Analysis of congested nonhomogeneous incompressible fluid flow models

D. Bresch[†], C. Burtea[§], P. Gonin-Joubert[*] and E. Süli[!]

March 11, 2026

[†] LAMA, UMR CNRS 5127, Université Savoie Mont Blanc,
Bât. Le Chablais, Campus Scientifique 73376 Le Bourget du Lac, France

[§] Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France
Batiment Sophie Germain, Bureau 727, 8 place Aurélie Nemours, 75013 Paris;

[*] Université Claude Bernard Lyon 1, CNRS, École Centrale de Lyon, INSA Lyon, Université Jean Monnet
ICJ UMR5208, 69622 Villeurbanne, France;

[!] Mathematical Institute,
University of Oxford, Woodstock Road, Oxford OX2 6GG, United Kingdom;

Abstract

This paper is dedicated to Professor Francisco Guillén-González (University of Sevilla) in celebration of his 60th birthday. The first author would like to express his friendship to Francisco (Kisco) by dedicating to him this short paper, written with friends and colleagues on a subject of his doctoral thesis, the incompressible Navier–Stokes equations with variable density; see, for instance, [FG93]. The main objective of the paper is to present recent mathematical developments for a simpler system than the compressible Navier–Stokes system, to highlight certain relevant techniques even when a more concise compactness method may be used in the incompressible setting. We first consider a nonhomogeneous incompressible Navier–Stokes system with a density-dependent viscosity coefficient, which may be singular close to the maximal admissible value of the density; this model is usually referred as the *soft formulation.* We then perform a limit passage to the so-called *hard formulation* and show how the diffusion may be altered by including an unilateral constraint related to maximal packing. In the final section we briefly describe an application to the modeling of atmospheric dispersion, where the transport of species occurs in all directions, driven by the velocity field, but diffusion is only present in one direction (usually the vertical direction); consequently, the system is degenerate.

1 Introduction

In this paper we present two mathematical results related to the incompressible Navier–Stokes equations with variable density. The first one (see Section 2) is aimed at illustrating the application of the new compactness method developed by the first author and P.-E. Jabin [BJ18] in the, more complicated, compressible setting. Then, in order to illustrate the limit passage from the soft to the hard description, we perform in Section 3 a limit passage with application to the modeling of atmospheric dispersion, where the transport of species occurs in all directions driven by the velocity field, but diffusion is only present in one direction (usually the vertical direction); consequently, the system is degenerate. It is important to note that recently soft approximations for congested systems have been studied by several authors in relation to various compressible flow models; see, for instance, Bresch, Perrin & Zatorska [BPZ14], Perrin & Zatorska [PZ15], Bresch, Nečasova & Perrin [BNP19].

In Section 2, we will start by focusing on the following PDE system. Suppose that $T > 0$ and let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ denote the d-dimensional flat torus, with $d \in \{2, 3\}$. We consider the system of incompressible Navier–Stokes equations with variable density ρ:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{on } (0,T] \times \mathbb{T}^d, & (1) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)\mathbb{D}(u)) + \nabla \mathrm{p} = 0 & \text{on } (0,T] \times \mathbb{T}^d, & (2) \\ \operatorname{div} u = 0 & \text{on } (0,T] \times \mathbb{T}^d, & (3) \end{cases}$$

where u and p are the velocity and the pressure of the fluid, subject to the initial conditions

$$\rho(0,x) = \rho_0(x) \quad \text{and} \quad (\rho u)(0,x) = \rho_0(x) u_0(x), \qquad x \in \mathbb{T}^d,$$

where, for some $\beta \in (0,1)$, $\gamma \geq 1$, and $\varepsilon \in (0,1)$,

$$\mu(\rho) = \frac{\varepsilon \rho^\gamma}{(1-\rho)^\beta} + 1. \tag{4}$$

We assume that ρ_0 is such that

$$0 \leq \rho_0(x) \leq 1 - \varepsilon^{1/\beta}, \qquad x \in \mathbb{T}^d; \tag{5}$$

then,

$$1 \leq \mu(\rho_0(x)) \leq 1 + (1 - \varepsilon^{1/\beta})^\gamma < 2, \qquad x \in \mathbb{T}^d. \tag{6}$$

We shall further suppose that there exists a positive constant C_0 such that

$$\int_{\mathbb{T}^d} \frac{1}{2} \rho_0(x) u_0^2(x) \, \mathrm{d}x \leq C_0.$$

Since μ depends on the parameter $\varepsilon > 0$, as does the upper bound on the initial density ρ_0, the same is true of ρ, u and p; for the sake of notational simplicity we shall not explicitly indicate this dependence on ε unless it is necessary to emphasize it. We are interested in passing to the limit $\varepsilon \to 0_+$ by means of a compactness argument. Because for $\varepsilon > 0$ the function ρ is nonnegative and strictly and uniformly in $(t,x) \in [0,T] \times \mathbb{T}^d$ bounded by $1 - \varepsilon^{1/\beta}$ (cf. (20) below), the stated initial-boundary-value problem has, for each $\varepsilon \in (0,1)$, a weak solution (cf., for example, [AK73], [AKM90], [Sim90], and [Lio96]) with

$$u \in L^2(0,T; W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)), \quad \rho \in L^\infty([0,T] \times \mathbb{T}^d; \mathbb{R}_{\geq 0}), \quad \mathrm{p} \in \mathcal{D}'((0,T) \times \mathbb{T}^d), \quad \rho|u|^2 \in L^\infty(0,T; L^1(\mathbb{T}^d)),$$

with additional regularity properties, which we shall not list here. We get, by letting $\varepsilon \to 0_+$, the limit system, called the *congested fluid model*:

$$\begin{cases} \partial_t \overline{\rho} + \operatorname{div}(\overline{\rho}\, \overline{u}) = 0 & \text{in } (0,T] \times \Omega, & (7) \\ \partial_t \overline{\mu} + \operatorname{div}(\overline{\mu}\, \overline{u}) = 0 & \text{in } (0,T] \times \Omega, & (8) \\ \partial_t(\overline{\rho}\, \overline{u}) + \operatorname{div}(\overline{\rho}\, \overline{u} \otimes \overline{u}) - \operatorname{div}(\overline{\mu}\, \mathbb{D}(\overline{u})) + \nabla \mathrm{p} = 0 & \text{in } (0,T] \times \Omega, & (9) \\ \operatorname{div} \overline{u} = 0 & \text{in } (0,T] \times \Omega, & (10) \\ (1 - \overline{\rho})\, (\overline{\mu} - 1) = 0 & \text{in } (0,T] \times \Omega, & (11) \end{cases}$$

subject to the initial conditions

$$\overline{\rho}(0,x) = \overline{\rho_0}(x), \quad \overline{\mu}(0,x) = \overline{\mu(\rho_0)}(0,x), \quad (\overline{\rho}\, \overline{u})(0,x) = \overline{\rho_0}(x)\, \overline{u_0}(x) \qquad \text{for } x \in \mathbb{T}^d.$$

In Section 4, we shall prove the existence of global weak solutions to the following system:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (12)\\ \partial_t\pi + \mathrm{div}(\pi u) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (13)\\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \Delta u + \nabla \mathrm{p} = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (14)\\ \mathrm{div}\, u = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (15)\\ \partial_t c + u\cdot\nabla c - \partial_z((1+\pi)\partial_z c) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (16)\end{cases}$$

with

$$\pi \geq 0, \qquad \pi(1-\rho) = 0, \qquad 0\leq\rho\leq 1 \qquad \text{on } (0,T]\times\mathbb{T}^d, \tag{17}$$

and subject to the initial conditions

$$\rho(0,x) = \rho_0(x), \quad (\rho u)(0,x) = \rho_0(x)u_0(x), \quad c(0,x) = c_0(x) \qquad \text{for } x\in\mathbb{T}^d.$$

The key idea is to regularize the evolution equation (16) for the concentration of the pollutant in a particular way using a parameter-dependent diffusion matrix A_ε, with $0<\varepsilon<1$, and then pass to the limit $\varepsilon\to 0_+$ with the regularization parameter ε. We shall define A_ε so that some of its components are small when the density of the flow is smaller than a certain value, and they are nonnegligible otherwise. Passing to the limit $\varepsilon\to 0_+$ will enable us to rigorously derive the degenerate reaction-diffusion equation (16) coupled with the nonhomogeneous incompressible Navier–Stokes system.

2 The soft approximation for congested incompressible Navier-Stokes

In this section, we consider the soft approximation for the congested incompressible Navier–Stokes equations. We first provide some formal estimates. Then we prove certain compactness results concerning $\mu(\rho)$ using the technique introduced by the first author and P.–E. Jabin; we believe that this is a helpful example through which the key ideas of the method can be easily explained, despite fact that in our simpler context of a divergence-free velocity field one can establish compactness in a more standard manner.

Remark 1. It is important to note that we can prove the compactness on $(\mu(\rho))_\varepsilon$ in $L^p((0,T)\times\mathbb{T}^d)$ for all $p\in[1,\infty)$ in a more standard manner. First we use the bound on ρ^ε to get a uniform bound on $\mu(\rho^\varepsilon)$ in $L^\infty((0,T)\times\mathbb{T}^d)$. We also obtain a bound on $\partial_t\mu(\rho^\varepsilon)$ in $L^\infty(0,T;W^{-1,2}(\mathbb{T}^d))$. Thus by the Aubin–Lions–Simon Lemma, there exists a subsequence such that $\mu(\rho^\epsilon)$ strongly converges to $\overline{\mu}$ in $L^2(0,T;W^{-1,2}(\mathbb{T}^d))$. As u^ε converges weakly to u in $L^2(0,T;W^{1.2}(\mathbb{T}^d))$, it follows that in the limit $\partial_t\overline{\mu} + \mathrm{div}(\overline{\mu}u) = 0$. One can also show that

$$\int_{\mathbb{T}^d} |\mu(\rho^\varepsilon(t,x))|^2\,\mathrm{d}x = \int_{\mathbb{T}^d} |\mu(\rho_0^\varepsilon(x))|^2\,\mathrm{d}x \to \int_{\mathbb{T}^d} |\overline{\mu}_0(x)|^2\,\mathrm{d}x = \int_{\mathbb{T}^d} |\overline{\mu}(t,x)|^2\,\mathrm{d}x, \qquad t\in[0,T],$$

which then implies the strong convergence of $\mu(\rho^\varepsilon)$ to $\overline{\mu}$ in $L^2((0,T)\times\mathbb{T}^d)$ and therefore in $L^p((0,T)\times\mathbb{T}^d)$ for all $p\in[1,\infty)$ thanks to the uniform bound on $\mu(\rho^\varepsilon)$ in $L^\infty((0,T)\times\mathbb{T}^d)$.

2.1 Some formal *a priori* estimates

Integrating (1) and (2) over $\mathbb{T}^d$ we get,

$$\int_{\mathbb{T}^d} \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} \rho_0(x)\,\mathrm{d}x =: \mathcal{M}_1 > 0 \qquad \text{for all } t\in[0,T], \tag{18}$$

and

$$\int_{\mathbb{T}^d} \rho(t,x)u(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} \rho_0(x)u_0(x)\,\mathrm{d}x =: \mathcal{C} \qquad \text{for all } t \in [0,T].$$

Testing (1) with $\rho_- := \min\{0,\rho\}$, thanks to (3) and the assumed nonnegativity of ρ_0 (cf. (5)) it follows that $\rho(t,\cdot) \geq 0$ for all $t \in [0,T]$. Similarly, testing (1) with $p\rho^{p-1}$, $p \in (1,\infty)$, again using (3) and (5), it follows, together with (18) and the nonnegativity of ρ that

$$\left(\int_{\mathbb{T}^d} |\rho(t,x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}^d} |\rho_0(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} =: \mathcal{M}_p \qquad \text{for all } t \in [0,T] \text{ and all } p \in [1,\infty). \tag{19}$$

The formal calculation above can be made rigorous by adapting the proof of Corollary II.1 in [DL89] to our current setting, with the spatial domain being $\mathbb{T}^d$ rather than the whole of $\mathbb{R}^d$, which guarantees the existence of a unique weak solution to the continuity equation (1) for each $\rho_0 \in L^\infty(\mathbb{T}^d)$ and each $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d;\mathbb{R}^d))$.

Passing to the limit $p \to \infty$ in (19) it follows that

$$\text{ess.sup}_{x\in\mathbb{T}^d}|\rho(t,x)| \leq \text{ess.sup}_{x\in\mathbb{T}^d}|\rho_0(x)| \qquad \text{for all } t \in [0,T],$$

and because $\rho(t,\cdot) \geq 0$ for all $t \in [0,T]$, by (5) also

$$0 \leq \text{ess.sup}_{x\in\mathbb{T}^d}\rho(t,x) \leq \text{ess.sup}_{x\in\mathbb{T}^d}\rho_0(x) \leq 1 - \varepsilon^{1/\beta} \qquad \text{for all } t \in [0,T]. \tag{20}$$

Taking the dot-product of (2) with u, integrating over $\mathbb{T}^d$, and integrating by parts using (3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \frac{1}{2}\,\rho(t,x)u^2(t,x)\,\mathrm{d}x + \int_{\mathbb{T}^d} \mu(\rho(t,x))|\mathbb{D}(u(t,x))|^2\,\mathrm{d}x = 0 \qquad \text{for all } t \in (0,T]. \tag{21}$$

From (4) and using Korn's inequality[1] we obtain

$$\text{ess.sup}_{t\in[0,T]}\left[\int_{\mathbb{T}^d} \frac{1}{2}\,\rho(t,x)u^2(t,x)\,\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d} |\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\right] \leq C_0. \tag{22}$$

Next, we note that

$$\begin{aligned} u(t,x) &= u(t,y) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}u(t,\theta x + (1-\theta)y)\,\mathrm{d}\theta \\ &= u(t,y) + \int_0^1 \nabla u(t,\theta x + (1-\theta)y)(x-y)\,\mathrm{d}\theta \qquad \text{for all } x,y \in \mathbb{T}^d. \end{aligned}$$

---

[1] Suppose that $v \in C^\infty(\overline{\mathbb{T}^d};\mathbb{R}^d)$. Then,

$$\begin{aligned} \|\mathbb{D}(v)\|^2_{L^2(\mathbb{T}^d)} &= \frac{1}{4}\sum_{i,j=1}^d \int_{\mathbb{T}^d} \left|\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right|^2 \mathrm{d}x = \frac{1}{4}\sum_{i,j=1}^d \int_{\mathbb{T}^d} \left|\frac{\partial v_i}{\partial x_j}\right|^2 + 2\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \left|\frac{\partial v_j}{\partial x_i}\right|^2 \mathrm{d}x \\ &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{T}^d} \left|\frac{\partial v_i}{\partial x_j}\right|^2 + \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\,\mathrm{d}x = \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{T}^d} \left|\frac{\partial v_i}{\partial x_j}\right|^2 + \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\,\mathrm{d}x = \frac{1}{2}\left(\|\nabla v\|^2_{L^2(\mathbb{T}^d)} + \|\mathrm{div}\, v\|^2_{L^2(\mathbb{T}^d)}\right). \end{aligned}$$

Therefore, $\|\mathbb{D}(v)\|^2_{L^2(\mathbb{T}^d)} \geq \frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{T}^d)}$. For Sobolev functions $v \in W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)$, the inequality then follows by a density argument in $W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)$; recall that in our case $u \in L^2(0,T;W^{1,2}(\mathbb{T}^d;\mathbb{R}^d))$. If, in addition $\mathrm{div}\, v = 0$, then $\|\mathbb{D}(v)\|^2_{L^2(\mathbb{T}^d)} = \frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{T}^d)}$.

Then, multiplying by $\rho(t,y)$ and integrating over $y \in \mathbb{T}^d$, we get

$$\begin{aligned}\mathcal{M}_1 u(t,x) = &\int_{\mathbb{T}^d} \rho(t,y) u(t,y)\,\mathrm{d}y \\ &+ \int_{\mathbb{T}^d} \rho(t,y) \int_0^1 \nabla u(t, \theta x + (1-\theta)y)\,(x-y)\,\mathrm{d}\theta\,\mathrm{d}y \qquad \text{for all } x \in \mathbb{T}^d.\end{aligned} \tag{23}$$

Now,

$$\begin{aligned}\mathrm{T}_1 &:= \left| \int_{\mathbb{T}^d} \rho(t,y) \int_0^1 \nabla u(t, \theta x + (1-\theta)y)(x-y)\,\mathrm{d}\theta\,\mathrm{d}y \right| \\ &\leq \int_{\mathbb{T}^d} |x-y|\,\rho(t,y) \int_0^1 |\nabla u(t, \theta x + (1-\theta)y)|\,\mathrm{d}\theta\,\mathrm{d}y \\ &\leq \left(\int_{\mathbb{T}^d} |x-y|^2\,[\rho(t,y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left| \int_0^1 |\nabla u(t, \theta x + (1-\theta)y)|\,\mathrm{d}\theta \right|^2 \mathrm{d}y \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{T}^d} |x-y|^2\,[\rho(t,y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left[\frac{1}{|x-y|} \int_0^{|x-y|} \left| \nabla u\left(t, y + s\frac{x-y}{|x-y|}\right) \right| \mathrm{d}s \right]^2 \mathrm{d}y \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{T}^d} |x-y|^2\,[\rho(t,y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left[\sup_{\tau>0} \frac{1}{\tau} \int_0^{\tau} \left| \nabla u\left(t, y + s\frac{x-y}{|x-y|}\right) \right| \mathrm{d}s \right]^2 \mathrm{d}y \right)^{\frac{1}{2}},\end{aligned}$$

where in the transition to the fourth line we performed the change of variable $\theta = s/|x-y|$. Let

$$g(t,y;\nu) := \sup_{\tau>0} \frac{1}{\tau} \int_0^{\tau} |\nabla u(t, y + s\nu)|\,\mathrm{d}s, \qquad \text{where } \nu \in \mathbb{R}^d \text{ and } |\nu| = 1.$$

Then,

$$\mathrm{T}_1 \leq \left(\int_{\mathbb{T}^d} |x-y|^2\,[\rho(t,y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left[g\left(t, y; \frac{x-y}{|x-y|}\right) \right]^2 \mathrm{d}y \right)^{\frac{1}{2}}, \qquad t \in [0,T].$$

Since g is the Hardy–Littlewood maximal function of $|\nabla u|$ in the direction ν, it follows (see, for example, [Cal72, Proof of Lemma 7] or [Ste93]) that

$$\left(\int_{\mathbb{T}^d} |g(t,y;\nu)|^p\,\mathrm{d}y \right)^{\frac{1}{p}} \leq c_p \left(\int_{\mathbb{T}^d} |\nabla u(t,y)|^p\,\mathrm{d}y \right)^{\frac{1}{p}}, \qquad t \in [0,T],\ \nu \in \mathbb{R}^d,\ |\nu| = 1.$$

Hence, with $p = 2$,

$$\begin{aligned}\mathrm{T}_1 &\leq c_2 \left(\int_{\mathbb{T}^d} |x-y|^2\,[\rho(t,y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} |\nabla u(t,y)|^2\,\mathrm{d}y \right)^{\frac{1}{2}} \\ &\leq c_2\,\mathrm{diam}(\mathbb{T}^d) \left(\int_{\mathbb{T}^d} [\rho_0(y)]^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} |\nabla u(t,y)|^2\,\mathrm{d}y \right)^{\frac{1}{2}}.\end{aligned}$$

We return with this bound to (23), to find that, for all $x \in \mathbb{T}^d$,

$$\begin{aligned}\mathcal{M}_1|u(t,x)| &\leq \left|\int_{\mathbb{T}^d} \rho(t,y)u(t,y)\,\mathrm{d}y\right| + \left|\int_{\mathbb{T}^d} \rho(t,y)\int_0^1 \nabla u(t,\theta x+(1-\theta)y)(x-y)\,\mathrm{d}\theta\,\mathrm{d}y\right| \\ &\leq |\mathcal{C}| + c_2\,\mathrm{diam}(\mathbb{T}^d)\left(\int_{\mathbb{T}^d}[\rho_0(y)]^2\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\mathbb{T}^d}|\nabla u(t,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &= |\mathcal{C}| + c_2\mathcal{M}_2\,\mathrm{diam}(\mathbb{T}^d)\left(\int_{\mathbb{T}^d}|\nabla u(t,y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}.\end{aligned}$$

Squaring this and integrating over $x \in \mathbb{T}^d$ and $t \in [0,T]$ then gives

$$\int_0^T\int_{\mathbb{T}^d}|u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{2T|\mathcal{C}|^2}{\mathcal{M}_1^2}|\mathbb{T}^d| + \frac{4c_2^2C_0\mathcal{M}_2^2}{\mathcal{M}_1^2}[\mathrm{diam}(\mathbb{T}^d)]^2|\mathbb{T}^d|.$$

As $|\mathbb{T}^d| = 1$ and $\mathrm{diam}(\mathbb{T}^d) = \sqrt{d}$, it follows that

$$\int_0^T\int_{\mathbb{T}^d}|u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{2T|\mathcal{C}|^2}{\mathcal{M}_1^2} + \frac{4c_2^2dC_0\mathcal{M}_2^2}{\mathcal{M}_1^2}. \tag{24}$$

Thus we have shown that $\|u\|_{L^2(0,T;L^2(\mathbb{T}^d))}$ is bounded by a constant, uniformly with respect to ε.

From (1), (3), (22) and renormalization theory, we get furthermore, for each function $f \in C^1([0,\infty);\mathbb{R})$,

$$\partial_t f(\rho) + u\cdot\nabla f(\rho) = 0.$$

Thus, if $f(\rho_0)$ is initially uniformly bounded, then these bounds are preserved during the course of evolution in time. In particular, we deduce from (5) and (6) that

$$0 \leq \rho(t,x) \leq 1, \tag{25}$$

$$1 \leq \mu(\rho(t,x)) \leq 2 \tag{26}$$

for all $(t,x) \in [0,T]\times\mathbb{T}^d$, and uniformly with respect to ε. It therefore follows that there exist $\overline{\rho}$, $\overline{\mu} \in L^\infty(0,T,L^\infty(\mathbb{T}^d))$ such that

$$\rho^\varepsilon,\mu^\varepsilon \underset{\varepsilon\to 0}{\overset{\star}{\rightharpoonup}} \overline{\rho},\overline{\mu} \quad \text{in } L^\infty(0,T,L^\infty(\mathbb{T}^d)), \tag{27}$$

where $\overline{\rho}$, $\overline{\mu}$ are weak-$\star$ limits to be identified in $L^\infty(0,T,L^\infty(\mathbb{T}^d))$.

2.2 Strong compactness of $\mu(\rho)$

To prove strong compactness of $(\mu(\rho))_{\varepsilon>0}$, we use the following two results. The first of the two is a compactness criterion in $L^1(\mathbb{T}^d)$, while the second result is a compactness criterion in $L^1(0,T;L^1(\mathbb{T}^d))$. For their proofs and additional details the reader is referred to the publications by Belgacem and Jabin [BJ13] and Bresch and Jabin [BJ18].

Theorem 2. *For $h>0$, we define $K_h \in L^1(\mathbb{T}^d)$ by*

$$K_h(z) = \frac{1}{(|z|+h)^d}, \quad z \in \mathbb{T}^d. \tag{28}$$

Suppose that $(f^n)_{n=1}^\infty \subset L^1(\mathbb{T}^d)$. Then, the following statements are equivalent:

(1) Up to a subsequence, $(f^n)_{n=1}^\infty$ converges strongly in $L^1(\mathbb{T}^d)$ as $n \to \infty$;

(2) The sequence $(f^n)_{n=1}^\infty$ is uniformly bounded in $L^1(\mathbb{T}^d)$ and

$$\limsup_{n\to+\infty} \left[\frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}} \int_{\mathbb{T}^d\times\mathbb{T}^d} K_h(x-y)|f^n(x)-f^n(y)|\,\mathrm{d}x\,\mathrm{d}y \right] \underset{h\to 0}{\longrightarrow} 0.$$

Theorem 3. *Let $(f^n)_{n=1}^\infty$ be a sequence that is uniformly integrable in $L^1((0,T)\times\mathbb{T}^d)$ and suppose that K_h is as in Theorem 2. Assume that $\partial_t f^n \in L^q(0,T,W^{-1,q}(\mathbb{T}^d))$ for some $q>1$, and bounded uniformly in n in the norm of $L^q(0,T,W^{-1,q}(\mathbb{T}^d))$. Let further*

$$\limsup_{n\to+\infty} \sup_{t\in[0,T]} \left[\frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}} \int_{\mathbb{T}^d\times\mathbb{T}^d} K_h(x-y)\,|f^n(t,x)-f^n(t,y)|\,\mathrm{d}x\,\mathrm{d}y \right] \underset{h\to 0}{\longrightarrow} 0;$$

then, the sequence $(f^n)_{n=1}^\infty$ is compact in $L^1((0,T)\times\mathbb{T}^d)$. Conversely if the sequence $(f^n)_{n=1}^\infty$ is compact in $L^1((0,T)\times\mathbb{T}^d)$ then the above quantity converges to 0 as h tends to zero.

Denoting $\mu_x(t) := \mu(\rho(t,x))$, $u_x(t) := u(t,x)$ and $\mu_y(t) := \mu(\rho(t,y))$, $u_y(t) := u(t,y)$ for all $x,y\in\mathbb{T}^d$ and $t\in[0,T]$, we have from (1) and (3) the following equations:

$$\partial_t \mu_x + \mathrm{div}_x(\mu_x u_x) = 0,$$

$$\partial_t \mu_y + \mathrm{div}_y(\mu_y u_y) = 0.$$

Hence, taking the difference and then using (3), we have

$$\partial_t(\mu_x-\mu_y) + u_x\cdot\nabla_x(\mu_x-\mu_y) + u_y\cdot\nabla_y(\mu_x-\mu_y) = 0.$$

Multiplying this by $(\mu_x-\mu_y)/|\mu_x-\mu_y|$ yields

$$\partial_t|\mu_x-\mu_y| + u_x\cdot\nabla_x|\mu_x-\mu_y| + u_y\cdot\nabla_y|\mu_x-\mu_y| = 0.$$

Dividing the last equation by $(|x-y|+h)^d$, we obtain

$$\partial_t \frac{|\mu_x-\mu_y|}{(|x-y|+h)^d} + u_x\cdot\nabla_x \frac{|\mu_x-\mu_y|}{(|x-y|+h)^d} + u_y\cdot\nabla_y \frac{|\mu_x-\mu_y|}{(|x-y|+h)^d} = -d\frac{u_x-u_y}{|x-y|}\cdot\frac{|\mu_x-\mu_y|(x-y)}{(|x-y|+h)^{d+1}}. \tag{29}$$

Integrating this over $\mathbb{T}^d\times\mathbb{T}^d$, we get using equation (3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{T}^d} K_h(x-y)|\mu_x-\mu_y|\,\mathrm{d}x\,\mathrm{d}y = -d\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{u_x-u_y}{|x-y|}\cdot\frac{|\mu_x-\mu_y|(x-y)}{(|x-y|+h)^{d+1}}\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, integrating in time, we obtain

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{T}^d} K_h(x-y)|\mu_x(t)-\mu_y(t)|\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{T}^d\times\mathbb{T}^d} K_h(x-y)|\mu_x^0-\mu_y^0|\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - d\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{u_x(s)-u_y(s)}{|x-y|}\cdot\frac{|\mu_x(s)-\mu_y(s)|(x-y)}{(|x-y|+h)^{d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{30}$$

To simplify matters, let us first assume that ∇u is bounded in $L^1(0,T,L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d}))$, uniformly with respect to ε, by some constant $A>0$. Hence,

$$|u_x(s)-u_y(s)| \le \|\nabla u(s)\|_{L^\infty(\mathbb{T}^d))}|x-y|, \qquad s\in[0,T] \tag{31}$$

and we obtain using (31) the following bound:

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{u_x-u_y}{|x-y|}\cdot\frac{|\mu_x-\mu_y|(x-y)}{(|x-y|+h)^{d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\right| &\le \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|u_x-u_y|\frac{|\mu_x-\mu_y|}{(|x-y|+h)^{d+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &\le \int_0^t\|\nabla u(s)\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|\mu_x-\mu_y|}{(|x-y|+h)^{d}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &= \int_0^t\|\nabla u(s)\|_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d\times\mathbb{T}^d}K_h(x-y)|\mu_x(s)-\mu_y(s)|\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}s,\end{aligned}\tag{32}$$

where in the transition to the second line we used the obvious inequality $|x-y|\le|x-y|+h$ for $x,y\in\mathbb{T}^d$. Thus, from (30), (32) and Grönwall's lemma we get

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}K_h(x-y)|\mu_x(t)-\mu_y(t)|\,\mathrm{d}x\,\mathrm{d}y\le\exp(dA)\int_{\mathbb{T}^d\times\mathbb{T}^d}K_h(x-y)|\mu_x^0-\mu_y^0|\,\mathrm{d}x\,\mathrm{d}y.\tag{33}$$

Hence, by assuming the compactness of $(\mu^{\varepsilon,0})_{\varepsilon>0}$ in $L^1(\mathbb{T}^d)$, it would follow from Theorem 2 that the right-hand side in (33) tends to 0 as $\varepsilon\to 0_+$ (via the implication 2 $\Rightarrow$ 1). The inequality (33) would then imply that the expression on its left-hand side converges to 0 as $\varepsilon\to 0_+$, and then using Theorem 3 we could deduce the compactness of $(\mu^\varepsilon)_{\varepsilon>0}$ in $L^1(0,T;L^1(\mathbb{T}^d))$.

Unfortunately, we can only ensure that ∇u belongs to the function space $L^2(0,T,L^2(\mathbb{T}^d;\mathbb{R}^{d\times d}))$ (see (22)) rather than $L^1(0,T,L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d}))$. In order to circumvent this difficulty, we introduce a weight function w in order to avoid subsets of $[0,T]\times\mathbb{T}^d$ over which $|\nabla u|$ is 'large'.

Definition 4. We define the weight function $w:[0,T]\times\mathbb{T}^d\to\mathbb{R}$ as the solution of the problem

$$\begin{cases} w(0,\cdot)=1 & \text{on } \mathbb{T}^d, \qquad (34)\\ \partial_t w+u\cdot\nabla w+\lambda\,\mathcal{M}(|\nabla u|)\,w=0 & \text{on } (0,T]\times\mathbb{T}^d, \qquad (35)\end{cases}$$

where $\lambda>0$ is to be fixed, and $\mathcal{M}$ is the Hardy–Littlewood maximal operator defined, for all $f\in L^1(\mathbb{T}^d)$, by

$$[\mathcal{M}(f)](x):=\sup_{r>0}\frac{1}{B(x,r)}\int_{B(x,r)}f(y)\,\mathrm{d}y,\qquad x\in\mathbb{T}^d,$$

where $B(x,r)$ is a unit ball or radius $r>0$ in $\mathbb{R}^d$ centred at x.

The following result is classical (cf. [Ste70, Theorem I.1]).

Lemma 5. For all $1<p\le\infty$, there exists a constant $C_1>0$ depending only on d and p such that

$$\|\mathcal{M}f\|_{L^p(\mathbb{T}^d)}\le C_1\|f\|_{L^p(\mathbb{T}^d)}\qquad \textit{for all } f\in L^1(\mathbb{T}^d).$$

The velocity field u has sufficient regularity (see (3), (22)) to deduce from the initial condition (34) the following result.

Proposition 6.

$$0\le w(t,x)\le 1\qquad \textit{for all } (t,x)\in[0,T]\times\mathbb{T}^d.\tag{36}$$

The proof is straightforward: it is based on first testing with $w_-:=\min\{0,w\}$ to prove that $w\ge 0$ on $[0,T]\times\mathbb{T}^d$. Then, testing with pw^{p-1} we prove that $\|w(t,\cdot)\|_{L^p(\mathbb{T}^d)}\le\|w(0,\cdot)\|_{L^p(\mathbb{T}^d)}$, using in both cases that $\operatorname{div}u=0$. Finally, passing to the limit $p\to\infty$ we infer that $0\le w(t,x)\le w(0,x)\equiv 1$ for all $(t,x)\in[0,T]\times\mathbb{T}^d$.

Remark 7. Equation (35) involves the transport term $\partial_t w + u\cdot\nabla w$ and the damping term $\lambda\mathcal{M}(|\nabla u|)w > 0$. Roughly speaking, if $|\nabla u|$ is large on some set, then the damping is large on that set, so w must be close to 0. Conversely, if $|\nabla u|$ is small, then the damping is weak, so the initial datum $w(0,\cdot)\equiv 1$ is transported with little or no damping, and $w(\leq 1)$ is therefore close to its initial value of 1.

Let $w_x := w(\cdot,x)$ and $w_y := w(\cdot,y)$ for $x,y\in\mathbb{T}^d$. Multiplying (29) by $w_x w_y$, we obtain

$$\begin{aligned}
&\partial_t \frac{w_x w_y|\mu_x-\mu_y|}{(|x-y|+h)^d} + u_x\cdot\nabla_x \frac{w_x w_y|\mu_x-\mu_y|}{(|x-y|+h)^d} + u_y\cdot\nabla_y \frac{w_x w_y|\mu_x-\mu_y|}{(|x-y|+h)^d}\\
&= -d w_x w_y \frac{u_x-u_y}{|x-y|}\cdot\frac{|\mu_x-\mu_y|(x-y)}{(|x-y|+h)^{d+1}} - \lambda(\mathcal{M}(|\nabla_x u_x|)+\mathcal{M}(|\nabla_y u_y|))w_x w_y \frac{|\mu_x-\mu_y|}{(|x-y|+h)^d}\\
&= w_x w_y \frac{|\mu_x-\mu_y|}{(|x-y|+h)^d}\left(-d\frac{u_x-u_y}{|x-y|}\cdot\frac{x-y}{|x-y|+h} - \lambda(\mathcal{M}(|\nabla_x u_x|)+\mathcal{M}(|\nabla_y u_y|))\right). \qquad (37)
\end{aligned}$$

By invoking the inequality (cf. [BH93, Theorem 3] with $m=1$ there)

$$|u_x-u_y| \leq C_2|x-y|(\mathcal{M}(|\nabla_x u_x|)+\mathcal{M}(|\nabla_y u_y|)) \qquad \text{for all } x,y\in\mathbb{T}^d,$$

where $C_2>0$ is a constant that depends only on d, we get

$$\left|-d\frac{u_x-u_y}{|x-y|}\cdot\frac{x-y}{|x-y|+h}\right| \leq d\frac{|u_x-u_y|}{|x-y|} \leq dC_2(\mathcal{M}(|\nabla_x u_x|)+\mathcal{M}(|\nabla_y u_y|)).$$

Thus, choosing

$$\lambda = dC_2,$$

the right-hand side of (37) becomes nonpositive. Hence, integrating (37) over $\mathbb{T}^d\times\mathbb{T}^d$ and recalling (3) and the definition (28) of K_h, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t)-\mu_y(t)|w_x w_y\,\mathrm{d}x\,\mathrm{d}y \leq 0.$$

By integrating this with respect to t and using (34), we obtain

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t)-\mu_y(t)|w_x w_y\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x^0-\mu_y^0|\,\mathrm{d}x\,\mathrm{d}y. \qquad (38)$$

Had we obtained (38) without the weights w_x and w_y that appear in the integrand on the left-hand side, the compactness of $(\mu^\varepsilon)_{\varepsilon>0}$ in $L^1(0,T;L^1(\mathbb{T}^d))$, would have directly followed from Theorem 2 and Theorem 3. Therefore, our objective now is to show that the weight function is strictly positive a.e. on $[0,T]\times\mathbb{T}^d$, i.e., (see Remark 7) that ∇u is not 'large' at almost all points in the set $[0,T]\times\mathbb{T}^d$. To this end, we shall prove the following result using the uniform (in ε) bound (22).

Lemma 8. There exists a constant $B>0$, depending only on d, T, and C_0 and C_1, such that

$$\text{ess.sup}_{t\in[0,T]}\int_{\mathbb{T}^d}|\log w(t,x)|\,\mathrm{d}x \leq B.$$

Remark 9. Lemma 8 implies that w is positive a.e. on $[0,T]\times\mathbb{T}^d$. Indeed, if $w\to 0$, then $|\log w|\to+\infty$.

Proof. Thanks to (36), $0 \le w(t,x) \le 1$ for a.e. $(t,x) \in [0,T] \times \mathbb{T}^d$. Let $0 < \alpha \ll 1$. Then, $0 < \alpha \le w(t,x) + \alpha \le 1 + \alpha$ for a.e. $(t,x) \in [0,T] \times \mathbb{T}^d$. Dividing (35) by $w + \alpha$ we obtain

$$\partial_t \log(w+\alpha) + u \cdot \nabla \log(w+\alpha) = -\frac{\lambda w}{w+\alpha}\mathcal{M}(|\nabla u|).$$

Integrating this over $\mathbb{T}^d$ and recalling (3) yields

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \log(w+\alpha)\,\mathrm{d}x = \lambda \int_{\mathbb{T}^d} \frac{w}{w+\alpha}\mathcal{M}(|\nabla u|)\,\mathrm{d}x \le \lambda \int_{\mathbb{T}^d} \mathcal{M}(|\nabla u|)\,\mathrm{d}x.$$

Integration of this inequality over $t \in [0,T]$ gives, using (34) and the fact that $|\mathbb{T}^d| = 1$,

$$\begin{aligned}
-\int_{\mathbb{T}^d} \log(w(t,x)+\alpha)\,\mathrm{d}x &\le -\int_{\mathbb{T}^d} \log(1+\alpha)\,\mathrm{d}x + \lambda \int_0^t \int_{\mathbb{T}^d} \mathcal{M}(|\nabla u|)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= -\log(1+\alpha) + \lambda \int_0^t \int_{\mathbb{T}^d} \mathcal{M}(|\nabla u|)(s,x)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Hence, by the Cauchy–Schwarz inequality, Lemma 5 with $p = 2$, and (22), we obtain

$$\begin{aligned}
-\int_{\mathbb{T}^d} \log(w(t,x)+\alpha)\,\mathrm{d}x &\le -\log(1+\alpha) + \lambda \int_0^t \int_{\mathbb{T}^d} \mathcal{M}(|\nabla u|)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\le -\log(1+\alpha) + \lambda \int_0^t \|\mathcal{M}(|\nabla u|)(s,\cdot)|\|_{L^2(\mathbb{T}^d)}\,\mathrm{d}s \\
&\le -\log(1+\alpha) + C_1\lambda \int_0^T \|\nabla u(s,\cdot)\|_{L^2(\mathbb{T}^d)}\,\mathrm{d}s \\
&\le -\log(1+\alpha) + C_1 T^{1/2}\lambda \left(\int_0^T \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{T}^d)}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le -\log(1+\alpha) + C_0^{1/2} C_1 T^{1/2}\lambda.
\end{aligned}$$

Hence,

$$\begin{aligned}
&-\int_{\{x\in\mathbb{T}^d\,:\,0\le w(t,x)\le 1-\alpha\}} \log(w(t,x)+\alpha)\,\mathrm{d}x \\
&\quad\le \int_{\{x\in\mathbb{T}^d\,:\,1-\alpha< w(t,x)\le 1\}} \log(w(t,x)+\alpha)\,\mathrm{d}x - \log(1+\alpha) + C_0^{1/2}C_1T^{1/2}\lambda \\
&\quad\le \int_{\{x\in\mathbb{T}^d\,:\,1-\alpha< w(t,x)\le 1\}} \log(1+\alpha)\,\mathrm{d}x - \log(1+\alpha) + C_0^{1/2}C_1T^{1/2}\lambda \\
&\quad\le \log(1+\alpha) - \log(1+\alpha) + C_0^{1/2}C_1T^{1/2}\lambda = C_0^{1/2}C_1T^{1/2}\lambda.
\end{aligned}$$

On the left-hand side we can now use the monotone convergence theorem to pass to the limit $\alpha \downarrow 0_+$:

$$\begin{aligned}
\lim_{\alpha\to 0_+} -\int_{\{x\in\mathbb{T}^d\,:\,0\le w(t,x)\le 1-\alpha\}} \log(w(t,x)+\alpha)\,\mathrm{d}x &= \lim_{\alpha\to 0_+}\int_{\mathbb{T}^d} \chi_{\{x\in\mathbb{T}^d\,:\,0\le w(t,x)\le 1-\alpha\}}(-\log(w(t,x)+\alpha))\,\mathrm{d}x \\
&= \int_{\mathbb{T}^d}(-\log w(t,x))\,\mathrm{d}x = \int_{\mathbb{T}^d} |\log w(t,x)|\,\mathrm{d}x,
\end{aligned}$$

thanks to the fact that the integrand

$$\chi_{\{x\in\mathbb{T}^d\,:\,0\le w(t,x)\le 1-\alpha\}}(-\log(w(t,x)+\alpha))$$

is a nonnegative function, which, for each fixed $(t, x) \in [0, T] \times \mathbb{T}^d$, is nondecreasing as $\alpha \downarrow 0_+$. Thus we deduce that

$$\text{ess.sup}_{t\in[0,T]} \int_{\mathbb{T}^d} |\log w(t,x)| \,\mathrm{d}x \le C_0^{1/2} C_1 T^{1/2} \lambda =: B,$$

with $\lambda = dC_2$, as required. □

Let us use Lemma 8 and (38) to prove the compactness of $(\mu^\varepsilon)_{\varepsilon>0}$. Let $\eta \in (0, 1)$, to be chosen. For $t \in [0, T]$, We consider the set $E_\eta(t) \subset \mathbb{T}^{2d}$ defined by

$$E_\eta(t) = \{(x, y) \in \mathbb{T}^d \times \mathbb{T}^d \mid (w_x(t) \le \eta) \vee (w_y(t) \le \eta)\}.$$

Remark 10. From (36), we get

$$\text{for all } (x, y) \in E_\eta, \quad |\log(w_x w_y)| = -\log(w_x) - \log(w_y) \ge -\log \eta = |\log \eta|, \tag{39}$$

$$\text{for all } (x, y) \notin E_\eta, \quad w_x w_y > \eta^2 > 0. \tag{40}$$

Thus we obtain from (39), (26) and (40)

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t) - \mu_y(t)| \,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_{E_\eta} K_h(x-y)|\mu_x(t) - \mu_y(t)| \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^{2d}\setminus E_\eta} K_h(x-y)|\mu_x(t) - \mu_y(t)| \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \frac{1}{|\log\eta|}\int_{E_\eta} K_h(x-y)|\log(w_x(t)w_y(t))| \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+ \frac{1}{\eta^2}\int_{\mathbb{T}^{2d}\setminus E_\eta} K_h(x-y)|\mu_x(t) - \mu_y(t)| w_x(t)w_y(t) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \frac{1}{|\log\eta|}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\log(w_x(t)w_y(t))| \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+ \frac{1}{\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t) - \mu_y(t)| w_x(t)w_y(t) \,\mathrm{d}x\,\mathrm{d}y, \quad t \in [0, T].
\end{aligned} \tag{41}$$

On the one hand, using Young's convolution inequality and Lemma 8,

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\log(w_x(t)w_y(t))| \,\mathrm{d}x\,\mathrm{d}y \\
&\quad= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)\log(w_x(t)) \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)\log(w_y(t)) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le 2\|K_h\|_{L^1(\mathbb{T}^d)} \int_{\mathbb{T}^d} |\log(w(t))| \,\mathrm{d}x \le 2B\|K_h\|_{L^1(\mathbb{T}^d)}, \quad t \in [0, T].
\end{aligned} \tag{42}$$

On the other hand, using (38), we get

$$\begin{aligned}
&\frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t) - \mu_y(t)| w_x(t)w_y(t) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \limsup_{\varepsilon\to 0} \frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x^0 - \mu_y^0| \,\mathrm{d}x\,\mathrm{d}y =: \varepsilon_h, \quad t \in [0, T],
\end{aligned} \tag{43}$$

where $\varepsilon_h \underset{h\to 0}{\longrightarrow} 0$ thanks to Theorem 2.

Finally, combining (41), (43) and (42) we arrive at the inequality

$$\frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t)-\mu_y(t)|\,\mathrm{d}x\,\mathrm{d}y \le \frac{2B}{|\log\eta|}+\frac{\varepsilon_h}{\eta^2},\quad t\in[0,T].$$

Choosing $\eta=\varepsilon_h^{1/4}\underset{h\to 0}{\longrightarrow}0$, we obtain

$$\limsup_{\varepsilon\to 0}\sup_{t\in[0,T]}\left[\frac{1}{\|K_h\|_{L^1(\mathbb{T}^d)}}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|\mu_x(t)-\mu_y(t)|\,\mathrm{d}x\,\mathrm{d}y\right]\le \frac{8B}{|\log\varepsilon_h|}+\varepsilon_h^{1/2}\underset{h\to 0}{\longrightarrow}0.$$

Hence, strong compactness of $(\mu^\varepsilon)_{\varepsilon>0}$ in $L^1(0,T;L^1(\mathbb{T}^d))$ using Theorem 2 will directly follow once we have shown that $(\partial_t\mu^\varepsilon)_{\varepsilon>0}$ is bounded in $L^q(0,T;W^{-1,q}(\mathbb{T}^d))$ for some $q>1$. Note to this end that, thanks to (3),

$$|\langle\partial_t\mu(\rho),\varphi\rangle| = |-\langle\operatorname{div}(u\mu(\rho)),\varphi\rangle| = \left|\int_{\mathbb{T}^d}\mu(\rho)u\cdot\nabla\varphi\,\mathrm{d}x\right| \le 2\int_{\mathbb{T}^d}|u||\nabla\varphi|\,\mathrm{d}x \le 2\|u\|_{L^q(\mathbb{T}^d)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\mathbb{T}^d)},$$

for all $\varphi\in W^{1,\frac{q}{q-1}}(\mathbb{T}^d)$. Hence,

$$\int_0^T \|\partial_t\mu(\rho(t))\|^q_{W^{-1,q}(\mathbb{T}^d)}\,\mathrm{d}t \le 2\int_0^T \|u(t)\|^q_{L^q(\mathbb{T}^d)}\,\mathrm{d}t.$$

Because of the bound (24), $q=2$ is a legitimate choice in the last inequality. Thus we have shown that $(\mu^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;W^{-1,2}(\mathbb{T}^d))$, and the strong compactness of the family $(\mu^\varepsilon)_{\varepsilon>0}$ in $L^1(0,T;L^1(\mathbb{T}^d))$ then directly follows from Theorem 3. As $1\le\mu(\cdot)\le 2$, it then further follows that the family $(\mu^\varepsilon)_{\varepsilon>0}$ is in fact strongly compact in $L^p(0,T;L^q(\mathbb{T}^d))$ for all $p,q\in[1,\infty)$ (and weak-$\star$ compact in $L^\infty(0,T;L^\infty(\mathbb{T}^d))$. By the uniqueness of the weak-$\star$ limit,

$$\overline{\mu}=\lim_{n\to\infty}\mu^\varepsilon,$$

in the norm $L^p(0,T;L^q(\mathbb{T}^d))$ for any $p,q\in[1,\infty)$ or indeed in the weak-$\star$ topology of $L^\infty(0,T;L^\infty(\mathbb{T}^d))$.

By an identical argument, the family $(\rho^\varepsilon)_{\varepsilon>0}$ is strongly compact in $L^p(0,T;L^q(\mathbb{T}^d))$ for all $p,q\in[1,\infty)$ (and weak-$\star$ compact in $L^\infty(0,T;L^\infty(\mathbb{T}^d))$, and

$$\overline{\rho}=\lim_{n\to\infty}\rho^\varepsilon,$$

in the norm $L^p(0,T;L^q(\mathbb{T}^d))$ for any $p,q\in[1,\infty)$ or indeed in the weak-$\star$ topology of $L^\infty(0,T;L^\infty(\mathbb{T}^d))$.

2.3 The congested fluid flow model

By rearranging the defining expression (4) for the viscosity, we have that

$$(1-\rho^\varepsilon)(\mu^\varepsilon-1)=\varepsilon(\rho^\varepsilon)^\gamma(1-\rho^\varepsilon)^{1-\beta} \tag{44}$$

Passing to the limit $\varepsilon\to 0$, using the strong compactness of $(\mu^\varepsilon)_{\varepsilon>0}$ and the fact that $\beta\le 1$, we get

$$(1-\overline{\rho})(\overline{\mu}-1)=0 \qquad \text{with } 0\le\overline{\rho}\le 1 \text{ and } 1\le\overline{\mu}\le 2. \tag{45}$$

Remark 11. In particular, $\overline{\mu}=1$ at any point in $[0,T]\times\mathbb{T}^d$ where $\overline{\rho}<1$.

As $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$ and $(\rho^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^\infty(\mathbb{T}^d))$, it follows that $(u^\varepsilon\rho^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$, and has therefore a weakly convergent subsequence (not indicated) in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$. Consequently, $\rho^\varepsilon u^\varepsilon \to \overline{\rho u}$ weakly in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$, and therefore also weakly in $L^1(0,T;L^1(\mathbb{T}^d;\mathbb{R}^d))$. On the other hand, $u^\varepsilon \rightharpoonup \overline{u}$ weakly in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$ and $\rho^\varepsilon \to \overline{\rho}$ strongly in $L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$. Therefore, $\rho^\varepsilon u^\varepsilon \rightharpoonup \overline{\rho}\,\overline{u}$ weakly in $L^1(0,T;L^1(\mathbb{T}^d;\mathbb{R}^d))$. By the uniqueness of the weak limit in $L^1(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$ it then follows that

$$\overline{\rho u} = \overline{\rho}\,\overline{u} \qquad \text{in } L^1(0,T;L^1(\mathbb{T}^d;\mathbb{R}^d)). \tag{46}$$

As both sides of the equality belong to $L^2(0,T;L^2(\mathbb{T}^d))$ we have,

$$\overline{\rho u} = \overline{\rho}\,\overline{u} \qquad \text{in } L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d)). \tag{47}$$

By an analogous argument as in [Sim90] (cf. p.1106 under the heading *Convergence properties*)

$$\overline{\rho u\otimes u} = \overline{\rho u}\otimes\overline{u} = \overline{\rho}\cdot\overline{u}\otimes\overline{u} \qquad \text{in } L^1(0,T;W^{-1,\frac{d}{d-1}}(\mathbb{T}^d;\mathbb{R}^{d\times d})). \tag{48}$$

As $\mathbb{D}$ is linear, we infer using the strong compactness of μ^ε and the weak compactness of $(\mathbb{D}(u^\varepsilon))_{\varepsilon>0}$ that

$$\overline{\mu\,\mathbb{D}(u)} = \overline{\mu}\,\mathbb{D}(\overline{u}) \qquad \text{in } L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})). \tag{49}$$

Finally, analogously to (47),

$$\overline{\mu u} = \overline{\mu}\,\overline{u} \qquad \text{in } L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d)). \tag{50}$$

Using these results, we can pass to the limit in (1), (2), (3), and recall (45), to arrive at the following, so-called, *congested fluid model*:

$$\begin{cases} \partial_t\overline{\rho} + \operatorname{div}(\overline{\rho}\,\overline{u}) = 0 & \text{in } (0,T]\times\Omega, & (51)\\ \partial_t\overline{\mu} + \operatorname{div}(\overline{\mu}\,\overline{u}) = 0 & \text{in } (0,T]\times\Omega, & (52)\\ \partial_t(\overline{\rho}\,\overline{u}) + \operatorname{div}(\overline{\rho}\,\overline{u}\otimes\overline{u}) - \operatorname{div}(\overline{\mu}\,\mathbb{D}(\overline{u})) + \nabla \mathrm{p} = 0 & \text{in } (0,T]\times\Omega, & (53)\\ \operatorname{div}\overline{u} = 0 & \text{in } (0,T]\times\Omega, & (54)\\ (1-\overline{\rho})\,(\overline{\mu}-1) = 0 & \text{in } (0,T]\times\Omega, & (55) \end{cases}$$

subject to the initial conditions

$$\overline{\rho}(0,x) = \overline{\rho_0}(x), \quad \overline{\mu}(0,x) = \overline{\mu(\rho_0)}(0,x), \quad (\overline{\rho}\,\overline{u})(0,x) = \overline{\rho_0}(x)\,\overline{u_0}(x) \qquad \text{for } x\in\mathbb{T}^d.$$

3 An application to modelling of atmospheric dispersion of a pollutant

The aim in this section is to focus on an application discussed in [PT09]. More precisely we want to justify the derivation of the following nonhomogeneous incompressible Navier–Stokes system coupled to a degenerate reaction-diffusion equation that arises in a model of atmospheric dispersion of ozone and other photochemically generated pollutants:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (56)\\ \partial_t\pi + \operatorname{div}(\pi u) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (57)\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \Delta u + \nabla\mathrm{p} = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (58)\\ \operatorname{div} u = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (59)\\ \partial_t c + u\cdot\nabla c - \partial_z((1+\pi)\partial_z c) = 0 & \text{in } (0,T]\times\mathbb{T}^d, & (60) \end{cases}$$

with

$$\pi \geq 0, \qquad \pi(1-\rho) = 0, \qquad 0 \leq \rho \leq 1 \qquad \text{on } (0,T] \times \mathbb{T}^d, \tag{61}$$

and subject to the initial conditions

$$\rho(0,x) = \rho_0(x), \quad (\rho u)(0,x) = \rho_0(x) u_0(x), \quad c(0,x) = c_0(x) \qquad \text{for } x \in \mathbb{T}^d.$$

The noteworthy feature of this model is that the diffusion in equation (60) acts only in one direction (usually the vertical direction), while the transport of species is in all directions, driven by the velocity field. Thus, the equation (60) modelling the evolution of the concentration c of the pollutant under consideration is a partial differential equation with nonnegative characteristic form [OR73], with degenerate diffusion.

The key idea is to regularize the evolution equation (60) for the concentration of the pollutant in a particular way using a parameter-dependent diffusion matrix A_ε, with $0 < \varepsilon < 1$, and then pass to the limit $\varepsilon \to 0$ with the regularization parameter ε. We shall define A_ε so that some of its components are small when the density of the flow is smaller than a certain value, and they are nonnegligible otherwise. Passing to the limit $\varepsilon \to 0$ will enable us to rigorously derive the degenerate reaction-diffusion equation (60) coupled with the nonhomogeneous incompressible Navier–Stokes system.

For the sake of simplicity, instead of a fully three-dimensional model, we consider the following model in two spatial dimensions, with x being a horizontal direction and z the vertical direction:

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 & \text{in } (0,T] \times \mathbb{T}^2, \quad (62) \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \Delta u^\varepsilon + \nabla \mathrm{p}^\varepsilon = 0 & \text{in } (0,T] \times \mathbb{T}^2, \quad (63) \\ \operatorname{div} u^\varepsilon = 0 & \text{in } (0,T] \times \mathbb{T}^2, \quad (64) \\ \partial_t c^\varepsilon + u^\varepsilon \cdot \nabla c^\varepsilon - \operatorname{div}(A_\varepsilon(\rho^\varepsilon) \nabla c^\varepsilon) = 0 & \text{in } (0,T] \times \mathbb{T}^2, \quad (65) \end{cases}$$

subject to the initial conditions

$$\rho^\varepsilon(0,x,z) = \rho_0^\varepsilon(x,z), \quad (\rho^\varepsilon u^\varepsilon)(0,x,z) = \rho_0^\varepsilon(x,z) u_0^\varepsilon(x,z), \quad c^\varepsilon(0,x,z) = c_0^\varepsilon(x,z), \qquad (x,z) \in \mathbb{T}^2,$$

where, for $\beta \in (0,1)$, $\gamma \geq 1$, and $\varepsilon \in (0,1)$,

$$A_\varepsilon(s) = \begin{pmatrix} \varepsilon A_{xx}(s) & \varepsilon A_{xz}(s) \\ \varepsilon A_{xz}(s) & 1 + \dfrac{\varepsilon s^\gamma}{(1-s)^\beta} \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},$$

and $A_{xx}, A_{xz} : [0,1] \to \mathbb{R}$ are bounded continuous functions, and $s \in [0,1)$. Suppose further that there exists an $\underline{a} > 0$ such that $A_{xx} \geq \underline{a}$. Then, there exists an $\overline{\varepsilon} = \overline{\varepsilon}(\overline{a}, \|A_{xx}\|_\infty, \|A_{xz}\|_\infty) > 0$ such that

$$A_{xx}(s)\left(1 + \frac{\varepsilon s^\gamma}{(1-s)^\beta}\right) \geq 2\varepsilon A_{xz}(s)^2 \qquad \text{for all } 0 < \varepsilon \leq \overline{\varepsilon} \text{ and all } s \in [0,1). \tag{66}$$

This then enables us to infer, for all $v = (v_x, v_z) \in \mathbb{R}^2$, $\varepsilon \in (0, \overline{\varepsilon}]$ and $s \in [0,1)$, the following two-sided bound:

$$\frac{1}{2}\varepsilon A_{xx}(s)|v_x|^2 + \frac{1}{2}\left(1 + \frac{\varepsilon s^\gamma}{(1-s)^\beta}\right)|v_z|^2 \leq A_\varepsilon(s) v \cdot v \leq \frac{3}{2}\varepsilon A_{xx}(s)|v_x|^2 + \frac{3}{2}\left(1 + \frac{\varepsilon s^\gamma}{(1-s)^\beta}\right)|v_z|^2. \tag{67}$$

In particular, $A_\varepsilon(s)$ is a positive definite matrix for all $0 < \varepsilon \leq \overline{\varepsilon}$ and all $s \in [0,1)$.

We shall suppose that the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, c_0^\varepsilon) \in L^\infty(\mathbb{T}^2) \times L^2(\mathbb{T}^2; \mathbb{R}^2) \times L^\infty(\mathbb{T}^2)$ satisfy

$$0 \leq \rho_0^\varepsilon \leq 1 - \varepsilon^{1/\beta}, \quad \operatorname{div} u_0^\varepsilon = 0, \quad 0 \leq c_0^\varepsilon \leq 1. \tag{68}$$

Finally, we define

$$\pi^\varepsilon := \frac{\varepsilon (\rho^\varepsilon)^\gamma}{(1-\rho^\varepsilon)^\beta}. \tag{69}$$

Theorem 12. *Let $T > 0$ and suppose that $0 < \varepsilon \leq \bar{\varepsilon}$. Then, the system (62)–(65) with initial condition $(\rho_0^\varepsilon, u_0^\varepsilon, c_0^\varepsilon) \in L^\infty(\mathbb{T}^2) \times L^2(\mathbb{T}^2; \mathbb{R}^2) \times L^\infty(\mathbb{T}^2)$ satisfying (68) admits a weak solution*

$$(\rho^\varepsilon, u^\varepsilon, c^\varepsilon) \in L^\infty([0,T] \times \mathbb{T}^2) \times L^2(0,T, W^{1,2}(\mathbb{T}^d; \mathbb{R}^2)) \times L^\infty([0,T] \times \mathbb{T}^2).$$

Moreover,

$$0 \leq \rho^\varepsilon(t,x,z) \leq 1, \quad 0 \leq \pi^\varepsilon(t,x,z) \leq 1 \qquad \text{for all } (t,x,z) \in [0,T] \times \mathbb{T}^2; \tag{70}$$

there exists some $C = C(\|u_0^\varepsilon\|_{L^2(\mathbb{T}^2)}, \gamma, \beta, T) > 0$ such that

$$\begin{aligned} \text{ess.sup}_{t\in[0,T]} \int_{\mathbb{T}^2} \frac{1}{2} \rho^\varepsilon(t,x,z) |u^\varepsilon(t,x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z + \int_0^T \int_{\mathbb{T}^2} |u^\varepsilon(t,x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}t \\ + \int_0^T \int_{\mathbb{T}^2} |\nabla u^\varepsilon(t,x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}t \leq C, \end{aligned} \tag{71}$$

$$\text{ess.sup}_{(t,x,z)\in[0,T]\times\mathbb{T}^2} |A_\varepsilon(\rho^\varepsilon(t,x,z))| \leq C, \tag{72}$$

and

$$0 \leq c^\varepsilon(t,x,z) \leq 1 \qquad \text{for all } (t,x,z) \in [0,T] \times \mathbb{T}^2, \tag{73}$$

$$\begin{aligned} \text{ess.sup}_{t\in[0,T]} \Bigg[\int_{\mathbb{T}^2} [c^\varepsilon(t,x,z)]^2 \, \mathrm{d}x \, \mathrm{d}z + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^2} A_{xx}(\rho^\varepsilon(s,x,z)) |\partial_x c^\varepsilon(s,x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}s \\ + \frac{1}{2} \int_0^t \int_{\mathbb{T}^2} |\partial_z c^\varepsilon(s,x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}s \Bigg] \leq 1. \end{aligned} \tag{74}$$

Proof. Using the method developed for the system (1)–(3), we obtain (70) and (71), while (72) is a straightforward consequence of (70). Multiplying (65) by c^ε then integrating over $[0,t] \times \mathbb{T}^2$, where $t \in (0,T]$, we obtain, using (64),

$$\begin{aligned} \text{ess.sup}_{t\in[0,T]} \Bigg[\int_{\mathbb{T}^2} [c^\varepsilon(t,x,z)]^2 \, \mathrm{d}x \, \mathrm{d}z + \int_0^t \int_{\mathbb{T}^2} A_\varepsilon(\rho^\varepsilon(s,x,z)) \nabla c^\varepsilon(s,x,z) \cdot \nabla c^\varepsilon(s,x,z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}s \Bigg] \\ = \int_{\mathbb{T}^2} |c_0^\varepsilon(x,z)|^2 \, \mathrm{d}x \, \mathrm{d}z \leq 1. \end{aligned} \tag{75}$$

In particular, if c_0^ε is not identically zero, then that is also true of c^ε. We then easily get (74) from (67) and (75), using the nonnegativity of π^ε.

More generally, after multiplying (65) by $f'(c^\varepsilon)$, where $f \in C^2(\mathbb{R})$ is a convex function, and integrating over $[0,t] \times \mathbb{T}^2$, where $t \in (0,T]$, we obtain

$$\int_{\mathbb{T}^2} f(c^\varepsilon(t,x,z)) \, \mathrm{d}x \, \mathrm{d}z \leq \int_{\mathbb{T}^2} f(c_0^\varepsilon(x,z)) \, \mathrm{d}x \, \mathrm{d}z \qquad \text{for all } t \in [0,T]. \tag{76}$$

Using an approximation argument, (76) is also valid for all convex function. Choosing f defined by

$$f(y) := \max\{0, y-1\}, \qquad y \in \mathbb{R}^d, \tag{77}$$

we get from (76) and (68) the bound

$$\int_{\mathbb{T}^2} \max\{0, c^\varepsilon(t,x,z) - 1\} \, \mathrm{d}x \, \mathrm{d}z \leq 0 \qquad \text{for all } t \in [0,T]; \tag{78}$$

hence $c^\varepsilon \leq 1$ a.e. on $[0,T] \times \mathbb{T}^2$. Similarly, choosing $f : y \mapsto -\min\{0, y\}$, we get $0 \leq c^\varepsilon$ a.e. on $[0,T] \times \mathbb{T}^2$. □

Theorem 13. *Let $T > 0$ and let $(\rho^\varepsilon, u^\varepsilon, c^\varepsilon) \in L^\infty([0,T]\times\mathbb{T}^2)\times L^2(0,T,W^{1,2}(\mathbb{T}^2;\mathbb{R}^2))\times L^\infty([0,T]\times\mathbb{T}^2)$ be a weak solution of (62)–(65) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, c_0^\varepsilon) \in L^\infty(\mathbb{T}^2)\times L^2(\mathbb{T}^2;\mathbb{R}^2)\times L^\infty(\mathbb{T}^2)$ satisfying (68). Suppose further that there exists a constant $C_0 > 0$ such that*

$$\|u_0^\varepsilon\|_{L^2(\mathbb{T}^2)} \le C_0 \qquad \text{for all } \varepsilon > 0. \tag{79}$$

Then, there exists a quadruple $(\rho,\pi,u,c) \in L^\infty([0,T]\times\mathbb{T}^2)^2\times L^2(0,T,W^{1,2}(\mathbb{T}^2;\mathbb{R}^2))\times L^\infty([0,T]\times\mathbb{T}^2)$ such that, up to a subsequence,

$$\rho^\varepsilon \underset{\varepsilon\to0}{\to} \rho \qquad \text{in } L^2(0,T,L^2(\mathbb{T}^2)), \tag{80}$$

$$\pi^\varepsilon \underset{\varepsilon\to0}{\to} \pi \qquad \text{in } L^2(0,T,L^2(\mathbb{T}^2)) \tag{81}$$

$$u^\varepsilon \underset{\varepsilon\to0}{\rightharpoonup} u \qquad \text{in } L^2(0,T,W^{1,2}(\mathbb{T}^2;\mathbb{R}^2)), \tag{82}$$

$$c^\varepsilon \underset{\varepsilon\to0}{\rightharpoonup^\star} c \qquad \text{in } L^\infty([0,T]\times\mathbb{T}^2), \tag{83}$$

$$c^\varepsilon \underset{\varepsilon\to0}{\to} c \qquad \text{in } L^2(0,T,W^{-1,2}(\mathbb{T}^2)), \tag{84}$$

$$\partial_z c^\varepsilon \underset{\varepsilon\to0}{\rightharpoonup} \partial_z c \quad \text{in } L^2(0,T,L^2(\mathbb{T}^2)). \tag{85}$$

Moreover, (ρ,π,u,c) is solution of (56)–(61).

Proof of Theorem 13. Analogously as in the case of the system (1)–(3), there exists a triple

$$(\rho,\pi,u) \in L^\infty([0,T]\times\mathbb{T}^2)^2\times L^2(0,T,W^{1,2}(\mathbb{T}^2;\mathbb{R}^2))$$

satisfying (80)–(82). Hence we obtain (56)–(59) and (61). The weak convergence results (83) and (85) directly follow, thanks to (73) and (74). Moreover, (64) and (65) give

$$\partial_t c^\varepsilon = \operatorname{div}(A_\varepsilon(\rho^\varepsilon)\nabla c^\varepsilon - u^\varepsilon c^\varepsilon). \tag{86}$$

As $A_\varepsilon(\rho^\varepsilon)$ is a symmetric positive definite matrix, it follows that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^2} |A_\varepsilon(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \\
&\le \operatorname{ess.sup}_{(t,x,z)\in[0,T]\times\mathbb{T}^2}|A_\varepsilon(\rho^\varepsilon(t,x,z))| \int_0^T\int_{\mathbb{T}^2} A_\varepsilon(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)\cdot\nabla c^\varepsilon(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \le C.
\end{aligned} \tag{87}$$

Furthermore,

$$\int_0^T\int_{\mathbb{T}^2} |u^\varepsilon(t,x,z)c^\varepsilon(t,x,z)|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \le \int_0^T\int_{\mathbb{T}^2} |u^\varepsilon(t,x,z)|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \le C. \tag{88}$$

Thus, using (86), (87) and (88), we get (84) by the Aubin–Lions lemma. Then, (84) and (82) give

$$\partial_t c^\varepsilon + u^\varepsilon\cdot\nabla c^\varepsilon = \partial_t c^\varepsilon + \operatorname{div}(u^\varepsilon c^\varepsilon) \underset{\varepsilon\to0}{\rightharpoonup} \partial_t c + \operatorname{div}(uc) = \partial_t c + u\cdot\nabla c \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \tag{89}$$

For $s\in[0,1)$, let $S(s)\in\mathbb{R}^{2\times2}$ denote the symmetric positive semi-definite matrix defined by

$$S(s) = \begin{pmatrix} A_{xx}(s) & A_{xz}(s) \\ A_{xz}(s) & 0 \end{pmatrix}. \tag{90}$$

It then follows that

$$A_\varepsilon(\rho^\varepsilon)\nabla c^\varepsilon = \varepsilon S(\rho^\varepsilon)\nabla c^\varepsilon + \begin{pmatrix} 0 \\ (1+\pi^\varepsilon)\,\partial_z c^\varepsilon \end{pmatrix} \tag{91}$$

and

$$\begin{aligned} \varepsilon \int_0^T \int_{\mathbb{T}^2} &|S(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \\ &\le \text{ess.sup}_{(t,x,z)\in[0,T]\times\mathbb{T}^2} |S(\rho^\varepsilon(t,x,z))| \int_0^T \int_{\mathbb{T}^2} \varepsilon(t,x,z) S(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)\cdot\nabla c^\varepsilon(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \\ &\le C \int_0^T \int_{\mathbb{T}^2} \varepsilon S(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)\cdot\nabla c^\varepsilon(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t. \end{aligned} \tag{92}$$

From (75), we then have

$$\begin{aligned} (0\le) \int_0^T \int_{\mathbb{T}^2} &\varepsilon S(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)\cdot\nabla c^\varepsilon(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}^2} A_\varepsilon(\rho^\varepsilon(t,x,z))\nabla c^\varepsilon(t,x,z)\cdot\nabla c^\varepsilon(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \\ &\quad - \int_0^T \int_{\mathbb{T}^2} (1+\pi^\varepsilon(t,x,z))|\partial_z c^\varepsilon(t,x,z)|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \le 1. \end{aligned} \tag{93}$$

Thus (92) and (93) imply that

$$\varepsilon \|S(\rho^\varepsilon)\nabla c^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^2))} \underset{\varepsilon\to 0}{\to} 0. \tag{94}$$

Moreover, from (81) and (85) we get

$$(1+\pi^\varepsilon)\partial_z c^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} (1+\pi)\partial_z c \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \tag{95}$$

Finally, (91), (94) and (95) give

$$\operatorname{div}(A_\varepsilon(\rho^\varepsilon)\nabla c^\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \partial_z((1+\pi)\partial_z c) \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \tag{96}$$

Hence we obtain (60). □

Acknowledgments. The first author gratefully acknowledges the partial support by the Agence Nationale pour la Recherche grant ANR-23-CE40-0014-01 (ANR Bourgeons). This work also benefited from the support of the ANR under France 2030 bearing the reference ANR-23-EXMA-004 (Complexflows project). The second author acknowledges the partial support by the Agence Nationale pour la Recherche grant CRISIS (ANR-20-CE40-0020-01).

References

[AK73] S. N. Antontsev and A. V. Kazhikhov. *Matematicheskie voprosy dinamiki neodnorodnykh zhidkosteĭ.* Lecture notes, Novosibirsk State University. Novosibirsk. Gosudarstv. Univ., Novosibirsk, 1973, p. 121.

[AKM90] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids.* Vol. 22. Studies in Mathematics and its Applications. Translated from the Russian. North-Holland Publishing Co., Amsterdam, 1990, pp. xii+309. ISBN: 0-444-88382-7.

[BJ13] F. B. Belgacem and P.-E. Jabin. "Compactness for nonlinear continuity equations". In: *J. Funct. Anal.* 264.1 (2013), pp. 139–168. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2012.10.005. URL: https://doi.org/10.1016/j.jfa.2012.10.005.

[BH93] B. Bojarski and P. Hajłasz. "Pointwise inequalities for Sobolev functions and some applications". In: *Studia Math.* 106.1 (1993), pp. 77–92. ISSN: 0039-3223.

[BJ18] D. Bresch and P.-E. Jabin. "Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor". In: *Ann. of Math. (2)* 188.2 (2018), pp. 577–684. ISSN: 0003-486X. DOI: 10.4007/annals.2018.188.2.4. URL: https://doi.org/10.4007/annals.2018.188.2.4.

[BNP19] D. Bresch, Š. Nečasová, and C. Perrin. "Compression Effects in Heterogeneous Media". In: *Journal de l'École polytechnique — Mathématiques* 6 (June 19, 2019), pp. 433–467. ISSN: 2270-518X. DOI: 10.5802/jep.98. URL: https://www.numdam.org/item/JEP_2019__6__433_0/ (visited on 02/18/2026).

[BPZ14] D. Bresch, C. Perrin, and E. Zatorska. "Singular Limit of a Navier–Stokes System Leading to a Free/Congested Zones Two-Phase Model". In: *Comptes Rendus Mathematique* 352.9 (Sept. 1, 2014), pp. 685–690. ISSN: 1631-073X. DOI: 10.1016/j.crma.2014.06.009. URL: https://www.sciencedirect.com/science/article/pii/S1631073X14001459 (visited on 04/13/2024).

[Cal72] A. P. Calderón. "Estimates for singular integral operators in terms of maximal functions". In: *Studia Math.* 44 (1972), pp. 563–582. ISSN: 0039-3223. DOI: 10.4064/sm-44-6-563-582. URL: https://doi.org/10.4064/sm-44-6-563-582.

[DL89] R. J. DiPerna and P.-L. Lions. "Ordinary differential equations, transport theory and Sobolev spaces". In: *Invent. Math.* 98.3 (1989), pp. 511–547. ISSN: 0020-9910. DOI: 10.1007/BF01393835. URL: https://doi.org/10.1007/BF01393835.

[FG93] E. Fernàndez-Cara and F. Guillén. "Some new existence results for the variable density Navier-Stokes equations". eng. In: *Annales de la Faculté des sciences de Toulouse : Mathématiques* 2.2 (1993), pp. 185–204. URL: http://eudml.org/doc/73318.

[Lio96] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1.* Vol. 3. Oxford Lecture Series in Mathematics and its Applications. Incompressible models, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996, pp. xiv+237. ISBN: 0-19-851487-5.

[OR73] O. A. Oleĭnik and E. V. Radkevič. *Second order equations with nonnegative characteristic form.* Translated from the Russian by Paul C. Fife. Plenum Press, New York-London, 1973, pp. vii+259. ISBN: 0-306-30751-0.

[PZ15] C. Perrin and E. Zatorska. "Free/Congested Two-Phase Model from Weak Solutions to Multi-Dimensional Compressible Navier-Stokes Equations". In: *Communications in Partial Differential Equations* 40.8 (Aug. 3, 2015), pp. 1558–1589. ISSN: 0360-5302. DOI: 10.1080/03605302.2015.1014560. URL: https://doi.org/10.1080/03605302.2015.1014560 (visited on 04/13/2024).

[PT09] M. Pierre and R. Texier-Picard. "Global existence for degenerate quadratic reaction-diffusion systems". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 26.5 (2009), pp. 1553–1568. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2008.06.003. URL: https://doi.org/10.1016/j.anihpc.2008.06.003.

[Sim90] J. Simon. "Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure". In: *SIAM J. Math. Anal.* 21.5 (1990), pp. 1093–1117. ISSN: 0036-1410. DOI: 10.1137/0521061. URL: https://doi.org/10.1137/0521061.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, NJ, 1970, pp. xiv+290.

[Ste93] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Vol. 43. Princeton Mathematical Series. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993, pp. xiv+695. ISBN: 0-691-03216-5.