{2}|u^m|^2\,\mathrm{d}x + \int_{\Omega\times D}\frac{1}{2}M(\hat{\psi}^m)^2\,\mathrm{d}q\,\mathrm{d}x + \nu\int_{\Omega_t}|\nabla_x u^m|^2\,\mathrm{d}x\\ + \int_0^t\int_{\Omega\times D} M\Big(\mu|\nabla_x\hat{\psi}^m|^2 + |\nabla_q\hat{\psi}^m|^2\Big)\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\\ \le \int_\Omega\frac{1}{2}|u_0^m|^2\,\mathrm{d}x + \int_{\Omega\times D}\frac{1}{2}M\hat{\psi}_0\,\mathrm{d}q\,\mathrm{d}x - \int_0^t\int_\Omega \tau^m : \nabla_x u^m\,\mathrm{d}x\,\mathrm{d}t'.\end{aligned} \tag{13}$$

Using equation (12), we can guarantee that the function $\hat{\psi}^m$ is nonnegative. To this end, we choose a function $\chi\in C_c^\infty([0,\infty))$ such that $\chi(s)=1$ for $0\le s<1$ and $\chi(s)=0$ for $s\ge 2$. For $R>0$ we define $\chi_R(q)=\chi\Big(\frac{|q|}{R}\Big)$. Then, using the function

$$\phi(t,x,q) := \hat{\psi}^m_-(t,x,q)\,\chi_R^2(q)$$

as a test function in (12), we obtain

$$\int_{\Omega\times D} M\partial_t\Big(\hat\psi^m_-\Big)^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x \le \int_{\Omega\times D} M\hat\psi^m\omega(u^m)q\cdot\Big[\chi_R^2\nabla_q\hat\psi^m_- + \hat\psi^m_-\nabla_q\chi_R^2\Big]\,\mathrm{d}q\,\mathrm{d}x \\ - \int_{\Omega\times D} M\Big|\nabla_q\hat\psi^m_-\Big|^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x - 2\int_{\Omega\times D} M\hat\psi^m_-\nabla_q\hat\psi^m_-\,\chi_R\nabla_q\chi_R\,\mathrm{d}q\,\mathrm{d}x.$$

Note that the convective term vanishes due to incompressibility. Integrating by parts, the first term on the right-hand side can be rewritten, and controlled, as

$$\left|\int_{\Omega\times D} M\hat\psi^m_-\omega(u^m)q\cdot\chi_R\nabla_q\chi_R\,\mathrm{d}q\,\mathrm{d}x\right| \lesssim \frac{1}{R}.$$

Next, we use Young's inequality to write

$$\left|\int_{\Omega\times D} M\hat\psi^m_-\nabla_q\hat\psi^m_-\,\chi_R\nabla_q\chi_R\,\mathrm{d}q\,\mathrm{d}x\right| \le \frac{C}{R} + \frac{1}{2}\int_{\Omega\times D} M\Big|\nabla_q\hat\psi^m_-\Big|^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x.$$

We deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega\times D} M\Big(\hat\psi^m_-\Big)^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\times D} M\Big|\nabla_q\hat\psi^m_-\Big|^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x \le \frac{C}{R},$$

whence,

$$\int_{\Omega\times D} M\Big(\hat\psi^m_-(t)\Big)^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x \le \frac{Ct}{R} + \int_{\Omega\times D} M\Big(\hat\psi^m_-(0)\Big)^2\chi_R^2\,\mathrm{d}q\,\mathrm{d}x.$$

Passing to the limit $R\to\infty$ we thus obtain

$$\int_{\Omega\times D} M\Big(\hat\psi^m_-(t)\Big)^2\,\mathrm{d}q\,\mathrm{d}x = 0,$$

and therefore $\hat\psi^m_- = 0$ a.e. in $[0,T]\times\Omega\times D$ uniformly in m.

We will now show that

$$\rho^m(t,x) := \int_D M\hat\psi^m\,\mathrm{d}q = 1 \quad \text{for a.e. } (t,x)\in[0,T]\times\Omega.$$

Notice that

$$\phi(t,x,q) := \bar\phi(t,x)\chi_R(q), \quad \bar\phi\in L^2(0,T;H^1(\Omega))$$

is a viable test function for equation (12). Since

$$|\nabla_q\phi| \lesssim \frac{1}{R},$$

it is easy to see that the terms of (12) involving q-gradients will vanish in the limit $R\to\infty$. Integrating in time over $(0,t)$ and passing to the limit with R we thus deduce

$$\int_\Omega \rho^m(t)\bar\phi\,\mathrm{d}x - \int_\Omega \rho^m(0)\bar\phi\,\mathrm{d}x = \int_{\Omega_t} \rho^m u^m\cdot\nabla_x\bar\phi\,\mathrm{d}x - \int_{\Omega_t} \nabla_x\rho^m\cdot\nabla_x\bar\phi\,\mathrm{d}x, \tag{14}$$

for all $\bar\phi\in L^2(0,T;H^1(\Omega))$. Notice that

$$\rho^m(0) = \int_D M\hat\psi_0\,\mathrm{d}q = 1,$$

and, using (10),

$$\rho^m \in L^2(0,T;H^1(\Omega)).$$

Using the divergence-free property of u^m, it is readily deduced that $\rho^m \equiv 1$ is the only solution of (14).

Now let $\beta_\delta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\beta_\delta(s) = s^p, \text{ for } |s| < \frac{1}{\delta}, \quad \beta_k(s) = \left(\frac{2}{\delta}\right)^p, \text{ for } |s| > \frac{2}{\delta}.$$

Then $\beta_\delta', \beta_\delta''$ are compactly supported, and $\beta'(\hat{\psi}^m) \in H^1(\Omega \times D)$, whereby it is a viable test function for the above Fokker–Planck equation. We thus deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega\times D} M\beta_\delta(\hat{\psi}^m)\,\mathrm{d}q\,\mathrm{d}x \le 0,$$

which, upon integration with respect to t and by the monotone convergence theorem, gives in the limit $\delta \to 0$ the inequality

$$\int_{\Omega\times D} M(\hat{\psi}^m)^p\,\mathrm{d}q\,\mathrm{d}x \le \int_{\Omega\times D} M\hat{\psi}_0^p\,\mathrm{d}q\,\mathrm{d}x.$$

With this uniform bound in hand, we can repeat the arguments in the proof of Theorem 2.6[1] to deduce existence of a pair $(u, \hat{\psi})$ satisfying all the requirements of Definition 2.1. Using an interpolation argument, similarly as in the case of the q-convective term, we deduce that

$$\int_{\Omega_t}\int_D \tau^m : \nabla_x u^m\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \longrightarrow \int_{\Omega_t}\int_D \tau : \nabla_x u\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t',$$

so that from (13) we conclude that the weak solution satisfies the energy inequality (3). This concludes the proof of Theorem 2.3. □

Remark 2.7. Note that since all the estimates in the above argument are uniform simultaneously in m and n, we could in principle take $m = n$ and pass to the limit at once. The only reason we do not do that is because we want to derive higher integrability of $\hat{\psi}$. In fact, by inspecting closely the above proofs, one can easily deduce existence of weak solutions to problem (1) when $\hat{\psi}_0 \in L^2_M(\Omega \times D)$ only; however, then one needs to restrict the class of test functions. In particular one needs to take $\epsilon = \infty$ for time integrability and replace M by M^α for some $0 < \alpha < 1$ in the definition of the space Y. This class still contains the spaces $C^\infty(\bar{\Omega}; C_c^\infty(D))$ and $H^s(\Omega \times D)$ for $s > d+1$; as well as functions of the form $\phi = q \otimes q : \varphi$, which are relevant for the macroscopic closure of the system (1) proved in Section 4.

3 Relative energy inequality and conditional uniqueness

Since the weak solutions of system (1) satisfy an energy inequality and since $\hat{\psi}$ has finite L^2_M norm, one might try to apply the relative energy method to obtain weak-strong uniqueness type results. This is indeed possible, as will be shown in the current section; however, additional assumptions on the Fokker–Planck solution have to be made on *both* solutions involved in our proof of (conditional) uniqueness.

[1]Strictly speaking, we use the weak formulation of the Fokker–Planck equation for $\hat{\psi}^m$ with test functions in $C^\infty(\bar{\Omega}; C_c^\infty(D))$ to pass to the limit; then extend to test functions as in Definition 2.1 by density.

Theorem 3.1. *Let $(u_i, \hat{\psi}_i)$, $i = 1, 2$, be two dissipative weak solutions of the corotational Navier–Stokes–Fokker–Planck system (1) subject to the same initial data $(u_0, \hat{\psi}_0)$, as in Definition 2.1, with $p > d$. Assume that $(u_2, \hat{\psi}_2)$ satisfies, for some $\alpha > 0$,*

$$u_2 \in L^r(0,T; L^s(\Omega)), \quad \frac{2}{r} + \frac{d}{s} = 1;$$
$$\left((t,x) \longmapsto \int_D M|q|^2 |\nabla_q \hat{\psi}_2|^2 \, \mathrm{d}q\right) \in L^{1+\alpha}(0,T; L^\infty(\Omega)). \tag{15}$$

Assume further that the pair $(u_1, \hat{\psi}_1)$ satisfies one of the following conditions:

1. *$\hat{\psi}_1$ satisfies (15);*
2. *$\hat{\psi}_1$ satisfies*
$$\left((t,x) \longmapsto \int_D M|q|^2 |\hat{\psi}_1|^2 \, \mathrm{d}q\right) \in L^\infty(0,T; L^\infty(\Omega));$$
3. *$\omega(u_1) \in L^2(0,T; L^{\frac{2p}{p-2}}(\Omega))$.*

Then $u_1 = u_2$ and $\hat{\psi}_1 = \hat{\psi}_2$ a.e. on $(0,T) \times \Omega$ and $(0,T) \times \Omega \times D$, respectively.

Proof. The theorem is proved by an application of the well-known relative energy method. As usual, the additional integrability of one of the velocities is used to justify using both of the velocities u_1 and u_2 as test functions in the weak formulations of the equations satisfied by u_2 and u_1, respectively. To justify such 'cross-testing' for the Fokker–Planck equation, one requires some additional conditions for both $\hat{\psi}_1$ and $\hat{\psi}_2$. Indeed, if $\hat{\psi}_2^\epsilon$ is an appropriate time-mollified approximate sequence for $\hat{\psi}_2$, then for the error term

$$\int_{\Omega_t} \int_D M \hat{\psi}_1 \nabla_q (\hat{\psi}_2^\epsilon - \hat{\psi}_2) \cdot \omega(u_1) q \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t'$$

to vanish when $\epsilon \to 0^+$ one still needs additional regularity of $\hat{\psi}_1$. Any of the assumptions in the statement of Theorem 3.1 suffices. We omit the details.

Using the energy inequality (3) we can write

$$\begin{aligned}
E(t) &:= \frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\Omega \times D} M |\hat{\psi}_1 - \hat{\psi}_2|^2 \, \mathrm{d}q \, \mathrm{d}x + \nu \int_0^t \int_{\Omega \times D} |\nabla_x (u_1 - u_2)|^2 \, \mathrm{d}x \, \mathrm{d}t' \\
&\quad + \int_0^t \int_{\Omega \times D} M \Big(\mu |\nabla_x (\hat{\psi}_1 - \hat{\psi}_2)|^2 + |\nabla_q (\hat{\psi}_1 - \hat{\psi}_2)|^2\Big) \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \\
&\leq \int_\Omega |u_0|^2 \, \mathrm{d}x + \int_{\Omega \times D} M \hat{\psi}_0^2 \, \mathrm{d}q \, \mathrm{d}x - \int_0^t \int_\Omega \tau_1 : \nabla_x u_1 - \int_0^t \int_\Omega \tau_2 : \nabla_x u_2 \\
&\quad - \int_\Omega u_1(s) \cdot u_2(s) \, \mathrm{d}x - \int_{\Omega \times D} M \hat{\psi}_1(s) \hat{\psi}_2(s) \, \mathrm{d}q \, \mathrm{d}x - 2\nu \int_0^t \int_\Omega \nabla_x u_1 : \nabla_x u_2 \, \mathrm{d}x \, \mathrm{d}t' \\
&\quad - 2\mu \int_0^t \int_\Omega M \nabla_x \hat{\psi}_1 \cdot \nabla_x \hat{\psi}_2 \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' - 2 \int_0^t \int_\Omega M \nabla_q \hat{\psi}_1 \cdot \nabla_q \hat{\psi}_2 \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t'.
\end{aligned}$$

Testing the weak formulation for $(u_1, \hat{\psi}_1)$ by $(u_2, \hat{\psi}_2)$, and vice versa, and rearranging terms appropriately, we arrive at

$$\begin{aligned}
E(t) &\leq \int_0^t \int_\Omega \nabla_x (u_1 - u_2) : ((u_1 - u_2) \otimes u_2) \, \mathrm{d}x \, \mathrm{d}t' + \int_0^t \int_\Omega (\tau_1 - \tau_2) : \nabla_x (u_2 - u_1) \, \mathrm{d}x \, \mathrm{d}t' \\
&\quad + \int_0^t \int_{\Omega \times D} M \hat{\psi}_2 \nabla_x (\hat{\psi}_1 - \hat{\psi}_2) \cdot (u_1 - u_2) \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t' \\
&\quad - \int_0^t \int_{\Omega \times D} M (\hat{\psi}_1 - \hat{\psi}_2) \nabla_q \hat{\psi}_2 \cdot (\omega(u_1) - \omega(u_2)) q \, \mathrm{d}q \, \mathrm{d}x \, \mathrm{d}t'.
\end{aligned} \tag{16}$$

The first term on the right-hand side is easily estimated using the L^rL^s integrability of u_2:

$$\int_0^t \int_\Omega \nabla_x(u_1-u_2) : ((u_1-u_2)\otimes u_2)\,\mathrm{d}x\,\mathrm{d}t' \\ \leq \delta \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)}\,\mathrm{d}t' + C_\delta \int_0^t \|u_2\|^r_{L^s(\Omega)}\|u_1-u_2\|^2_{L^2(\Omega)}\,\mathrm{d}t'.$$

Next, we observe that

$$\int_0^t \int_\Omega (\tau_1-\tau_2) : \nabla_x(u_2-u_1)\,\mathrm{d}x\,\mathrm{d}t' = \int_0^t \int_{\Omega\times D} M(\hat\psi_1-\hat\psi_2)\, q\otimes q : \nabla_x(u_2-u_1)\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ \leq \left(\int_0^t \int_{\Omega\times D} M|q|^4|\nabla_x(u_2-u_1)|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/2}\left(\int_0^t \int_{\Omega\times D} M|\hat\psi_1-\hat\psi_2|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/2} \\ \leq \delta \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)}\,\mathrm{d}t' + C_\delta \int_0^t \int_{\Omega\times D} M|\hat\psi_1-\hat\psi_2|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'.$$

For the remaining two terms we write, in turn,

$$\int_0^t \int_{\Omega\times D} M(\hat\psi_1-\hat\psi_2)\,\nabla_q\hat\psi_2\cdot(\omega(u_1)-\omega(u_2))\,q\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ \leq \delta \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)}\,\mathrm{d}t' \\ + C_\delta \int_0^t \left\|\int_D M|q|^2|\nabla_q\hat\psi_2|^2\,\mathrm{d}q\right\|_{L^\infty(\Omega)} \int_{\Omega\times D} M|\hat\psi_1-\hat\psi_2|^2\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t',$$

and, using the Gagliardo–Nirenberg inequality we find that

$$\int_0^t \int_{\Omega\times D} M\hat\psi_2\nabla_x(\hat\psi_1-\hat\psi_2)\cdot(u_1-u_2)\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ \leq \int_0^t \left\|\nabla_x(\hat\psi_1-\hat\psi_2)\right\|_{L^2_M(\Omega\times D)}\left\|\hat\psi_2\right\|_{L^p_M(\Omega\times D)}\left\|M^{\frac{p-2}{2p}}(u_1-u_2)\right\|_{L^{\frac{2p}{p-2}}(\Omega\times D)}\,\mathrm{d}t' \\ \leq \delta \int_0^t \left\|\nabla_x(\hat\psi_1-\hat\psi_2)\right\|^2_{L^2_M(\Omega\times D)}\,\mathrm{d}t' \\ + C_\delta \int_0^t \left\|\hat\psi_2\right\|^2_{L^p_M(\Omega\times D)}\|u_1-u_2\|^{\frac{2(p-d)}{p}}_{L^2(\Omega)}\|\nabla_x(u_1-u_2)\|^{\frac{2d}{p}}_{L^2(\Omega)}\,\mathrm{d}t' \\ \leq \delta \int_0^t \left\|\nabla_x(\hat\psi_1-\hat\psi_2)\right\|^2_{L^2_M(\Omega\times D)}\,\mathrm{d}t' + \epsilon \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)}\,\mathrm{d}t' \\ + C_{\delta,\epsilon} \int_0^t \left(\int_{\Omega\times D} M\hat\psi^p\,\mathrm{d}q\,\mathrm{d}x\right)^{\frac{2}{p-d}}\|u_1-u_2\|^2_{L^2(\Omega)}\,\mathrm{d}t'.$$

Choosing $\delta > 0$ and $\epsilon > 0$ small enough, so as to absorb the gradient terms into the left-hand side of (16), we deduce the following Gronwall inequality:

$$\frac{1}{2}\int_\Omega |u_1(s)-u_2(s)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\times D} M|\hat\psi_1(s)-\hat\psi_2(s)|^2\,\mathrm{d}q\,\mathrm{d}x \\ \leq \int_0^t a(t)\left(\int_\Omega |u_1(s)-u_2(s)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\times D} M|\hat\psi_1(s)-\hat\psi_2(s)|^2\,\mathrm{d}q\,\mathrm{d}x\right)\mathrm{d}t',$$

where $a \in L^1(0,T)$ and $a \geq 0$. It follows that $u_1 = u_2$ for a.e. $(t,x) \in (0,T)\times\Omega$ and $\hat\psi_1 = \hat\psi_2$ for a.e. $(t,x,q) \in (0,T)\times\Omega\times D$. □

4 Macroscopic closure of the corotational Hookean model

For most of the remaining part of the paper we will consider macroscopic corotational models, which we shall refer to as corotational Oldroyd-B models. More precisely, here we consider the following system of equations:

$$\begin{aligned} \partial_t u + (u \cdot \nabla_x) u - \nu \Delta_x u + \nabla_x p &= \mathrm{div}_x\, \tau, \\ \mathrm{div}_x\, u &= 0, \\ \partial_t \tau + (u \cdot \nabla_x)\tau - \mu \Delta_x \tau + \tau \omega(u) - \omega(u)\tau + 2\tau &= 0, \end{aligned} \tag{17}$$

posed on $(0, T) \times \Omega$ with initial data

$$u_0 \in L^2_{\mathrm{div}}(\Omega), \quad \tau_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$$

(where $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the space of $d \times d$ symmetric tensors); and boundary conditions

$$u = 0 \ \text{ on } (0, T] \times \partial\Omega, \quad \nabla_x \tau_{ij} \cdot \hat{n} = 0 \ \text{ on } (0, T] \times \partial\Omega.$$

By a weak solution to this model we shall mean a pair (u, τ) such that

$$\begin{aligned} &u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)) \cap L^2(0, T; V), \\ &\tau \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{d\times d}) \cap L^2(0, T; H^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \end{aligned}$$

which satisfies (17) in the sense of distributions. Model (17) can be considered to be a special case of model (23) with $b = 0$, and an existence result can be obtained in a similar way as in the more general case (discussed in the next section). However, for this special case we can make a rigorous connection with the corotational Navier–Stokes–Fokker–Planck model discussed before. More precisely, we will prove in this section the following macroscopic closure theorem

Theorem 4.1. *Let $(u, \hat{\psi})$ be a weak solution of the corotational Navier–Stokes–Fokker–Planck system (1) with initial data $(u_0, \hat{\psi}_0)$. Define*

$$\tau_0 := \int_D M\hat{\psi}_0\, q \otimes q \,\mathrm{d}q - \left(\int_D M\hat{\psi}_0 \,\mathrm{d}q\right) \mathrm{Id}, \quad \tau := \int_D \psi(t, x, q)\, q \otimes q \,\mathrm{d}q - \left(\int_D \psi(t, x, q) \,\mathrm{d}q\right) \mathrm{Id}\,.$$

Then, the pair (u, τ) is a weak solution of system (17) with data (u_0, τ_0).

Proof. Recall the weak formulation for the Fokker–Planck equation in terms of the normalised probability density $\hat{\psi}$:

$$\begin{aligned} \int_{\Omega_t}\int_D M\hat{\psi}\, \partial_t \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t' + \int_{\Omega\times D} M\hat{\psi}(0)\phi(0) \,\mathrm{d}q \,\mathrm{d}x - \int_{\Omega\times D} M\hat{\psi}(t)\phi(t) \,\mathrm{d}q \,\mathrm{d}x \\ = -\int_{\Omega_t}\int_D M\hat{\psi} u \cdot \nabla_x \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t' - \int_{\Omega_t}\int_D M\hat{\psi}\, \omega(u)\, q \cdot \nabla_q \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t' \\ + \int_{\Omega_t}\int_D M\nabla_x \hat{\psi} \cdot \nabla_x \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t' + \int_{\Omega_t}\int_D M\nabla_q \hat{\psi} \cdot \nabla_q \phi \,\mathrm{d}q \,\mathrm{d}x \,\mathrm{d}t', \end{aligned}$$

for any $\phi = \phi(t, x, q)$, $\phi \in \left\{\phi \in W^{1,1}(0, T; L^2_M(\Omega \times D)) \mid \exists \epsilon > 0 : \phi \in L^{2+\epsilon}(0, T; Y)\right\}$. To derive the macroscopic closure of the corotational Navier–Stokes–Fokker–Planck model, we need to test the Fokker–Planck equation with

$$\phi(t, x, q) = q \otimes q : \varphi(t, x),$$

where $\varphi \in C^1([0,T]; C^2(\bar{\Omega}; \mathbb{R}^{d\times d}))$ is a matrix-valued test function, and integrate. Notice that this testing is admissible – indeed, one easily checks that ϕ belongs to the above test space.

We now substitute the above choice of test function into the weak formulation and derive the macroscopic model term-by-term, as follows. The first term becomes

$$\begin{aligned} -\int_{\Omega_t} \left(\int_D M\hat{\psi} q \otimes q \,\mathrm{d}q \right) : \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t' &= -\int_{\Omega_t} \tau : \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t} \mathrm{Id} : \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t' \\ &= -\int_{\Omega_t} \tau : \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega} \mathrm{tr}\,\varphi(0)\,\mathrm{d}x - \int_{\Omega} \mathrm{tr}\,\varphi(t)\,\mathrm{d}x, \end{aligned}$$

while the terms on the right-hand side become

$$\begin{aligned} \int_{\Omega\times D} M\hat{\psi}(0)\phi(0)\,\mathrm{d}q\,\mathrm{d}x - \int_{\Omega\times D} M\hat{\psi}(t)\phi(t)\,\mathrm{d}q\,\mathrm{d}x &= \int_{\Omega} \tau(0)\varphi(0)\,\mathrm{d}x - \int_{\Omega} \tau(t)\varphi(t)\,\mathrm{d}x \\ &\quad + \int_{\Omega} \mathrm{tr}\,\varphi(0)\,\mathrm{d}x - \int_{\Omega} \mathrm{tr}\,\varphi(t)\,\mathrm{d}x. \end{aligned}$$

For the terms involving the spatial derivative we write, in turn

$$\begin{aligned} \int_{\Omega_t}\int_D Mu \cdot \nabla_x \hat{\psi}\, \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &= \int_{\Omega_t}\int_D u_k \partial_{x_k} \Big(M\hat{\psi}\, q \otimes q \Big) : \varphi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} u_k \partial_{x_k} \left(\int_D M\hat{\psi}\, q \otimes q \,\mathrm{d}q \right) : \varphi \,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} (u \cdot \nabla_x)\tau : \varphi \,\mathrm{d}x\,\mathrm{d}t', \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_t}\int_D M\nabla_x \hat{\psi} \cdot \nabla_x \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &= \int_{\Omega_t}\int_D \partial_{x_k} \Big(M\hat{\psi}\, q \otimes q \Big) : \partial_{x_k}\varphi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} \partial_{x_k} \left(\int_D M\hat{\psi}\, q \otimes q \,\mathrm{d}q \right) : \partial_{x_k}\varphi \,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} \nabla_x \tau :: \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t'. \end{aligned}$$

The main technical step in these manipulations is to interchange the D-integral with the spatial derivative. This is allowed, however, by a standard measure theoretical argument – indeed, the partial derivatives in question are both integrable functions.

Finally, we deal with the terms involving gradients in the q direction. Simple algebraic manipulations, using that $\omega(u)^T = -\omega(u)$, lead to

$$\begin{aligned} -\int_{\Omega_t}\int_D M\hat{\psi}\,\omega(u)\,q \cdot \nabla_q \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &= \int_{\Omega_t}\int_D M\hat{\psi}\,\omega_{kl}\,q_l(q_i\delta_{jk} + q_j\delta_{ik})\,\varphi_{ij}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} \left(\int_D M\hat{\psi}\, q \otimes q \,\mathrm{d}q \right) \omega(u) : \varphi \,\mathrm{d}x\,\mathrm{d}t' \\ &\quad - \int_{\Omega_t} \omega(u) \left(\int_D M\hat{\psi}\, q \otimes q \,\mathrm{d}q \right) : \varphi \,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t} (\tau\omega(u) - \omega(u)\tau) : \varphi \,\mathrm{d}x\,\mathrm{d}t'. \end{aligned}$$

The q-diffusion term is the most delicate, because an additional integration by parts is required. First we write

$$\begin{aligned}\int_{\Omega_t}\int_D M\nabla_q\hat{\psi}\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &= \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}\,(q_i\delta_{jk}+q_j\delta_{ik})\,\varphi_{ij}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ &= \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}\,q_i\,\varphi_{ik}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}\,q_j\,\varphi_{kj}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'.\end{aligned} \tag{18}$$

We now split the D domain of integration into the interior and exterior of a ball of radius $R > 0$, and then pass to the limit. Since the integrands in question are integrable (which can be seen by using the L^2 bound on $M^{1/2}\hat{\psi}$), we readily deduce that

$$\int_{\Omega_t}\int_{D\setminus B_R(0)} M\partial_{q_k}\hat{\psi}q_i\varphi_{ik}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \longrightarrow 0, \quad \text{as } R\to\infty.$$

For the complementary integral we can apply the divergence theorem to write

$$\begin{aligned}\int_{\Omega_t}\int_{B_R(0)} M\partial_{q_k}\hat{\psi}\,q_i\,\varphi_{ik}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' = &-\int_{\Omega_t}\int_{B_R(0)}\partial_{q_k}(Mq_i)\,\hat{\psi}\,\varphi_{ik}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\ &+\int_{\Omega_t}\int_{\partial B_R(0)} M\hat{\psi}\,q_i\,\varphi_{ik}\,\hat{n}_k\,\mathrm{d}S(q)\,\mathrm{d}x\,\mathrm{d}t'.\end{aligned}$$

Using the product rule in the first integral, together with the property of the Maxwellian $\partial_{q_k}M = -q_kM$, we obtain

$$-\int_{\Omega_t}\int_{B_R(0)}\partial_{q_k}(Mq_i)\,\hat{\psi}\,\varphi_{ik}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' = \int_{\Omega_t}\varphi : \left(\int_{B_R(0)} M\hat{\psi}\,q\otimes q\,\mathrm{d}q - \int_{B_R(0)} M\hat{\psi}\,\mathrm{d}q\right)\mathrm{d}x\,\mathrm{d}t'.$$

For the remaining boundary term we have the estimate

$$\left|\int_{\Omega_t}\int_{\partial B_R(0)} M\hat{\psi}\,q_i\,\varphi_{ik}\,\hat{n}_k\,\mathrm{d}S(q)\,\mathrm{d}x\,\mathrm{d}t'\right| \lesssim R\int_{\Omega_t}\int_{\partial B_R(0)} M\hat{\psi}\,\mathrm{d}S(q)\,\mathrm{d}x\,\mathrm{d}t'. \tag{19}$$

The last integral vanishes in the limit $R\to\infty$ by virtue of Lemma 3.2 in [2]. Since the other term in (18) can be treated in exactly the same manner, we deduce the following equality

$$\int_{\Omega_t}\int_D M\nabla_q\hat{\psi}\cdot\nabla_q\phi\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' = 2\int_{\Omega_t}\tau : \varphi\,\mathrm{d}x\,\mathrm{d}t'.$$

Putting all the terms together we obtain that the extra stress tensor τ satisfies the following identity, for any $\varphi \in C^1([0,T];C^2(\bar{\Omega};\mathbb{R}^{d\times d}))$,

$$\begin{aligned}-\int_{\Omega_t}\tau : \partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t}(u\cdot\nabla_x)\tau : \varphi\,\mathrm{d}x\,\mathrm{d}t' + \mu\int_{\Omega_t}\nabla_x\tau :: \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t' + 2\int_{\Omega_t}\tau : \varphi\,\mathrm{d}x\,\mathrm{d}t' \\ + \int_{\Omega_t}(\tau\omega(u) - \omega(u)\tau) : \varphi\,\mathrm{d}x\,\mathrm{d}t' = \int_\Omega \tau(0) : \varphi(0)\,\mathrm{d}x - \int_\Omega \tau(t) : \varphi(t)\,\mathrm{d}x.\end{aligned}$$

That is, (u,τ) is a weak solution of equation (17). □

4.1 An energy inequality

Of course, given a weak solution (u, τ) to system (17) it is not clear whether it satisfies any form of an energy inequality (for similar reasons as with weak solutions of the Navier–Stokes equations), even given that $(u, \hat{\psi})$ satisfies the energy inequality (3). However, as it happens, by returning to the construction used in Theorem 2.3 we can derive an additional energy inequality satisfied by the extra stress tensor τ.

As before, choose $\chi \in C_c^\infty([0,\infty))$ such that $\chi(s) = 1$ for $0 \le s < 1$ and $\chi(s) = 0$ for $s \ge 2$. Fix $R > 0$ and let $\chi_R(q) = \chi\Big(\frac{|q|}{R}\Big)$, so that $|\nabla_q \chi_R| \lesssim \frac{1}{R}$. Consider the approximate extra stress tensor

$$\tau^m(t,x) := \int_D M\hat{\psi}^m q \otimes q \,\mathrm{d}q - \left(\int_D M\hat{\psi}^m \,\mathrm{d}q\right) \mathrm{Id}\,.$$

By virtue of the uniform estimates for $\hat{\psi}^m$ it can be seen that $\tau^m \in L^2(0,T;H^1(\Omega))$ uniformly. Now let $\tau^{m,k}$ be a sequence of compactly supported smooth functions such that $\tau^{m,k} \to \tau^m$, as $k \to \infty$, in $L^2(0,T;H^1(\Omega))$. One can easily verify that the function

$$\phi_{m,R,k}(t,x,q) := \tau^{m,k}(t,x) : \chi_R(q)\, q \otimes q$$

is a valid test function for the Fokker–Planck equation (12). (Observe that the truncation in q is necessary.) Integrating in time we obtain

$$\begin{aligned}
&\int_{\Omega_t}\int_D M\hat{\psi}^m \partial_t \phi_{m,R,k} \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega\times D} M\hat{\psi}^m(0)\phi_{m,R,k}(0)\,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega\times D} M\hat{\psi}^m(t)\phi_{m,R,k}(t)\,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = -\int_{\Omega_t}\int_D M\hat{\psi}^m u^m \cdot \nabla_x \phi_{m,R,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t}\int_D M\hat{\psi}^m \omega(u^m) q \cdot \nabla_q \phi_{m,R,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\
&\qquad + \int_{\Omega_t}\int_D M\nabla_x \hat{\psi}^m \cdot \nabla_x \phi_{m,R,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t}\int_D M\nabla_q \hat{\psi}^m \cdot \nabla_q \phi_{m,R,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned} \tag{20}$$

We would like to pass to the limit $R \to \infty$ in this identity. The terms involving q-gradients require most care; for instance,

$$\begin{aligned}
\int_{\Omega_t}\int_D M\nabla_q \hat{\psi}^m \cdot \nabla_q \phi_{m,R,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' &= \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}^m\, \chi_R(q)\, \partial_{q_k}(q_i q_j)\, \tau_{ij}^{m,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\
&\quad + \int_{\Omega_t}\int_D M\nabla_q \hat{\psi}^m \cdot \nabla_q \chi_R(q)\; q\otimes q : \tau^{m,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned}$$

For the first term on the right-hand side we observe that

$$\begin{aligned}
&\left| \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}^m\, \chi_R(q)\, \partial_{q_k}(q_i q_j)\, \tau_{ij}^{m,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t}\int_D M\partial_{q_k}\hat{\psi}^m\, \partial_{q_k}(q_i q_j)\, \tau_{ij}^{m,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \right| \\
&\le \int_{\Omega_t}\int_D M|\nabla_q \hat{\psi}^m|\,|q|\,|\tau^{m,k}|\,(1-\chi_R(q))\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \longrightarrow 0, \quad \text{as } R \to \infty,
\end{aligned}$$

by monotone convergence; while for the second term we have

$$\begin{aligned}
&\left| \int_{\Omega_t}\int_D M\nabla_q \hat{\psi}^m \cdot \nabla_q \chi_R(q)\; q\otimes q : \tau^{m,k}\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \right| \\
&\quad \le \int_{\Omega_t}\int_D M|q|^2\,|\nabla_q \hat{\psi}^m|\,|\nabla_q \chi_R(q)|\,|\tau^{m,k}|\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \\
&\quad \lesssim \frac{1}{R}\|\tau^{m,k}\|_{L^2(0,T;L^2(\Omega\times D))}\|\nabla_q \hat{\psi}^m\|_{L^2_M(\Omega\times D)},
\end{aligned}$$

which vanishes as $R \to \infty$. The other terms can be treated similarly. We can therefore pass to the limit in (20). Consequently, we are now in a position to repeat the same argument as in the proof of Theorem 4.1. We thus obtain

$$
\begin{aligned}
-\int_{\Omega_t} \tau^m : \partial_t \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (u^m \cdot \nabla_x)\tau^m : \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' + \mu \int_{\Omega_t} \nabla_x \tau^m :: \nabla_x \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' \\
+ 2\int_{\Omega_t} \tau^m : \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (\tau^m \omega(u^m) - \omega(u^m)\tau^m) : \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' \\
= \int_{\Omega} \tau^m(0) : \tau^{m,k}(0) \,\mathrm{d}x - \int_{\Omega} \tau^m(t) : \tau^{m,k}(t) \,\mathrm{d}x.
\end{aligned}
$$

Let us point out that the approximation of τ^m by bounded functions was necessary to justify the vanishing of the boundary term, as in (19). Moreover, using the spatial $W^{1,\infty}$ regularity of the velocity field u^m, we observe that the equality (12) can be extended to test functions $\phi \in L^2(0,T; H^1_{M^\alpha}(\Omega \times D))$ for any fixed $\alpha \in (0,1)$. This makes $\phi(t,x,q) = \varphi(t,x)\, q \otimes q$, with $\varphi \in L^2(0,T; H^1(\Omega; \mathbb{R}^{d\times d}))$, a valid test function. Consequently, we have

$$
\left| \int_{\Omega_t} \partial_t \tau^m : \varphi \,\mathrm{d}x\,\mathrm{d}t' \right| = \left| \int_{\Omega_t} \int_D M \partial_t \hat{\psi}^m \phi \,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' \right| \le \|\phi\|_{L^2(0,T;H^1_{M^\alpha}(\Omega\times D))} \le \|\varphi\|_{L^2(0,T;H^1(\Omega))},
$$

so that $\partial_t \tau^m \in L^2(0,T;[H^1(\Omega)]')$ for each m (although not uniformly). In particular the sequence $\tau^{m,k}$ can be chosen so that $\partial_t \tau^{m,k}$ converges to $\partial_t \tau^m$ in $L^2(0,T;[H^1(\Omega)]')$.[2] Thus we can write

$$
\begin{aligned}
-\int_{\Omega_t} \tau^m : \partial_t \tau^{m,k} \,\mathrm{d}x\,\mathrm{d}t' = -\int_{\Omega_t} \tau^m : \partial_t \tau^m \,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t} \tau^m : (\partial_t \tau^m - \partial_t \tau^{m,k}) \,\mathrm{d}x\,\mathrm{d}t' \\
\xrightarrow[k\to\infty]{} -\frac{1}{2}\int_{\Omega} |\tau^m(t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Omega} |\tau^m(0)|^2 \,\mathrm{d}x.
\end{aligned}
$$

Passing to the limit $k \to \infty$ we therefore deduce that

$$
\frac{1}{2}\int_{\Omega} |\tau^m|^2 \,\mathrm{d}x + \mu \int_{\Omega_t} |\nabla_x \tau^m|^2 \,\mathrm{d}x\,\mathrm{d}t' + 2\int_{\Omega_t} |\tau^m|^2 \,\mathrm{d}x\,\mathrm{d}t' = \frac{1}{2}\int_{\Omega} |\tau^m(0)|^2 \,\mathrm{d}x.
$$

Furthermore, from (11) we have that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \frac{1}{2}|u^m|^2 \,\mathrm{d}x + \nu \int_{\Omega} |\nabla_x u^m|^2 \,\mathrm{d}x = -\int_{\Omega} \tau^m : \nabla_x u^m \,\mathrm{d}x. \tag{21}
$$

Passing to the limit $m \to \infty$ (and using lower semicontinuity for the weak limits), we obtain that the solution $(u, \hat{\psi})$ of the corotational Navier–Stokes–Fokker–Planck system constructed in the proof of Theorem 2.3 satisfies

$$
\begin{aligned}
\int_{\Omega} \frac{1}{2}|u(t)|^2 + \frac{1}{2}|\tau(t)|^2 \,\mathrm{d}x + 2\int_0^t \int_{\Omega} |\tau|^2 \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_0^t \int_{\Omega} |\nabla_x u|^2 \,\mathrm{d}x\,\mathrm{d}t' + \mu \int_0^t \int_{\Omega} |\nabla_x \tau|^2 \,\mathrm{d}x\,\mathrm{d}t' \\
\le \int_{\Omega} \frac{1}{2}|u_0|^2 + \frac{1}{2b}|\tau_0|^2 \,\mathrm{d}x - \int_0^t \int_{\Omega} \tau : \nabla_x u \,\mathrm{d}x\,\mathrm{d}t',
\end{aligned} \tag{22}
$$

for any $t \in (0,T]$.

We can therefore formulate the following result.

[2] One can take, for example, $\tau^{m,k} := \sum_{i=1}^k (\tau^m(t), w_i)_{H^1(\Omega)} w_i$ (where (w_i) is a basis for the Hilbert space $H^1(\Omega)$ consisting of elements of $C_c^\infty(\Omega)$). Then, $\partial_t \tau^{m,k} = \sum_{i=1}^k \langle \partial_t \tau^m, w_i \rangle_{(H^1(\Omega),(H^1(\Omega))')} w_i$ has the required convergence property. Additionally, $\tau^m, \tau^{m,k}$ can be identified with continuous functions $[0,T] \to H^1(\Omega)$, so that we have the convergence $\tau^{m,k}(t) \to \tau^m(t)$ for every $t \in [0,T]$.

Corollary 4.2. *There exists a weak solution $(u, \hat{\psi})$ of the corotational Navier–Stokes–Fokker–Planck system (1) (according to Definition (2.1)), which satisfies the energy inequality (3), and such that the corresponding extra stress tensor*

$$\tau(t,x) := \int_D \psi(t,x,q) q \otimes q \,\mathrm{d}q - \left(\int_D \psi(t,x,q) \,\mathrm{d}q \right) \mathrm{Id}$$

satisfies the energy inequality (22). Moreover, the pair (u, τ) solves the corotational Oldroyd-B system (17).

Analogously to dissipative weak solutions, we can refer to weak solutions of (1) satisfying (22) as *stress-dissipative solutions.* This concept seems to be weaker than dissipative solutions in the following sense: when $d = 2$, any weak solution of (1) is stress-dissipative (indeed, by interpolation $u \in L^4(0,T;L^4(\Omega))$ and $\tau \in L^4(0,T;L^4(\Omega))$, which is sufficient for the purpose of testing the weak formulation by the solution). However, it is not clear whether such an arbitrary weak solution will also satisfy the energy inequality (3), which, as indicated in Theorem 3.1, would either require additional integrability of the velocity gradient or an $L^\infty(0,T;L^\infty(\Omega))$ bound on the second q-moment of the probability density $\hat{\psi}$.

5 Decay estimates for the corotational Hookean dumbbell model

In this short section we derive equilibration bounds for weak solutions of the corotational system (1), as constructed in the previous section.

Theorem 5.1. *Let $(u, \hat{\psi})$ be a dissipative weak solution of (1), as constructed in the proof of Theorem 2.3. The following equilibration estimates hold for any $t > 0$:*

1. $\|\tau(t)\|_{L^2(\Omega)} \leq \mathrm{e}^{-2t} \|\tau(0)\|_{L^2(\Omega)}$;
2. $\|u(t)\|_{L^2(\Omega)} \leq C(u_0, \hat{\psi}_0) \, \mathrm{e}^{-\frac{C_P \nu}{2} t}$;
3. $\|\hat{\psi} - 1\|_{L^1_M(\Omega \times D)} \leq C(\hat{\psi}_0) \, \mathrm{e}^{-\frac{t}{2}}$,

where C_P is the constant in the Poincaré inequality on Ω.

Proof. As shown in Section 4, the approximate solutions $(u^m, \hat{\psi}^m)$ satisfy the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} |\tau^m|^2 \,\mathrm{d}x + \mu \int_\Omega |\nabla_x \tau^m|^2 \,\mathrm{d}x + 2 \int_\Omega |\tau^m|^2 \,\mathrm{d}x = 0,$$

so that, in particular,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\tau^m|^2 \,\mathrm{d}x + 4 \int_\Omega |\tau^m|^2 \,\mathrm{d}x \leq 0,$$

and hence

$$\int_\Omega |\tau^m(t)|^2 \,\mathrm{d}x \leq \mathrm{e}^{-4t} \int_\Omega |\tau^m(0)|^2 \,\mathrm{d}x.$$

Passing to the limit, $m \to \infty$, this proves the first assertion of the theorem. Next, we consider (21):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} |u^m|^2 \,\mathrm{d}x + \nu \int_\Omega |\nabla_x u^m|^2 \,\mathrm{d}x &= - \int_\Omega \tau^m : \nabla_x u^m \,\mathrm{d}x \\ &\leq \frac{\nu}{2} \|\nabla_x u^m\|^2_{L^2(\Omega)} + \frac{1}{2\nu} \|\tau^m\|^2_{L^2(\Omega)}. \end{aligned}$$

Using the Poincaré inequality and the previous estimate, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^m|^2\,\mathrm{d}x + C_P\nu\int_\Omega |u^m|^2\,\mathrm{d}x \leq 2\mathrm{e}^{-4t}\|\tau(0)\|^2_{L^2(\Omega)},$$

whence

$$\|u(t)\|_{L^2(\Omega)} \leq C(u_0,\hat{\psi}_0)\mathrm{e}^{-\frac{C_P\nu}{2}t}.$$

For the last estimate, we need to derive a logarithmic bound on $\hat{\psi}$. To this end, we test the approximate Fokker–Planck equation (12) with the test function

$$\phi_{R,\delta}(t,x,q) := \chi_R(q)\ln(\hat{\psi}+\delta),$$

where $\delta > 0$ is arbitrary and χ_R is a cut-off function as used before. Using this test function in (12), and passing to the limit $R \to \infty$ and $\delta \to 0$ we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times D} M\mathcal{F}(\hat{\psi})\,\mathrm{d}q\,\mathrm{d}x + \int_{\Omega\times D} M\left(\mu\left|\nabla_x\sqrt{\hat{\psi}}\right|^2 + \left|\nabla_q\sqrt{\hat{\psi}}\right|^2\right)\mathrm{d}q\,\mathrm{d}x = 0,$$

where $\mathcal{F}(s) := s\ln s - s + 1$. Recall that

$$\int_D M\hat{\psi}\,\mathrm{d}q = 1 \quad \text{a.e. on } (0,T)\times\Omega.$$

Gross' logarithmic Sobolev inequality [21] then implies that

$$\int_D M\hat{\psi}\ln\hat{\psi}\,\mathrm{d}q \leq \int_D M\left|\nabla_q\sqrt{\hat{\psi}}\right|^2\mathrm{d}q.$$

Moreover, for a.e. $x \in \Omega$

$$\int_D M\mathcal{F}(\hat{\psi})\,\mathrm{d}q = \int_D M\hat{\psi}\ln\hat{\psi}\,\mathrm{d}q - \int_D M\hat{\psi}\,\mathrm{d}q + \int_D M\,\mathrm{d}q = \int_D M\hat{\psi}\ln\hat{\psi}\,\mathrm{d}q.$$

Consequently, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times D} M\mathcal{F}(\hat{\psi})\,\mathrm{d}q\,\mathrm{d}x + \int_{\Omega\times D} M\mathcal{F}(\hat{\psi})\,\mathrm{d}q\,\mathrm{d}x \leq 0,$$

and hence, using the Csiszár–Kullback inequality [16]

$$\left\|\hat{\psi}-1\right\|^2_{L^1_M(\Omega\times D)} = \|\psi - M\|^2_{L^1(\Omega\times D)} \leq \int_{\Omega\times D} M\mathcal{F}(\hat{\psi})\,\mathrm{d}q\,\mathrm{d}x \leq \mathrm{e}^{-t}\int_{\Omega\times D} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}q\,\mathrm{d}x.$$

□

6 Corotational Oldroyd-B model with stress diffusion

Consider the following corotational version of the well-known Oldroyd-B model for polymeric flows:

$$\begin{aligned} \partial_t u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p &= \operatorname{div}_x\tau,\\ \operatorname{div}_x u &= 0,\\ \partial_t\tau + (u\cdot\nabla_x)\tau + \tau\omega(u) - \omega(u)\tau + a\tau &= bD(u) + \mu\Delta_x\tau, \end{aligned} \tag{23}$$

posed on $[0,T]\times\Omega$, where again $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded open set with Lipschitz boundary[3] and $T>0$ is any fixed time. We recall the notation

$$\omega(u) := \frac{1}{2}\big(\nabla_x u - \nabla_x^T u\big), \quad D(u) := \frac{1}{2}\big(\nabla_x u + \nabla_x^T u\big),$$

for the vorticity tensor and deformation tensor, respectively. The system is supplemented with the following Dirichlet and no-flux boundary conditions:

$$u = 0 \ \text{ on } (0,T]\times\partial\Omega, \quad \nabla_x\tau_{ij}\cdot\hat{n} = 0 \ \text{ on } (0,T]\times\partial\Omega, \tag{24}$$

and initial conditions

$$u(0,x) = u_0(x) \in L^2_{\text{div}}(\Omega), \quad \tau_0(0,x) = \tau_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}}). \tag{25}$$

The constant $b\geq 0$ relates the relaxation and retardation times, and viscosity properties of the fluid, see [11].

In the absence of stress diffusion ($\mu=0$), Lions and Masmoudi [28] showed existence of global weak solutions for (23) for $d=2$ and $d=3$. Rigorous derivation of the (noncorotational) Hookean Navier–Stokes–Fokker–Planck system from Brownian dynamics leads to a centre-of-mass diffusion term in the Fokker–Planck equation (cf. [33]). Consequently, the stress evolution equation in the Oldroyd-B model, that results from the Hookean Fokker–Planck equation upon multiplication by $q\otimes q$ integration over $D=\mathbb{R}^d$ and formal manipulations, should contain a stress diffusion term. On the other hand, there is physical evidence that the diffusivity coefficient μ is very small compared to the viscosity of the fluid (in [10] it is estimated to typically be of order 10^{-9} to 10^{-7}). Nevertheless, from the point of view of mathematical analysis, it is advantageous to retain the stress-diffusion term. It is an interesting open problem whether solutions of (23) converge to solutions of the system without stress diffusion as $\mu\to 0$. The existence problem for (23) is significantly easier when $\mu>0$, and follows essentially the same line of argument as that applied for the corotational Navier–Stokes–Fokker–Planck equation. In the following subsection we outline the main steps in this argument, omitting full detail. Then, we discuss the issue of (conditional) uniqueness.

6.1 Existence and energy inequality

Definition 6.1. A pair (u,τ) is a dissipative weak solution of system (23) with initial data $(u_0,\tau_0)\in L^2_{\text{div}}(\Omega)\times L^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})$ if

$$\begin{aligned} &u \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;V) \cap W^{1,\frac{4}{3}}(0,T;V')),\\ &\tau \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d})) \cap L^2(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})), \end{aligned}$$

and the following identities hold for a.e. $t\in(0,T]$:

$$\begin{aligned} \int_{\Omega_t} u\cdot\partial_t\vartheta \,\mathrm{d}x\,\mathrm{d}t' &+ \int_{\Omega_t} (u\cdot\nabla_x)u\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t' + \nu\int_{\Omega_t}\nabla_x u : \nabla_x\vartheta\,\mathrm{d}x\,\mathrm{d}t'\\ &= \int_{\Omega_t}\tau:\nabla_x\vartheta\,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega u(t)\cdot\vartheta(t)\,\mathrm{d}x - \int_\Omega u_0\cdot\vartheta(0)\,\mathrm{d}x,\\ &\forall\,\vartheta\in\Big\{\vartheta\in L^2(0,T;V) \mid \partial_t\vartheta\in L^1(0,T;L^2(\Omega;\mathbb{R}^d))\Big\}, \end{aligned}$$

[3] In fact, all the arguments in this section are valid also in the periodic case, and in the whole space $\mathbb{R}^d$ (with appropriate decay conditions).

and

$$\int_{\Omega_t} \tau : \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t' + \int_{\Omega_t} (u \cdot \nabla_x)\tau : \phi \, \mathrm{d}x \, \mathrm{d}t' + \mu \int_{\Omega_t} \nabla_x \tau :: \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t'$$
$$+ \int_{\Omega_t} (\tau\omega(u) - \omega(u)\tau) : \phi \, \mathrm{d}x \, \mathrm{d}t' + a \int_{\Omega_t} \tau : \phi \, \mathrm{d}x \, \mathrm{d}t'$$
$$= b \int_{\Omega_t} D(u) : \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t' + \int_\Omega \tau(t) : \phi(t) \, \mathrm{d}x - \int_\Omega \tau_0 : \phi(0) \, \mathrm{d}x,$$
$$\forall \phi \in \Big\{ \phi \in L^2(0,T; H^1(\Omega; \mathbb{R}^{d\times d})) \mid \partial_t \phi \in L^1(0,T; L^2(\Omega; \mathbb{R}^{d\times d})) \Big\}.$$

Furthermore we require that the following energy inequality is satisfied for each $t \in (0,T]$:

$$\int_\Omega \frac{1}{2}|u(t)|^2 + \frac{1}{2b}|\tau(t)|^2 \, \mathrm{d}x + \frac{a}{b}\int_0^t \int_\Omega |\tau|^2 \, \mathrm{d}x \, \mathrm{d}t' + \nu \int_0^t \int_\Omega |\nabla_x u|^2 \, \mathrm{d}x \, \mathrm{d}t' + \frac{\mu}{b}\int_0^t \int_\Omega |\nabla_x \tau|^2 \, \mathrm{d}x \, \mathrm{d}t' \leq \int_\Omega \frac{1}{2}|u_0|^2 + \frac{1}{2b}|\tau_0|^2 \, \mathrm{d}x, \tag{26}$$

when $b > 0$, and

$$\int_\Omega \frac{1}{2}|u(t)|^2 + \frac{1}{2}|\tau(t)|^2 \, \mathrm{d}x + a\int_0^t \int_\Omega |\tau|^2 \, \mathrm{d}x \, \mathrm{d}t' + \nu \int_0^t \int_\Omega |\nabla_x u|^2 \, \mathrm{d}x \, \mathrm{d}t' + \mu\int_0^t \int_\Omega |\nabla_x \tau|^2 \, \mathrm{d}x \, \mathrm{d}t' \leq \int_\Omega \frac{1}{2}|u_0|^2 + \frac{1}{2b}|\tau_0|^2 \, \mathrm{d}x - \int_0^t \int_\Omega \tau : \nabla_x u \, \mathrm{d}x \, \mathrm{d}t', \tag{27}$$

when $b = 0$.

Note that when $d = 2$ then the energy inequality can be deduced directly from the weak formulation. Clearly the main issue in showing existence of solutions defined in this way is to pass to the limit, for appropriate approximations, in the nonlinear terms

$$\int_{\Omega_t} \tau\omega(u) : \phi \, \mathrm{d}x \, \mathrm{d}t', \quad \text{and} \quad \int_{\Omega_t} \tau : \nabla_x u \, \mathrm{d}x \, \mathrm{d}t'.$$

This requires strong compactness of the stress tensor in $L^2(0,T; L^2(\Omega))$, which is a straightforward consequence of the Aubin–Lions lemma. Indeed, from the energy inequality one deduces formally that τ is uniformly bounded in $L^\infty(0,T; L^2(\Omega))$ and $L^2(0,T; H^1(\Omega))$. Consequently, a uniform bound on the time derivative $\partial_t \tau$ in $L^{4/3}(0,T; [H^1(\Omega)]')$ follows easily. One possible approximation with these properties is, for instance, a Galerkin approximation as outlined previously in this paper. In this way one obtains the following existence result.

Theorem 6.2. *Let* $(u_0, \tau_0) \in L^2_{\mathrm{div}}(\Omega) \times L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. *There exists at least one dissipative weak solution of the corotational Oldroyd-B model* (23), (24), (25).

6.2 Weak-strong uniqueness

We shall now derive a relative energy inequality for the system (23), and use it to prove a conditional uniqueness result for dissipative weak solutions. As for the Newtonian case, in dimension two this will actually imply uniqueness, which will have some bearing on the corotational Navier–Stokes–Fokker–Planck system (1).

Relative energy inequality

We shall assume here that $b > 0$. The case for $b = 0$ follows with minor modifications (importantly, using energy inequality (27) in place of (26)). Let us take two solutions (u_1, τ_1), (u_2, τ_2) of (23), and assume additionally that

$$u_2 \in L^r(0, T; L^s(\Omega; \mathbb{R}^d)), \quad \tau_2 \in L^r(0, T; L^s(\Omega; \mathbb{R}^{d\times d})), \quad \text{for} \quad \frac{d}{s} + \frac{2}{r} = 1, \;\; s > d.$$

Under this additional integrability it is a routine matter to verify that one can 'cross-test' the weak formulation for one solution by the other solution. We omit the details for brevity and refer the reader to [20, Chapter 4].

Define the following relative energy

$$E_{rel}(t) = \frac{1}{2}\int_\Omega |u_1 - u_2|^2 \,\mathrm{d}x + \frac{1}{2b}\int_\Omega |\tau_1 - \tau_2|^2 \,\mathrm{d}x.$$

Expanding the squares, and using the energy inequality, we obtain

$$\begin{aligned}
E_{rel}(t) &+ \frac{a}{b}\int_{\Omega_t} |\tau_1 - \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_{\Omega_t} |\nabla_x u_1 - \nabla_x u_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \frac{\mu}{b}\int_{\Omega_t} |\nabla_x \tau_1 - \nabla_x \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\le \frac{1}{2}\int_\Omega |(u_1)_0|^2 + |(u_2)_0|^2 \,\mathrm{d}x + \frac{1}{2b}\int_\Omega |(\tau_1)_0|^2 + |(\tau_2)_0|^2 \,\mathrm{d}x \\
&\quad - \int_\Omega u_1(t)\cdot u_2(t)\,\mathrm{d}x - \frac{1}{b}\int_\Omega \tau_1(t) : \tau_2(t)\,\mathrm{d}x - \frac{2a}{b}\int_{\Omega_t} \tau_1 : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad - 2\nu \int_{\Omega_t} \nabla_x u_1 : \nabla_x u_2 \,\mathrm{d}x\,\mathrm{d}t' - \frac{2\mu}{b}\int_{\Omega_t} \nabla_x \tau_1 :: \nabla_x \tau_2 \,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned} \tag{28}$$

Now test the weak formulation for u_1, τ_1 with test functions u_2 and τ_2 to write

$$\begin{aligned}
&-\int_\Omega u_1(t)\cdot u_2(t)\,\mathrm{d}x - \nu\int_{\Omega_t} \nabla_x u_1 : \nabla_x u_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = -\int_\Omega (u_1)_0\cdot(u_2)_0\,\mathrm{d}x - \int_{\Omega_t} u_1\cdot\partial_t u_2 \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (u_1\cdot\nabla_x)u_1\cdot u_2 \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} \tau_1 : \nabla_x u_2 \,\mathrm{d}x\,\mathrm{d}t'
\end{aligned}$$

and

$$\begin{aligned}
&-\int_\Omega \tau_1(t) : \tau_2(t)\,\mathrm{d}x - \mu\int_{\Omega_t} \nabla_x\tau_1 :: \nabla_x\tau_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = -\int_\Omega (\tau_1)_0 : (\tau_2)_0\,\mathrm{d}x - \int_{\Omega_t} \tau_1 : \partial_t\tau_2 \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (u_1\cdot\nabla_x)\tau_1 : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad\quad + a\int_{\Omega_t} \tau_1 : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} (\tau_1\omega(u_1) - \omega(u_1)\tau_1) : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' - b\int_{\Omega_t} D(u_1) : \tau_2 \,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned}$$

For the terms involving the time derivative, we further note the following identities:

$$\begin{aligned}
&-\int_{\Omega_t} u_1\cdot\partial_t u_2 \,\mathrm{d}x\,\mathrm{d}t' - \nu\int_{\Omega_t} \nabla_x u_1 : \nabla_x u_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = \int_{\Omega_t} \partial_t u_1\cdot u_2 \,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega (u_1)_0\cdot(u_2)_0\,\mathrm{d}x - \int_\Omega u_1(t)\cdot u_2(t)\,\mathrm{d}x - \nu\int_{\Omega_t} \nabla_x u_1 : \nabla_x u_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = \int_{\Omega_t} (u_2\cdot\nabla_x)u_2\cdot u_1 \,\mathrm{d}x\,\mathrm{d}t' + \int_{\Omega_t} \tau_2 : \nabla_x u_1 \,\mathrm{d}x\,\mathrm{d}t'
\end{aligned}$$

and

$$
\begin{aligned}
&-\frac{1}{b}\int_{\Omega_t} \tau_1 : \partial_t \tau_2 \,\mathrm{d}x\,\mathrm{d}t' - \frac{\mu}{b}\int_{\Omega_t} \nabla_x \tau_1 :: \nabla_x \tau_2 \,\mathrm{d}x\,\mathrm{d}t' - \frac{a}{b}\int_{\Omega_t} \tau_1 : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad = \frac{1}{b}\int_{\Omega_t} (u_2 \cdot \nabla_x)\tau_2 : \tau_1 \,\mathrm{d}x\,\mathrm{d}t' + \frac{1}{b}\int_{\Omega_t} (\tau_2 \omega(u_2) - \omega(u_2)\tau_2) : \tau_1 \,\mathrm{d}x\,\mathrm{d}t' - \int_{\Omega_t} D(u_2) : \tau_1 \,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned}
$$

Furthermore, we observe that, thanks to τ being a symmetric tensor,

$$
\int_{\Omega_t} \tau_1 : \nabla_x u_2 + \tau_2 : \nabla_x u_1 - D(u_1) : \tau_2 - D(u_2) : \tau_1 \,\mathrm{d}x\,\mathrm{d}t' = 0.
$$

Substituting these identities into inequality (28), we obtain

$$
\begin{aligned}
E_{rel}(t) &+ \frac{a}{b}\int_{\Omega_t} |\tau_1 - \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_{\Omega_t} |\nabla_x u_1 - \nabla_x u_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \frac{\mu}{b}\int_{\Omega_t} |\nabla_x \tau_1 - \nabla_x \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\le \frac{1}{2}\int_{\Omega} |(u_1)_0 - (u_2)_0|^2 \,\mathrm{d}x + \frac{1}{2b}\int_{\Omega} |(\tau_1)_0 - (\tau_2)_0|^2 \,\mathrm{d}x \\
&\quad + \int_{\Omega_t} (u_1 \cdot \nabla_x) u_1 \cdot u_2 + (u_2 \cdot \nabla_x) u_2 \cdot u_1 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad + \frac{1}{b}\int_{\Omega_t} (u_1 \cdot \nabla_x)\tau_1 : \tau_2 + (u_2 \cdot \nabla_x)\tau_2 : \tau_1 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad + \frac{1}{b}\int_{\Omega_t} (\tau_1 \omega(u_1) - \omega(u_1)\tau_1) : \tau_2 + (\tau_2 \omega(u_2) - \omega(u_2)\tau_2) : \tau_1 \,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned}
$$

Finally, after some routine manipulations (taking advantage of the solenoidality of the velocity fields and symmetry of the stress tensors), we arrive at the following relative energy inequality:

$$
\begin{aligned}
E_{rel}(t) &+ \frac{a}{b}\int_{\Omega_t} |\tau_1 - \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_{\Omega_t} |\nabla_x u_1 - \nabla_x u_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \frac{\mu}{b}\int_{\Omega_t} |\nabla_x \tau_1 - \nabla_x \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\le \frac{1}{2}\int_{\Omega} |(u_1)_0 - (u_2)_0|^2 \,\mathrm{d}x + \frac{1}{2b}\int_{\Omega} |(\tau_1)_0 - (\tau_2)_0|^2 \,\mathrm{d}x \\
&\quad + \int_{\Omega_t} \nabla_x (u_1 - u_2) : ((u_1 - u_2) \otimes u_2) \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad + \frac{1}{b}\int_{\Omega_t} ((u_1 - u_2) \cdot \nabla_x)(\tau_1 - \tau_2) : \tau_2 \,\mathrm{d}x\,\mathrm{d}t' \\
&\quad + \frac{1}{b}\int_{\Omega_t} \nabla_x (u_1 - u_2)\tau_2 : (\tau_1 - \tau_2) + \nabla_x (u_1 - u_2)(\tau_1 - \tau_2) : \tau_2 \,\mathrm{d}x\,\mathrm{d}t'.
\end{aligned} \tag{29}
$$

Weak-strong uniqueness

Using the relative energy inequality derived in the last section, we can prove the following version of the classical Prodi–Serrin–Ladyzhenskaya weak-strong uniqueness property. In particular, in the two dimensional case we can actually infer uniqueness of weak solutions.

Theorem 6.3. *Let $d \in \{2, 3\}$ and let (u_i, τ_i), $i = 1, 2$, be two solutions emanating from the same initial data $u_0 \in L^2_{\mathrm{div}}(\Omega)$, $\tau_0 \in L^2(\Omega; \mathbb{R}^{d\times d})$. Suppose additionally that*

$$
u_2 \in L^r(0, T; L^s(\Omega; \mathbb{R}^d)), \quad \tau_2 \in L^r(0, T; L^s(\Omega; \mathbb{R}^{d\times d})), \quad \text{for} \quad \frac{d}{s} + \frac{2}{r} = 1, \;\; s > d.
$$

Then the two solutions coincide, i.e., $(u_1, \tau_1) = (u_2, \tau_2)$ a.e. in $(0, T) \times \Omega$.

Proof. Let I_1, I_2, and I_3, denote the terms on the right-hand side of the relative energy inequality (29). (The terms involving initial data are zero in the current setting.) The first two terms are bounded upon a straightforward application of the inequalities of Hölder, Gagliardo–Nirenberg, and Young (exactly as in the Newtonian case):

$$|I_1| \le \delta \int_{\Omega_t} |\nabla_x(u_1 - u_2)|^2 \,\mathrm{d}x\,\mathrm{d}t' + C_\delta \int_0^t \|u_2\|^r_{L^s(\Omega)} \int_\Omega |u_1 - u_2|^2 \,\mathrm{d}x\,\mathrm{d}t',$$

and

$$|I_2| \le \delta \int_{\Omega_t} |\nabla_x(\tau_1 - \tau_2)|^2 \,\mathrm{d}x\,\mathrm{d}t' + C_\delta \int_0^t \|\tau_2\|^r_{L^s(\Omega)} \int_\Omega |\tau_1 - \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t'.$$

The last term is treated very similarly:

$$\begin{aligned}
|I_3| &\le \int_0^t \|\nabla_x(u_1-u_2)\|_{L^2(\Omega)} \|\tau_1-\tau_2\|_{L^{2s/s-2}(\Omega)} \|\tau_2\|_{L^s(\Omega)} \,\mathrm{d}t' \\
&\lesssim \int_0^t \|\nabla_x(u_1-u_2)\|_{L^2(\Omega)} \|\tau_1-\tau_2\|^{1-d/s}_{L^2(\Omega)} \|\nabla_x(\tau_1-\tau_2)\|^{d/s}_{L^2(\Omega)} \|\tau_2\|_{L^s(\Omega)} \,\mathrm{d}t' \\
&\lesssim \delta \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)} \,\mathrm{d}t' + C_\delta \int_0^t \|\tau_1-\tau_2\|^{4/r}_{L^2(\Omega)} \|\nabla_x(\tau_1-\tau_2)\|^{2d/s}_{L^2(\Omega)} \|\tau_2\|^2_{L^s(\Omega)} \,\mathrm{d}t' \\
&\lesssim \delta \int_0^t \|\nabla_x(u_1-u_2)\|^2_{L^2(\Omega)} \,\mathrm{d}t' + \epsilon \int_0^t \|\nabla_x(\tau_1-\tau_2)\|^2_{L^2(\Omega)} \,\mathrm{d}t' \\
&\quad + C_{\delta,\epsilon} \int_0^t \|\tau_2\|^r_{L^s(\Omega)} \|\tau_1-\tau_2\|^2_{L^2(\Omega)} \,\mathrm{d}t'.
\end{aligned}$$

Choosing $\delta > 0$ and $\epsilon > 0$ small enough, we can absorb the gradient terms into the relative energy. We then end up with the following Gronwall-type inequality

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |u_1 - u_2|^2(t) \,\mathrm{d}x + \frac{1}{2b} \int_\Omega |\tau_1 - \tau_2|^2(t) \,\mathrm{d}x \\
&\qquad \lesssim \int_0^t \|\tau_2\|^r_{L^s(\Omega)} \int_\Omega |\tau_1 - \tau_2|^2 \,\mathrm{d}x\,\mathrm{d}t' + \int_0^t \|u_2\|^r_{L^s(\Omega)} \int_\Omega |u_1 - u_2|^2 \,\mathrm{d}x\,\mathrm{d}t',
\end{aligned}$$

from which we infer that $u_1 = u_2$ and $\tau_1 = \tau_2$ almost everywhere. ☐

Corollary 6.4. *Let $d = 2$. Then there is a unique dissipative weak solution to* (23).

Proof. Using the bounds in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for u and τ and a standard interpolation argument, we see that the conditions of the previous theorem are satisfied by any dissipative weak solution. ☐

6.3 The case $b = 0$

Let us consider now the special case of system (23) with $b = 0$. If additionally $\tau_0 = 0$, then $\tau \equiv 0$ solves the stress equation, and the system reduces to the classical Navier–Stokes equations of a viscous Newtonian fluid. If, however, the stress equation is nontrivial, the system can be considered as a particular case of the Oldroyd-B model, albeit a fairly peculiar one. On the one hand it can be rigorously shown to be the macroscopic closure of a corresponding corotational Navier–Stokes–Fokker–Planck model (as proved in the previous section), but on the other hand such a macroscopic model corresponding to $b = 0$ does not appear to be properly justified from the point of view of constitutive theory of viscoelastic fluids.

From the mathematical viewpoint, the main feature of the system with $b = 0$ is that there is no cancellation in the energy identity for the term arising from the right-hand side of the Navier–Stokes equation. There is, nevertheless, an energy balance satisfied by dissipative weak solutions (as an equality in the two-dimensional case and inequality in three-dimensions):

$$\int_\Omega \frac{1}{2}|u(t)|^2 + \frac{1}{2}|\tau(t)|^2 \,\mathrm{d}x + a\int_{\Omega_t} |\tau|^2 \,\mathrm{d}x\,\mathrm{d}t' + \nu \int_{\Omega_t} |\nabla_x u|^2 \,\mathrm{d}x\,\mathrm{d}t' + \mu \int_{\Omega_t} |\nabla_x \tau|^2 \,\mathrm{d}x\,\mathrm{d}t'$$
$$\leq \int_\Omega \frac{1}{2}|u_0|^2 + \frac{1}{2}|\tau_0|^2 \,\mathrm{d}x - \int_{\Omega_t} \tau : \nabla_x u \,\mathrm{d}x\,\mathrm{d}t'.$$

With this energy balance incorporated into the definition of a weak solution, the conclusions of Theorem 6.3 and Corollary 6.4 remain true, thus providing weak-strong uniqueness when $d = 3$ and uniqueness when $d = 2$.

Let us now focus on the two-dimensional case and discuss the ramifications of the uniqueness theorem. First, we can state a 'representation' result. Of course, given a symmetric 2×2 matrix τ_0, it is not in general true that there exists a nonnegative density ψ_0 such that $\tau_0 = \int_D \psi_0\, q \otimes q \,\mathrm{d}q - \mathrm{Id}$ (think, for example, of a diagonal matrix with one entry equal to zero). However, we have the following result.

Corollary 6.5. *Let $u_0 \in L^2_{\mathrm{div}}(\Omega)$ and suppose $\tau_0 \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ admits a representation as*

$$\tau_0(x) = \int_D \psi_0(x, q) q \otimes q \,\mathrm{d}q - \mathrm{Id},$$

for some nonnegative ψ_0 with $\int_D \psi_0(x) \,\mathrm{d}q = 1$ for a.e. $x \in \Omega$. Then, the solution (u, τ) of system (17) with initial data (u_0, τ_0) satisfies

$$\tau(t, x) = \int_D M(q)\hat{\psi}(t, x, q) q \otimes q \,\mathrm{d}q - \mathrm{Id}, \quad \textit{for a.e. } (t, x) \in (0, T) \times \Omega,$$

where $(u, \hat{\psi})$ is a dissipative weak solution of the corotational Navier–Stokes–Fokker–Planck equation (1).

Proof. Given (u_0, τ_0) as above, Theorem 6.2 provides a dissipative weak solution $(u_{\mathrm{OB}}, \tau_{\mathrm{OB}})$ of the Oldroyd-B model (17). On the other hand, Theorem 2.3 gives a weak solution $(u_{\mathrm{FP}}, \hat{\psi}_{\mathrm{FP}})$ of (1) with initial data $(u_0, \hat{\psi}_0)$. Now, Theorem 4.1 gives another solution $(\bar{u}_{\mathrm{OB}}, \bar{\tau}_{\mathrm{OB}})$ for (17) with

$$u_{\mathrm{FP}} = \bar{u}_{\mathrm{OB}}, \quad \bar{\tau}_{\mathrm{OB}} = \int_D M \hat{\psi}_{\mathrm{FP}}\, q \otimes q \,\mathrm{d}q - \mathrm{Id}\,.$$

Finally, the uniqueness assertion of Corollary 6.4 implies that $u_{\mathrm{OB}} = \bar{u}_{\mathrm{OB}}$ and $\tau_{\mathrm{OB}} = \bar{\tau}_{\mathrm{OB}}$. □

Admittedly, the above statement is not particularly practical, since explicitly identifying such a scalar density $\hat{\psi}_0$ for a given symmetric matrix-valued function τ_0 seems extremely difficult, even when $\hat{\psi}_0$ is known to exist. Note also that such a representation of τ in terms of $\hat{\psi}$ might not be unique, since we have no uniqueness result for the system (1). On the other hand, any two solutions to (1) give rise to the same τ. We can guarantee uniqueness of $\hat{\psi}$ for smooth initial data. Let $H^1_n(\Omega)$ denote the completion of $\{\zeta \in C^\infty(\bar{\Omega}) \mid \nabla_x \zeta \cdot \hat{n} = 0 \text{ on } \partial\Omega\}$ in the norm of $H^1(\Omega)$.

Theorem 6.6. *Let $d = 2$. Let $u_0 \in V$ and $\hat{\psi}_0 \in L^p_M(\Omega \times D)$, $p > 2$, be such that*

$$\sigma_0(x) := \int_D M \hat{\psi}_0\, q \otimes q \,\mathrm{d}q \in H^1_n(\Omega).$$

Then, there is a unique weak solution to the corotational Navier–Stokes–Fokker–Planck model (1) with initial data $(u_0, \hat{\psi}_0)$.

Proof. Suppose $(u_1, \hat{\psi}_1)$ and $(u_2, \hat{\psi}_2)$ are two solutions of (1) with the same initial data $(u_0, \hat{\psi}_0)$. Then, by Corollary 6.4, they give rise to the same dissipative weak solution (u, τ) of the corotational Oldroyd-B system (17). It follows that $u_1 = u_2 = u$. However, we need additional regularity to ensure that the Fokker–Planck solutions agree. This can be done by applying standard elliptic regularity theory. We refer the reader to Section 3 and Theorem 3.2 in [9], which implies, in particular, that

$$\nabla_x u \in L^2(0,T;L^r(\Omega)), \quad \text{for any } r \in [2,\infty).$$

This regularity is sufficient to justify testing the Fokker–Planck equation by its solution. Subtracting the weak formulation for $\hat{\psi}_2$ from that for $\hat{\psi}_1$ and using the difference $\hat{\psi}_1 - \hat{\psi}_2$ as a test function we obtain

$$\int_{\Omega\times D} M|\hat{\psi}_1 - \hat{\psi}_2|^2 \,\mathrm{d}q\,\mathrm{d}x + \int_{\Omega_t}\int_D M\Big(\mu|\nabla_x(\hat{\psi}_1 - \hat{\psi}_2)|^2 + |\nabla_q(\hat{\psi}_1 - \hat{\psi}_2)|^2\Big)\,\mathrm{d}q\,\mathrm{d}x\,\mathrm{d}t' = 0,$$

which implies that $\hat{\psi}_1 = \hat{\psi}_2$ almost everywhere. □

Remark 6.7. Notice that we require $p > 2$ in the initial data because from elliptic regularity we get that $\nabla_x u \in L^{\frac{4}{3}}(0,T;L^\infty(\Omega))$, so we are still lacking the required integrability with respect to t. It would be interesting to know if one can prove the assertion of the theorem without requiring integrability of the initial velocity, and deducing uniqueness of $\hat{\psi}$ just from the gain in regularity of the associated symmetric stress tensor. Finally, let us observe that the $H^1(\Omega)$ regularity of σ_0 is guaranteed by requiring that $\nabla_x\hat{\psi}_0 \in L^2_M(\Omega\times D)$. (That is, only regularity of $\hat{\psi}_0$ with respect to x is required.)

6.4 Zero stress diffusion, $\mu = 0$

In the absence of a stress diffusion term (i.e., when $\mu = 0$), we are not able to recover the above weak-strong uniqueness/uniqueness results, because the energy does not contain the L^2 norm of the gradient of the stress tensor τ, which was crucial to estimate the remainder terms in the relative energy inequality. To formulate a weak-strong uniqueness theorem, we therefore require additional assumptions on the stress of the strong solution:

$$d = 3, \quad \tau_2 \in L^\infty((0,T)\times\Omega), \quad \nabla_x\tau_2 \in L^\infty(0,T;L^3) \;\; (\text{or } \;\nabla_x\tau_2 \in L^8(0,T;L^4)),$$

and

$$d = 2, \quad \tau_2 \in L^\infty((0,T)\times\Omega), \quad \nabla_x\tau_2 \in L^\infty(0,T;L^\infty).$$

The first assumption comes from term I_3:

$$\begin{aligned}|I_3| &\le \int_0^t\int_\Omega |\nabla_x(u_1 - u_2)||\tau_2||\tau_1 - \tau_2|\,\mathrm{d}x\,\mathrm{d}t' \\ &\le \|\tau_2\|_{L^\infty((0,T)\times\Omega)}\left(\delta\int_0^t \|\nabla_x(u_1 - u_2)\|^2_{L^2(\Omega)}\right)\mathrm{d}t' + C_\delta\int_0^t \|\tau_1 - \tau_2\|^2_{L^2(\Omega)}\,\mathrm{d}t',\end{aligned}$$

while the second assumption is needed to control I_2:

$$|T_2| \le \int_0^t \underbrace{\|u_1 - u_2\|_{L^6(\Omega)}}_{\le \|\nabla_x(u_1-u_2)\|_{L^2(\Omega)}} \|\nabla_x\tau_2\|_{L^3(\Omega)}\|\tau_1 - \tau_2\|_{L^2(\Omega)}\,\mathrm{d}t'.$$

7 Some concluding remarks

Modification of the Fokker–Planck equation

A notable drawback of the corotational Oldroyd-B model (23) (with $b > 0$) is that it does not seem to arise as the macroscopic closure of any Fokker–Planck equation. This is true of the degenerate case $b = 0$ and of the general noncorotational case (at least formally, or for smooth initial data when $d = 2$). To generate this equation from a coupled Navier–Stokes–Fokker–Planck system, one could introduce an extra term on the left-hand side of the Fokker–Planck equation:

$$\mathrm{div}_q(D(u)qM). \tag{30}$$

The motivation for this could be to simplify the general case in the following way

$$(\nabla u)q\psi = \omega(u)q\psi + D(u)q\psi \approx \omega(u)q\psi + D(u)qM.$$

(So, roughly speaking, we consider the action of the vorticity tensor on the 'full' ψ, while for the action of the deformation tensor we assume that any ψ behaves like the Maxwellian.) Notice that, upon introducing the term (30) into the Fokker–Planck equation, the Maxwellian M is no longer a stationary solution of the resulting equation (M is not, in fact, a stationary solution in the general noncorotational case either unless u is identically 0; M is only a stationary solution of the Fokker–Planck equation for an arbitrary u in the case of the original corotational model).

Clearly, the addition of the extra term does not change anything in our proof of existence of solutions to system (1). (One qualitative difference is that there is a restriction on the propagation of the $L^p_M(\Omega \times D)$ norm: one has to require that $p \in (2, 2 + \frac{4}{d})$.) Additionally, the 'augmented' system enjoys an energy equality thanks to a cancellation between the newly introduced term and the divergence of the extra-stress tensor appearing on the right-hand side of the Navier–Stokes equation – analogously to how the $bD(u)$ term in the corotational Oldroyd-B equation cancels with the $\mathrm{div}_x\,\tau$ term.

One can show that any weak solution of the modified system gives rise to a weak solution of the following corotational Oldroyd-B type system:

$$\begin{aligned}\partial_t u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p &= \mathrm{div}_x\,\tau,\\ \partial_t\tau + (u\cdot\nabla_x)\tau - \mu\Delta_x\tau + \tau\omega(u) - \omega(u)\tau + 2\tau &= (BD(u) + D(u)B),\end{aligned}$$

where B is a constant symmetric tensor. (In fact $B = \int_D Mq\otimes q\,\mathrm{d}q$.) This system can be then reduced to the corotational Oldroyd-B model by introducing the simplification that $B = b\,\mathrm{Id}$ for some $b > 0$. (Alternatively, if one wanted to recover the cancellation property in the energy equality, one would have to modify the stress τ in the Navier–Stokes equation and write $B\tau + \tau B$ instead.)

There is, however, one serious flaw of this programme, which is that the introduction of the term (30) obstructs guaranteeing that the Fokker–Planck equation preserves nonnegativity of the probability density function ψ (or, at least the derivation of a lower bound on ψ). Indeed, the new term acts as an external forcing and has no sign. The problem of preserving positivity is present also in the corotational Oldroyd-B equation (23). Consider for a moment the general Oldroyd-B model. While it cannot be guaranteed that the stress tensor τ is positive semi-definite, one can expect preservation of positive semi-definiteness for the conformation tensor $\sigma \approx \tau + \mathrm{Id}$. (The $D(u)$ term is then 'hidden' in the upper convective derivative of σ.) This is especially evident when a solution is obtained through a solution of the noncorotational Navier–Stokes–Fokker–Planck system, as then $\sigma = \int_D M\hat{\psi}\,q\otimes q\,\mathrm{d}q$. We refer the reader to [22] for further details. Let us point out that positivity of the conformation tensor is a desirable property from the point of view of numerical simulations. It is a notorious phenomenon in computational rheology that numerical simulations of Oldroyd fluids break down at 'frustratingly low' values of the Weissenberg number (this is known as the 'high Weissenberg number problem') – a useful technique is then to consider the log-conformation representation of the conformation tensor equation. This is, naturally, only defined when σ is a positive definite tensor, see [18, 19]

On the other hand, the model (23) does not enjoy this property. The equation satisfied by the conformation tensor σ is then

$$\partial_t \sigma + (u \cdot \nabla_x)\sigma + \sigma\omega(u) - \omega(u)\sigma + a(\sigma - \mathrm{Id}) = bD(u) + \mu\Delta_x\sigma,$$

that is, the forcing $D(u)$ persists and propagation of positivity from the initial datum is unclear.

The corotational derivative

As remarked above, there does not seem to exist any micro-macro model which gives rise to the corotational Oldroyd-B model (23). One could alternatively look for a justification of this model in the constitutive theory of viscoelastic fluids. Consider the general equation for the balance of linear momentum

$$\rho\frac{Du}{Dt} = \mathrm{div}\, \tau_{tot},$$

where $\rho \equiv$ const. is the constant density of the fluid, $\frac{D}{Dt}$ denotes the material derivative and τ_{tot} is the total stress tensor

$$\tau_{tot} = -p\,\mathrm{Id} + 2\nu D(u) + \tau,$$

where τ is the (viscoelastic) extra stress tensor, which is assumed to satisfy the generalised Maxwell equation

$$\tau + \lambda_r \frac{\delta\tau}{\delta t} = \nu D(u),$$

where λ_r is the relaxation time and $\frac{\delta\tau}{\delta t}$ is an objective time derivative (Maxwell actually proposed $\frac{\partial}{\partial t}$ which is not frame-invariant). The classical Oldroyd-B model arises by taking the upper convective derivative

$$\frac{\delta\tau}{\delta t} = \partial_t\tau + (u \cdot \nabla_x)\tau - \left(\nabla u\tau + \tau\nabla_x^T u\right),$$

but one can consider more general objective derivatives

$$\frac{\delta\tau}{\delta t} = \frac{D\tau}{Dt} + \tau\omega(u) - \omega(u)\tau - \epsilon(\tau D(u) + D(u)\tau), \tag{31}$$

for $\epsilon \in [-1, 1]$. The value $\epsilon = 1$ then corresponds to the Oldroyd-B model, while the value $\epsilon = 0$ gives the corotational model. While both models have predictive merit, there seems to be agreement in the literature that the upper convective case is the most relevant for polymeric flows, see for instance [11]. Let us point out, however, that $\epsilon = 0$ is the only value for which (31) has the following properties: the unit tensor has zero derivative and the derivative of a symmetric tensor is symmetric.

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References

[1] J. W. Barrett and S. Boyaval. Existence and approximation of a (regularized) Oldroyd-B model. *Math. Models Methods Appl. Sci.*, 21(9):1783–1837, 2011.

[2] J. W. Barrett, C. Schwab, and E. Süli. Existence of global weak solutions for some polymeric flow models. *Mathematical Models and Methods in Applied Sciences*, 15(06):939–983, 2005.

[3] J. W. Barrett and E. Süli. Existence of global weak solutions to coupled Navier–Stokes–Fokker–Planck systems: a brief survey. *Novi Sad J. Math.*, 38(3):7–14, 2008.

[4] J. W. Barrett and E. Süli. Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains. *Math. Models Methods Appl. Sci.*, 21(6):1211–1289, 2011.

[5] J. W. Barrett and E. Süli. Existence and equilibration of global weak solutions to kinetic models for dilute polymers II: Hookean-type models. *Math. Models Methods Appl. Sci.*, 22(5):1150024, 84, 2012.

[6] J. W. Barrett and E. Süli. Existence of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers with variable density and viscosity. *J. Differential Equations*, 253(12):3610–3677, 2012.

[7] J. W. Barrett and E. Süli. Finite element approximation of finitely extensible nonlinear elastic dumbbell models for dilute polymers. *ESAIM Math. Model. Numer. Anal.*, 46(4):949–978, 2012.

[8] J. W. Barrett and E. Süli. Reflections on Dubinskiĭ's nonlinear compact embedding theorem. Publ. Inst. Math. (Belgrade) (N.S.) 91(105): 95-110, 2012.

[9] J. W. Barrett and E. Süli. Existence of global weak solutions to the kinetic Hookean dumbbell model for incompressible dilute polymeric fluids. *Nonlinear Anal. Real World Appl.*, 39:362–395, 2018.

[10] A. V. Bhave, R. C. Armstrong, and R. A. Brown. Kinetic theory and rheology of dilute, nonhomogeneous polymer solutions. *The Journal of Chemical Physics*, 95(4):2988–3000, 2021/12/13 1991.

[11] R. Bird, C. Curtiss, A. R., and O. Hassanger. *Dynamics of Polymeric Liquids*. John Wiley and Sons, 1987.

[12] M. Bulíček, J. Málek, and E. Süli. Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous dilute polymers. *Comm. Partial Differential Equations*, 38(5):882–924, 2013.

[13] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84–112, 2001.

[14] P. Constantin and M. Kliegl. Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress. *Arch. Ration. Mech. Anal.*, 206(3):725–740, 2012.

[15] P. Constantin and W. Sun. Remarks on Oldroyd-B and related complex fluid models. *Commun. Math. Sci.*, 10(1):33–73, 2012.

[16] I. Csiszár. Information-type measures of difference of probability distributions. *Stud. Sc. Math. Hung.*, 1:227–230, 1966.

[17] D. Fang and R. Zi. Global solutions to the Oldroyd-B model with a class of large initial data. *SIAM J. Math. Anal.*, 48(2):1054–1084, 2016.

[18] R. Fattal and R. Kupferman. Constitutive laws for the matrix-logarithm of the conformation tensor. *Journal of Non-Newtonian Fluid Mechanics*, 123(2):281–285, 2004.

[19] R. Fattal and R. Kupferman. Time-dependent simulation of viscoelastic flows at high Weissenberg number using the log-conformation representation. *Journal of Non-Newtonian Fluid Mechanics*, 126(1):23–37, 2005.

[20] G. P. Galdi. *An Introduction to the Navier–Stokes Initial-Boundary Value Problem*, pages 1–70. Birkhäuser Basel, Basel, 2000.

[21] L. Gross. Logarithmic Sobolev Inequalities. *American Journal of Mathematics*, 97(4):1061-–1083, 1975.

[22] D. Hu and T. Lelièvre. New entropy estimates for the Oldroyd-B model and related models. *Communications in Mathematical Sciences*, 5(4):909–916, 2007.

[23] X. Hu and F. Lin. Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data. *Comm. Pure Appl. Math.*, 69(2):372–404, 2016.

[24] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.*, 209(1):162–193, 2004.

[25] J. La. On diffusive 2D Fokker–Planck–Navier–Stokes systems. *Arch. Ration. Mech. Anal.*, 235(3):1531–1588, 2020.

[26] Z. Lei, C. Liu, and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.

[27] F.-H. Lin, C. Liu, and P. Zhang. On a micro-macro model for polymeric fluids near equilibrium. *Comm. Pure Appl. Math.*, 60(6):838–866, 2007.

[28] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[29] N. Masmoudi. Global existence of weak solutions to macroscopic models of polymeric flows. *J. Math. Pures Appl. (9)*, 96(5):502–520, 2011.

[30] N. Masmoudi. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Invent. Math.*, 191(2):427–500, 2013.

[31] N. Masmoudi, P. Zhang, and Z. Zhang. Global well-posedness for 2D polymeric fluid models and growth estimate. *Phys. D*, 237(10-12):1663–1675, 2008.

[32] J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. Roy. Soc. London Ser. A*, 200:523–541, 1950.

[33] E. Süli and G. Yahiaoui. McKean–Vlasov diffusion and the well-posedness of the Hookean bead-spring-chain model for dilute polymeric fluids: small-mass limit and equilibration in momentum space. Available from *arXiv:1802.06268*, 2018.