3. (a) [5 marks] The θ-scheme has the form

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + [\theta U_j^{m+1} + (1-\theta)U_j^m] = \theta \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2},$$

for $j \in \mathbb{Z}$, $m = 0, \ldots, M-1$, where $\Delta x > 0$ and $\Delta t = T/M$, $M \geqslant 1$, and

$$U_j^0 = u_0(j\Delta t), \qquad j \in \mathbb{Z}.$$

[Bookwork.] **[5 marks]**

(b) [10 marks] Let $\mu = \Delta t/(\Delta x)^2$, and rewrite the scheme as

$$(1 + \theta\Delta t + 2\theta\mu)U_j^{m+1} = \theta\mu(U_{j+1}^{m+1} + U_{j-1}^{m+1}) + (1-\theta)\mu(U_{j+1}^m + U_{j-1}^m) + (1 - (1-\theta)\Delta t - 2(1-\theta)\mu)U_j^m.$$

Suppose that $1 - (1-\theta)\Delta t - 2(1-\theta)\mu \geqslant 0$. Then, since both $\theta \geqslant 0$ and $1 - \theta \geqslant 0$, we have that

$$(1 + \theta\Delta t + 2\theta\mu)|U_j^{m+1}| \leqslant 2\theta\mu\|U^{m+1}\|_{\ell_\infty} + 2(1-\theta)\mu\|U^m\|_{\ell_\infty} + (1 - (1-\theta)\Delta t - 2(1-\theta)\mu)\|U^m\|_{\ell_\infty}.$$

Taking the maximum over all $j \in \mathbb{Z}$,

$$(1 + \theta\Delta t + 2\theta\mu)\|U^{m+1}\|_{\ell_\infty} \leqslant 2\theta\mu\|U^{m+1}\|_{\ell_\infty} + 2(1-\theta)\mu\|U^m\|_{\ell_\infty} + (1 - (1-\theta)\Delta t - 2(1-\theta)\mu)\|U^m\|_{\ell_\infty},$$

and hence, $\|U^{m+1}\|_{\ell_\infty} \leqslant [(1 - (1-\theta)\Delta t)/(1 + \theta\Delta t)]\|U^m\|_{\ell_\infty}$. Thus we have shown that if

$$1 - (1-\theta)\Delta t - 2(1-\theta)\mu \geqslant 0$$

i.e., if $A(\theta)\Delta t \leqslant (\Delta x)^2/(2 + (\Delta x)^2)$ where $A(\theta) = 1 - \theta$, then

$$\begin{aligned}\|U^{m+1}\|_{\ell_\infty} &\leqslant [(1 - (1-\theta)\Delta t)/(1 + \theta\Delta t)]\|U^m\|_{\ell_\infty} \\ &\leqslant \cdots \leqslant [(1 - (1-\theta)\Delta t)/(1 + \theta\Delta t)]^{m+1}\|U^0\|_{\ell_\infty}.\end{aligned}$$

As $A(1) = 0$, the implicit (backward) Euler scheme, corresponding to $\theta = 1$ is unconditionally stable in the $\|\cdot\|_{\ell_\infty}$ norm.

For the Crank–Nicolson scheme, corresponding to $\theta = 1/2$, we have $A(1/2) = 1/2$, so we have conditional stability, provided that $\Delta t \leqslant 2(\Delta x)^2/(2 + (\Delta x)^2)$.

[Extension of bookwork to an unseen example.] **[10 marks]**

(c) [10 marks] Upon taking the (semi-discrete) Fourier transform of the finite difference scheme, with $\hat{U}^m(k)$ denoting the semi-discrete Fourier transform of the mesh-function U_j^m, we get, after some simplification,

$$\frac{\hat{U}^{m+1}(k) - \hat{U}^m(k)}{\Delta t} + \theta\hat{U}^{m+1}(k) + (1-\theta)\hat{U}^m(k) = \theta\hat{U}^{m+1}(k)\frac{\mathrm{e}^{k\Delta x} - 2 + \mathrm{e}^{-k\Delta x}}{(\Delta x)^2} + (1-\theta)\hat{U}^m(k)\frac{\mathrm{e}^{k\Delta x} - 2 + \mathrm{e}^{-k\Delta x}}{(\Delta x)^2},$$

where $k \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]$. Hence,

$$(1+\theta\Delta t)\hat{U}^{m+1}(k) - \hat{U}^m(k)$$
$$= -4\mu\theta\hat{U}^{m+1}(k)\sin^2\frac{k\Delta x}{2} - 4\mu(1-\theta)\hat{U}^m(k)\sin^2\frac{k\Delta x}{2} - (1-\theta)\Delta t\hat{U}^m(k)\,,$$

where $\mu = \Delta t/(\Delta x)^2$. This gives

$$\hat{U}^{m+1}(k) = \frac{1-(1-\theta)\Delta t - 4\mu(1-\theta)\sin^2\frac{k\Delta x}{2}}{1+\theta\Delta t + 4\mu\theta\sin^2\frac{k\Delta x}{2}}\,\hat{U}^m(k) \equiv \lambda(k)\hat{U}^m(k).$$

Let $t = \sin^2\frac{k\Delta x}{2} \in [0,1]$. Define $g(t) = \frac{1-(1-\theta)\Delta t - 4\mu(1-\theta)t}{1+\theta\Delta t+4\mu\theta t}$. Now $|\lambda(k)| \leqslant 1$ if, and only if, $|g(t)| \leqslant 1$; the last inequality holds:

a) if $\theta \in [1/2, 1]$ without any conditions on Δt and Δx, — including, in particular, the implicit Euler and Crank–Nicolson schemes; or

b) if $\theta \in [0, 1/2)$ and $(\Delta t + 4\mu t)\mu(1-2\theta) \leqslant 2$ for all $t \in [0,1]$, i.e. if $B(\theta)\Delta t \leqslant \frac{2(\Delta x)^2}{4+(\Delta x)^2}$ with $B(\theta) = 1-2\theta$.

Either way,

$$|\hat{U}^m(k)| \leqslant |\hat{U}^0(k)| \qquad \forall k \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right] \equiv \mathcal{I}.$$

Therefore,

$$\|\hat{U}^m\|_{L_2(\mathcal{I})} \leqslant \|\hat{U}^0\|_{L_2(\mathcal{I})},$$

and the desired inequality then follows by Parseval's identity.

[Extension of bookwork to an unseen example.] **[10 marks]**

4. (a) [5 marks] The explicit Euler scheme for the initial-boundary-value problem is defined as follows:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \kappa \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}, \qquad \begin{cases} j = 1, 2, \dots, J-1, \\ m = 0, 1, \dots, M, \end{cases}$$

with $U_0^{m+1} = A(t_{m+1})$, $U_J^{m+1} = B(t_{m+1})$, $m = 0, 1, \dots, M-1$, and $U_j^0 = u_0(x_j)$, $j = 0, 1, \dots, J$.

[Bookwork.] **[5 points]**

(b) [10 marks] We define $\mu := \kappa \Delta t/(\Delta x)^2$. Thus,

$$U_j^{m+1} = (1 - 2\mu)U_j^m + \mu(U_{j+1}^m + U_{j-1}^m).$$

Suppose that

$$0 < \mu \leqslant \mu_0 =: \frac{1}{2}.$$

Then, $1 - 2\mu \geqslant 0$, and therefore coefficients multiplying the U's on the right-hand side are non-negative.

Therefore,

$$U_j^{m+1} \leqslant \max\{U_j^m, U_{j+1}^m, U_{j-1}^m\} \quad \text{for all} \begin{cases} j = 1, 2, \dots, J-1, \\ m = 0, 1, \dots, M. \end{cases} \qquad (**)$$

If $U_{\max}$ is attained at one of the mesh points on one of the 'boundary segments' (viz. on $x = 0$, $x = 1$ or $t = 0$), the proof is complete. Otherwise, we will show that if the maximum value of U is attained at an internal mesh point, then it is also attained at a mesh point that lies on one of the three boundary segments, and that will then complete the proof.

Suppose, therefore, that there exist $j_0 \in \{1, 2, \dots, J-1\}$ and $m_0 \in \{0, 1, \dots, M\}$ such that $U_{j_0}^{m_0+1} = U_{\max}$ is largest in the set of solution values at all mesh points. Define

$$U_* := \max\{U_{j_0}^{m_0}, U_{j_0+1}^{m_0}, U_{j_0-1}^{m_0}\}.$$

Thus,

$$U_{j_0}^{m_0+1} \leqslant U_*.$$

On the other hand, since by definition $U_{j_0}^{m_0+1}$ is the largest possible value of U over the mesh, also $U_* \leqslant U_{j_0}^{m_0+1}$. Thus we have shown that $U_{j_0}^{m_0+1} = U_*$. By the definition of U_*, this means that $U_{j_0}^{m_0+1}$ is equal to one of $U_{j_0}^{m_0}, U_{j_0+1}^{m_0}, U_{j_0-1}^{m_0}$. As a matter of fact, all of these three U values are equal to U_*; for if one of them were strictly smaller than U_*, then the inequality in $(**)$ would be strict, whereby we would then have that $U_{j_0}^{m_0+1} < U_*$, and this would contradict to what we have already proved (i.e. that $U_{j_0}^{m_0+1} = U_*$). We thereby conclude that all four values $U_{j_0}^{m_0}, U_{j_0+1}^{m_0}, U_{j_0-1}^{m_0}, U_{j_0}^{m_0+1}$ are equal to U_*.

We can repeat this procedure until we reach either the left boundary of the domain $[0, 1] \times [0, T]$ at $x = 0$, or the right boundary at $x = 1$, or the bottom boundary at $t = 0$. Once this occurs, we will have shown that the value U_* is also taken at one of the mesh points that lies on one of the three boundary segments. Hence, $U_{\max} = U_* = \max\{\max_{0 \leqslant m \leqslant M} A(t_m), \max_{0 \leqslant m \leqslant M} B(t_m), \max_{0 \leqslant j \leqslant J} u_0(x_j)\}$.

[Variation on bookwork: in the lectures the discrete maximum principle is discussed for the θ-scheme, with the explicit and implicit Euler schemes corresponding to $\theta = 0$ and $\theta = 1$ omitted as special cases when the six-point scheme collapses to a four-point scheme.]

[10 points]

(c) [10 marks] It follows from the definition of the consistency error for the scheme that

$$u(x_j, t_{m+1}) = (1 - 2\mu)u(x_j, t_m) + \mu(u(x_{j+1}, t_m) + u(x_{j-1}, t_m)) + \Delta t \cdot T_j^{m+1}.$$

Hence, by defining $e_j^m := u(x_j, t_m) - U_j^m$, we deduce from the last equality and the definition of the explicit Euler scheme that

$$e_j^{m+1} = (1 - 2\mu)e_j^m + \mu(e_{j+1}^m + e_{j-1}^m) + \Delta t \cdot T_j^{m+1},$$

for $j = 1, 2, \ldots, J - 1$, $m = 0, 1, \ldots, M - 1$, and with zero initial value (at $m = 0$) and zero boundary values (at $j = 0$ and $j = J$). This is the required recursion for the error. Assuming that $0 < \mu \leqslant \frac{1}{2} = \mu_0$, it follows from this recursion that

$$|e_j^{m+1}| \leqslant \max\{|e_j^m|, |e_{j+1}^m|, |e_{j-1}^m|\} + \Delta t |T_j^{m+1}|.$$

Let $E^m := \max_{0 \leqslant j \leqslant J} |e_j^m|$. We deduce that

$$|e_j^{m+1}| \leqslant E^m + \Delta t \max_{1 \leqslant j \leqslant J-1} |T_j^{m+1}|.$$

Taking the maximum over all $j \in \{0, 1, \ldots, J\}$ and letting $T^{m+1} := \max_{1 \leqslant j \leqslant J-1} |T_j^{m+1}|$, this gives

$$E^{m+1} \leqslant E^m + \Delta t \cdot T^{m+1}.$$

Summing over m yields, on noting that $E^0 = 0$, that

$$E^m \leqslant \Delta t \sum_{k=1}^{m} T^k \leqslant (m\Delta t) \max_{1 \leqslant k \leqslant m} T^k \leqslant T \cdot \max_{1 \leqslant k \leqslant m} T^k.$$

[Extension of bookwork to an unseen example.] **[10 points]**