Mathematical Modelling and Scientific Computing

B1 Numerical Linear Algebra and Numerical Solution of Differential Equations
Hilary Term 2020

Do not turn this page until you are told that you may do so

1. Consider the initial-value problem $y' = f(x, y)$, $y(0) = 1$, where f is a real-valued twice continuously differentiable function of its arguments such that $|\frac{\partial f}{\partial y}(x, y)| \leqslant L$ for all $(x, y) \in \mathbb{R}^2$, where L is a positive real number. Suppose further that the unique solution y of this initial-value problem is a three times continuously differentiable function of x on the interval $[0, 1]$ of the real line. Let N be a positive integer, $h := 1/N$, $x_n := nh$ for $n = 0, 1, \ldots, N$, and let y_n be an approximation to $y(x_n)$, $n = 0, 1, \ldots, N$, defined successively by the explicit one-step method

$$y_{n+1} := y_n + hf(x_n + \beta h, y_n + \beta h f(x_n, y_n)), \quad n = 0, 1, \ldots, N-1, \qquad y_0 := 1,$$

where $\beta \in [0, 1]$ is a parameter.

(a) [5 marks] Show that the method is consistent for any value of β.

(b) [12 marks] Show that the consistency error T_n of the method can be expressed as

$$T_n = h\left(\frac{1}{2} - \beta\right) y''(x_n) + \mathcal{O}(h^2).$$

Deduce that if $\beta \neq \frac{1}{2}$ then the method is first-order accurate, and that if $\beta = \frac{1}{2}$ then it is at least second-order accurate.

Show further by deriving a bound on the global error of the method in terms of the consistency error that if $\beta = \frac{1}{2}$ then the order of convergence of the method is at least 2.

(c) [8 marks] Apply the method, with $\beta = \frac{1}{2}$, to the initial-value problem $y' = y$, $y(0) = 1$, and show that

$$y_n = \left(1 + h + \frac{1}{2}h^2\right)^n, \qquad n = 0, 1, \ldots, N.$$

By using the identity $a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^{n-k-1}b^k$, where a and b are arbitrary real numbers and n is a positive integer, show that

$$y(x_n) - y_n \leqslant \frac{1}{3!}h^2 x_n \mathrm{e}^{x_n}, \qquad n = 0, 1, \ldots, N.$$

Show further that

$$y(x_n) - y_n \geqslant \frac{1}{3!}h^2 x_n, \qquad n = 0, 1, \ldots, N.$$

Hence deduce that for $\beta = 1/2$ the order of convergence of the method is equal to 2.

2. Consider the ordinary differential equation $y' = f(x, y)$, where f is a real-valued continuous function defined for all $(x, y) \in \mathbb{R}^2$, and let $x_0, y_0 \in \mathbb{R}$.

   (a) [2 marks] State the general form of a linear k-step method for the numerical solution of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the mesh $\{x_n : x_n = x_0 + nh, n = 0, 1, \ldots\}$ of uniform spacing $h > 0$.

   (b) [6 marks] Define the *consistency error* of a linear k-step method. What is meant by saying that a linear k-step method is *consistent*? What is meant by saying that a linear multistep method is *second-order accurate*?

   (c) [6 marks] What is meant by saying that a linear k-step method is *zero-stable*? Formulate an equivalent characterisation of zero-stability in terms of the roots of a certain polynomial of degree k.

   (d) [11 marks] Consider the three-parameter family of linear two-step methods defined by

$$y_{n+2} - ay_{n+1} + by_n = h\,c\,f_{n+2},$$

   where $f_j = f(x_j, y_j)$, and a, b and c are real numbers. Show that there exists a unique choice of a, b and c such that the method is second-order accurate; show further that, for these values of a, b and c, the method is second-order convergent.

   [If Dahlquist's Theorem is used, it must be stated carefully.]

3. Consider the initial-value problem

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < t \leqslant T,$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty,$$

where T is a fixed real number, and u_0 is a real-valued continuous function of $x \in (-\infty, \infty)$.

(a) [5 marks] Formulate the θ scheme for the numerical solution of this initial-value problem on a mesh with uniform spacings $\Delta x > 0$ and $\Delta t = T/M$ in the x and t co-ordinate directions, respectively, where M is a positive integer. You should state the scheme so that $\theta = 1$ corresponds to the implicit (backward) Euler scheme.

(b) [10 marks] Let U_j^m denote the θ-scheme-approximation to $u(j\Delta x, m\Delta t)$, $0 \leqslant m \leqslant M$, $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Let $\|U^m\|_{\ell_\infty} := \max_{j\in\mathbb{Z}} |U_j^m|$, and suppose that $\|U^0\|_{\ell_\infty}$ is finite. Show that if $\theta \in [0,1]$ then

$$\|U^m\|_{\ell_\infty} \leqslant \left(\frac{1-(1-\theta)\Delta t}{1+\theta\Delta t}\right)^m \|U^0\|_{\ell_\infty}$$

for all m, $1 \leqslant m \leqslant M$, provided that $A(\theta)\Delta t \leqslant \frac{(\Delta x)^2}{2+(\Delta x)^2}$, where $A(\theta)$ is a constant, depending on the choice of θ, which you should determine.

Deduce that the implicit (backward) Euler scheme is *unconditionally stable* in the $\|\cdot\|_{\ell_\infty}$ norm. Show, further, that the Crank–Nicolson scheme is *conditionally stable* in the $\|\cdot\|_{\ell_\infty}$ norm and state the condition on Δt and Δx that ensures stability.

(c) [10 marks] Let U_j^m denote the θ-scheme-approximation to $u(j\Delta x, m\Delta t)$, $0 \leqslant m \leqslant M$, $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Let $\|U^m\|_{\ell_2} := \left(\Delta x \sum_{j\in\mathbb{Z}} |U_j^m|^2\right)^{1/2}$ and suppose that $\|U^0\|_{\ell_2}$ is finite. Show that if $\theta \in [\frac{1}{2}, 1]$, then

$$\|U^m\|_{\ell_2} \leqslant \|U^0\|_{\ell_2}$$

for all m, $1 \leqslant m \leqslant M$, for any Δt and Δx.

Now, suppose that $\theta \in [0, \frac{1}{2})$. Show that $\|U^m\|_{\ell_2} \leqslant \|U^0\|_{\ell_2}$ for all m, $1 \leqslant m \leqslant M$, provided that $B(\theta)\Delta t \leqslant \frac{2(\Delta x)^2}{4+(\Delta x)^2}$, where $B(\theta)$ is a constant, depending on the choice of θ, which you should determine.

Deduce that the implicit (backward) Euler scheme and the Crank–Nicolson scheme are *unconditionally stable* in the $\|\cdot\|_{\ell_2}$ norm.

4. Consider the finite difference mesh $\mathcal{M} := \{(x_j, t_m) : j = 0, 1, \ldots, J,\ m = 0, 1, \ldots, M\}$, where $x_j := j\Delta x$ and $t_m := m\Delta t$, with $\Delta x := 1/J$, $\Delta t := T/M$, $J \geqslant 2$, $M \geqslant 1$, and $T > 0$.

   (a) [5 marks] Formulate the explicit Euler scheme on $\mathcal{M}$ for the numerical solution of the initial-boundary-value problem

$$u_t = \kappa u_{xx}, \quad x \in (0,1), \quad t \in (0,T];$$

$$u(0,t) = A(t), \quad u(1,t) = B(t), \quad t \in (0,T]; \qquad u(x,0) = u_0(x), \quad x \in [0,1],$$

   where κ is a positive real number, A, B are continuous real-valued functions defined on $[0,T]$, and u_0 is a continuous real-valued function defined on $[0,1]$ with $u_0(0) = A(0)$ and $u_0(1) = B(0)$.

   (b) [10 marks] Show that U_j^m, the approximation to $u(x_j, t_m)$ computed from the explicit Euler scheme, is bounded above by $U_{\max}$, where

$$U_{\max} := \max\left\{ \max_{0 \leqslant m \leqslant M} A(t_m),\ \max_{0 \leqslant m \leqslant M} B(t_m),\ \max_{0 \leqslant j \leqslant J} u_0(x_j) \right\},$$

   provided that a stability condition of the form

$$0 < \mu \leqslant \mu_0$$

   is satisfied with $\mu := \frac{\kappa \Delta t}{(\Delta x)^2}$, where μ_0 is a positive real number, independent of Δt and Δx, which you should determine.

   (c) [10 marks] Write down the recurrence relation satisfied by the *global error* at the mesh points, defined by

$$e_j^m := u(x_j, t_m) - U_j^m.$$

   Assuming that the initial and boundary conditions for the explicit Euler scheme are exact, and Δt and Δx are such that $0 < \mu \leqslant \mu_0$, derive a bound on

$$E^m := \max_{0 \leqslant j \leqslant J} |e_j^m|, \quad 0 \leqslant m \leqslant M,$$

   in terms of

$$T^m := \max_{1 \leqslant j \leqslant J-1} |T_j^m|, \quad 0 \leqslant m \leqslant M,$$

   where T_j^m is the *consistency error* of the explicit Euler scheme at the mesh point (x_j, t_m).