3. (a) [5 marks] Suppose that u and v are two solutions to the initial-boundary-value problem subject to the same initial condition. Then

$$\frac{\partial}{\partial t}(u-v) + (u^3 - v^3) - \frac{\partial^2}{\partial x^2}(u-v) = 0 \quad \text{on } (0,1) \times (0,T],$$

where $u-v$ satisfies homogeneous initial and boundary conditions. Multiplying the above equality by $(u-v)$, integrating over $(0,1)$ and performing integration by parts in the third term on the left-hand side, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u(x,t)-v(x,t))^2\,\mathrm{d}x + \int_0^1 (u^3(x,t)-v^3(x,t))(u(x,t)-v(x,t))\,\mathrm{d}x$$
$$+ \int_0^1 (u_x(x,t)-v_x(x,t))^2\,\mathrm{d}x = 0 \quad \text{for all } t \in (0,T].$$

The third term on the left-hand side is clearly nonnegative, and the second term on the left-hand side is also nonnegative, because $z \in \mathbb{R} \mapsto z^3 \in \mathbb{R}$ is a monotonically increasing function, whereby $(a^3-b^3)(a-b) \geqslant 0$ for all $a, b \in \mathbb{R}$. Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u(x,t)-v(x,t))^2\,\mathrm{d}x \leqslant 0 \quad \text{for all } t \in (0,T].$$

Therefore, because $\int_0^1 (u(x,0)-v(x,0))^2\,\mathrm{d}x = 0$ it follows that $\int_0^1 (u(x,t)-v(x,t))^2\,\mathrm{d}x = 0$ for all $t \in (0,T]$. Consequently, $u(x,t) = v(x,t)$ for all $(x,t) \in [0,1] \times [0,T]$; i.e. the solution, if it exists, must be unique.

[Unseen modification of bookwork to a nonlinear PDE.] **[5 marks]**

(b) [10 marks] The implicit Euler approximation of the initial-boundary-value problem has the form

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + [U_j^{m+1}]^3 = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},$$

for $j = 1, \dots, N-1$, $m = 0, \dots, M-1$, where $\Delta x = 1/N$ and $\Delta t = T/M$, $M \geqslant 1$, and

$$U_j^0 = u_0(j\Delta t), \qquad j = 0, \dots, N.$$

Suppose that the solution U to this approximation scheme exists, and that V is another solution to this scheme. Then, defining $\mu := \Delta t/(\Delta x)^2$, we have that

$$(1+2\mu)(U_j^{m+1} - V_j^{m+1}) + \Delta t([U_j^{m+1}]^3 - [V_j^{m+1}]^3)$$
$$= (U_j^m - V_j^m) + \mu(U_{j+1}^{m+1} - V_{j+1}^{m+1}) + \mu(U_{j-1}^{m+1} - V_{j-1}^{m+1})$$

for $j = 1, \dots, N-1$, $m = 0, \dots, M-1$. Equivalently,

$$\left(1 + 2\mu + \Delta t([U_j^{m+1}]^2 + U_j^{m+1} V_j^{m+1} + [V_j^{m+1}]^2)\right) (U_j^{m+1} - V_j^{m+1})$$
$$= (U_j^m - V_j^m) + \mu(U_{j+1}^{m+1} - V_{j+1}^{m+1}) + \mu(U_{j-1}^{m+1} - V_{j-1}^{m+1})$$

for $j = 1, \dots, N-1$, $m = 0, \dots, M-1$. Let $\|W\|_\infty := \max_{j=1,\dots,N-1} |W_j|$ and note that $U_0^k - V_0^k = 0$ and $U_N^k - V_N^k = 0$ for all $k \in \{0, 1, \dots, M\}$. Therefore,

$$\left(1 + 2\mu + \Delta t([U_j^{m+1}]^2 + U_j^{m+1} V_j^{m+1} + [V_j^{m+1}]^2)\right) |U_j^{m+1} - V_j^{m+1}|$$
$$\leqslant \|U^m - V^m\|_\infty + \mu\|U^{m+1} - V^{m+1}\|_\infty + \mu\|U^{m+1} - V^{m+1}\|_\infty$$

for $j = 1, \ldots, N-1$, $m = 0, \ldots, M-1$. Note that the prefactor of $|U_j^{m+1} - V_j^{m+1}|$ appearing on the left-hand side is positive and is bounded below by $1 + 2\mu$ (observe the elementary inequality $a^2 + ab + b^2 \geqslant \frac{1}{2}(a^2 + b^2) \geqslant 0$ which follows from $(a-b)^2 \geqslant 0$). Therefore, and by taking the maximum over all $j = 0, \ldots, N$, we have that

$$(1 + 2\mu)\,\|U^{m+1} - V^{m+1}\|_\infty \leqslant \|U^m - V^m\|_\infty + \mu\|U^{m+1} - V^{m+1}\|_\infty + \mu\|U^{m+1} - V^{m+1}\|_\infty$$

for $m = 0, \ldots, M-1$. Equivalently,

$$\|U^{m+1} - V^{m+1}\|_\infty \leqslant \|U^m - V^m\|_\infty, \qquad m = 0, \ldots, M-1.$$

As $U_j^0 - V_j^0 = u_0(j\Delta x) - u_0(j\Delta x) = 0$, it follows that $\|U^0 - V^0\|_\infty = 0$, and therefore $\|U^m - V^m\|_\infty = 0$ for all $m = 0, 1, \ldots, M$, meaning that $U_j^m = V_j^m$ for all $j = 0, \ldots, N$ and all $m = 0, \ldots, M$. That completes the proof of the uniqueness of the solution U to the scheme (assuming its existence).

[Unseen modification of bookwork to a nonlinear PDE.] **[10 marks]**

(c) [10 marks] The implicit Euler approximation of the initial-boundary-value problem has the form

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + [U_j^{m+1}]^3 = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},$$

for $j = 1, \ldots, N-1$, $m = 0, \ldots, M-1$, where $\Delta x = 1/N$ and $\Delta t = T/M$, $M \geqslant 1$, and

$$U_j^0 = u_0(j\Delta t), \qquad j = 0, \ldots, N.$$

The consistency error T_j^m of the scheme is defined by

$$T_j^m := \frac{u_j^{m+1} - u_j^m}{\Delta t} + [u_j^{m+1}]^3 - \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2},$$

where $u_j^m := u(j\Delta x, m\Delta t)$. Subtracting the definition of the scheme from this equality yields

$$\begin{aligned}(1 + 2\mu)(u_j^{m+1} - U_j^{m+1}) &+ \Delta t([u_j^{m+1}]^3 - [U_j^{m+1}]^3) \\ &= (u_j^m - U_j^m) + \mu(u_{j+1}^{m+1} - U_{j+1}^{m+1}) + \mu(u_{j-1}^{m+1} - U_{j-1}^{m+1}) + \Delta t T_j^m\end{aligned}$$

for $j = 1, \ldots, N-1$, $m = 0, \ldots, M-1$. Therefore, by an identical argument as in the previous part of the question,

$$\|u^{m+1} - U^{m+1}\|_\infty \leqslant \|u^m - U^m\|_\infty + \Delta t\|T^m\|_\infty$$

for $m = 0, \ldots, M-1$. As $\|u^0 - U^0\|_\infty$, it follows by summing the above inequalities through $m = 0, \ldots, k-1$ for any $k \in \{1, \ldots, M\}$ that

$$\|u^k - U^k\|_\infty \leqslant \Delta t \sum_{m=0}^{k-1} \|T^m\|_\infty \leqslant \Delta t\, M \max_{m=0,\ldots,M-1} \|T^m\|_\infty = T \max_{m=0,\ldots,M-1} \|T^m\|_\infty,$$

because $\Delta t = T/M$. It remains to bound $\max_{m=0,\ldots,M-1} \|T^m\|_\infty$.
As, from the partial differential equation,

$$[u_j^{m+1}]^3 = -\frac{\partial u}{\partial t}(j\Delta x, (m+1)\Delta t) + \frac{\partial^2 u}{\partial x^2}(j\Delta x, (m+1)\Delta t),$$

inserting this into the definition of the consistency error

$$T_j^m := \frac{u_j^{m+1} - u_j^m}{\Delta t} + [u_j^{m+1}]^3 - \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2}$$

gives

$$T_j^m := \left[\frac{u_j^{m+1} - u_j^m}{\Delta t} - \frac{\partial u}{\partial t}(j\Delta x, (m+1)\Delta t)\right] + \left[\frac{\partial^2 u}{\partial x^2}(j\Delta x, (m+1)\Delta t) - \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2}\right].$$

By Taylor expansion about the point $(j\Delta x, (m+1)\Delta t)$ in both expressions appearing in the square brackets we have that

$$T_j^m := -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(j\Delta x, \eta^m) - \frac{(\Delta x)^2}{24}\frac{\partial^4 u}{\partial x^4}(\xi_j, (m+1)\Delta t),$$

where $\xi_j \in ((j-1)\Delta x, (j+1)\Delta x)$ and $\eta^m \in (m\Delta t, (m+1)\Delta t)$. Thus,

$$\max_{m=0,\dots,M-1} \|T^m\|_\infty \leqslant \frac{\Delta t}{2} \max_{(x,t)\in[0,1]\times[0,T]} \left|\frac{\partial^2 u}{\partial t^2}\right| + \frac{(\Delta x)^2}{12} \max_{(x,t)\in[0,1]\times[0,T]} \left|\frac{\partial^4 u}{\partial x^4}\right|.$$

Inserting this into the bound on the global error $\|u^k - U^k\|_\infty$, $k = 1, \dots, M$, in terms of the consistency error we have that

$$\max_{1\leqslant k\leqslant M} \|u^k - U^k\|_\infty \leqslant C(\Delta t + (\Delta x)^2),$$

where

$$C := T \max\left(\frac{1}{2} \max_{(x,t)\in[0,1]\times[0,T]} \left|\frac{\partial^2 u}{\partial t^2}\right|, \frac{1}{12} \max_{(x,t)\in[0,1]\times[0,T]} \left|\frac{\partial^4 u}{\partial x^4}\right|\right).$$

[Unseen modification of bookwork to a nonlinear PDE.] **[10 marks]**

4. (a) [4 marks] Suppose that v is a real-valued function, defined and three times continuously differentiable on $(-\infty, \infty)$ with a bounded third derivative. Then, by Taylor expansion with remainder, for each $x \in (-\infty, \infty)$, we have

$$v(x \pm \Delta x) = v(x) \pm \Delta x\, v'(x) + \frac{(\Delta x)^2}{2} v''(x) \pm \frac{(\Delta x)^3}{6} v'''(\xi^{\pm}),$$

where $\xi^- \in (x - \Delta x, x)$ and $\xi^+ \in (x, x + \Delta x)$. Hence,

$$\frac{v(x + \Delta x) - v(x - \Delta x)}{2\Delta x} = v'(x) + \frac{(\Delta x)^2}{6} \frac{v'''(\xi^+) + v'''(\xi^-)}{2}.$$

As v''' has been assumed to be a continuous function on $(-\infty, \infty)$, thanks to the intermediate value theorem there exists a real number $\xi \in (\xi^-, \xi^+) \subset (x - \Delta x, x + \Delta x)$ such that

$$\frac{v'''(\xi^+) + v'''(\xi^-)}{2} = v'''(\xi).$$

Hence,

$$\frac{v(x + \Delta x) - v(x - \Delta x)}{2\Delta x} = v'(x) + \frac{(\Delta x)^2}{6} v'''(\xi).$$

[Elementary real analysis.] **[4 points]**

(b) [7 marks] Let $a \in \mathbb{R}$ and κ a positive real number. Consider the time-dependent advection-diffusion equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2}$$

on the space-time domain $(-\infty, \infty) \times (0, T]$, where $T > 0$, subject to the initial condition $u(x, 0) = \mathrm{e}^{-x^2}$. The finite difference approximation of this problem with the required properties is

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + a \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} = \kappa \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}, \qquad j \in \mathbb{Z}, \quad m = 0, 1, \dots, M-1,$$

subject to the initial condition $U_j^0 = u_0(x_j) = \exp(-x_j^2)$ for $j \in \mathbb{Z}$.

[Note: The question is (intentionally) not explicitly asking candidates to show that the consistency error of this scheme is $\mathcal{O}(\Delta t + (\Delta x)^2)$, because for the special case of $a = 0$ when the equation is the heat equation, this is covered in the lecture notes, and based on the previous part of the question the inclusion in the consistency error analysis of the approximation of the convection term $a\partial u/\partial x$ is then trivial.]

[Extension of bookwork.] **[7 points]**

(c) [14 marks] We are now ready to embark on the stability analysis of the explicit Euler scheme. Let $\nu = a\Delta t/\Delta x$ and $\mu = \kappa \Delta t/(\Delta x)^2$.

We wish to show that if $\nu^2 \leqslant 2\mu \leqslant 1$ then the scheme is practically stable. By inserting

$$U_j^m = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x} \hat{U}^m(k)\, \mathrm{d}k$$

into the scheme we deduce that

$$\frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x} \frac{\hat{U}^{m+1}(k) - \hat{U}^m(k)}{\Delta t}\, \mathrm{d}k + \frac{a}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \frac{\mathrm{e}^{\imath k(j+1)\Delta x} - \mathrm{e}^{\imath k(j-1)\Delta x}}{2\Delta x} \hat{U}^m(k)\, \mathrm{d}k$$
$$= \frac{\kappa}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \frac{\mathrm{e}^{\imath k(j+1)\Delta x} - 2\mathrm{e}^{\imath kj\Delta x} + \mathrm{e}^{\imath k(j-1)\Delta x}}{(\Delta x)^2} \hat{U}^m(k)\, \mathrm{d}k.$$

Therefore, we have that

$$\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\frac{\hat{U}^{m+1}(k)-\hat{U}^m(k)}{\Delta t}\,\mathrm{d}k + \frac{a}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\frac{\mathrm{e}^{\imath k\Delta x}-\mathrm{e}^{-\imath k\Delta x}}{2\Delta x}\,\hat{U}^m(k)\,\mathrm{d}k$$
$$= \frac{\kappa}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\frac{\mathrm{e}^{\imath k\Delta x}-2+\mathrm{e}^{-\imath k\Delta x}}{(\Delta x)^2}\,\hat{U}^m(k)\,\mathrm{d}k.$$

Recalling that the semidiscrete Fourier transform and its inverse are one-to-one mappings, by comparing the left-hand side with the right-hand side we deduce that

$$\hat{U}^{m+1}(k) = \hat{U}^m(k) - \frac{\nu}{2}(\mathrm{e}^{\imath k\Delta x}-\mathrm{e}^{-\imath k\Delta x})\hat{U}^m(k) + \mu(\mathrm{e}^{\imath k\Delta x}-2+\mathrm{e}^{-\imath k\Delta x})\hat{U}^m(k)$$

for all wave numbers $k \in [-\pi/\Delta x,\, \pi/\Delta x]$, and we thus deduce that

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k),$$

where

$$\lambda(k) = 1 - \frac{\nu}{2}(\mathrm{e}^{\imath k\Delta x}-\mathrm{e}^{-\imath k\Delta x}) + \mu(\mathrm{e}^{\imath k\Delta x}-2+\mathrm{e}^{-\imath k\Delta x})$$

is called the amplification factor. By the discrete Parseval identity we have that

$$\begin{aligned}
\|U^{m+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{m+1}\|_{L_2}\\
&= \frac{1}{\sqrt{2\pi}}\,\|\lambda\hat{U}^m\|_{L_2}\\
&\leqslant \frac{1}{\sqrt{2\pi}}\max_k|\lambda(k)|\;\|\hat{U}^m\|_{L_2}\\
&= \max_k|\lambda(k)|\;\|U^m\|_{\ell_2}.
\end{aligned}$$

Practical stability requires that

$$\|U^{m+1}\|_{\ell_2} \leqslant \|U^m\|_{\ell_2}, \qquad m = 0,1,\ldots,M-1.$$

Thus we demand that

$$\max_k|\lambda(k)| \leqslant 1,$$

i.e., that

$$\max_k\left|1 - \frac{\nu}{2}(\mathrm{e}^{\imath k\Delta x}-\mathrm{e}^{-\imath k\Delta x}) + \mu(\mathrm{e}^{\imath k\Delta x}-2+\mathrm{e}^{-\imath k\Delta x})\right| \leqslant 1.$$

Using Euler's formula

$$\mathrm{e}^{\imath\varphi} = \cos\varphi + \imath\sin\varphi$$

and the trigonometric identity

$$1-\cos\varphi = 2\sin^2\frac{\varphi}{2}$$

we can restate this as follows:

$$\max_k\left|1 - \nu\imath\sin(k\Delta x) - 4\mu\sin^2\left(\frac{k\Delta x}{2}\right)\right| \leqslant 1.$$

Clearly,

$$\left|1 - \nu\imath\sin(k\Delta x) - 4\mu\sin^2\left(\frac{k\Delta x}{2}\right)\right|^2 = \left(1-4\mu\sin^2\left(\frac{k\Delta x}{2}\right)\right)^2 + \nu^2\sin^2(k\Delta x)$$
$$= 1 - 8\mu\sin^2\left(\frac{k\Delta x}{2}\right) + 16\mu^2\sin^4\left(\frac{k\Delta x}{2}\right) + 4\nu^2\sin^2\left(\frac{k\Delta x}{2}\right)\left(1-\sin^2\left(\frac{k\Delta x}{2}\right)\right).$$

Writing $S := \sin\left(\frac{k\Delta x}{2}\right)$, we thus need to ensure that

$$1 - 8\mu S^2 + 16\mu^2 S^4 + 4\nu^2 S^2(1 - S^2) \leqslant 1 \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x].$$

Since the inequality trivially holds for $S = 0$, we can assume without loss of generality that $S \neq 0$. Cancelling 1 on the left- and right-hand sides and dividing the resulting inequality by $4S^2 \neq 0$, we have that

$$(4\mu^2 - \nu^2)S^2 \leqslant 2\mu - \nu^2 \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x].$$

For this range of k we have that $S^2 \in [0, 1]$, and therefore a sufficient condition for the last inequality to hold is that (corresponding to $S^2 = 0$ and $S^2 = 1$, respectively,) we have

$$0 \leqslant 2\mu - \nu^2 \quad \text{and} \quad 4\mu^2 - \nu^2 \leqslant 2\mu - \nu^2.$$

Equivalently,

$$\nu^2 \leqslant 2\mu \leqslant 1.$$

Thus we have shown that, if $\nu^2 \leqslant 2\mu \leqslant 1$, then the scheme is practially stable.

[Unseen extension of bookwork.] **[15 points]**