DISCONTINUOUS GALERKIN FINITE ELEMENT APPROXIMATION OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS II: STRONGLY MONOTONE QUASI-NEWTONIAN FLOWS

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Abstract. In this article we develop both the *a priori* and *a posteriori* error analysis of *hp*–version interior penalty discontinuous Galerkin finite element methods for strongly monotone quasi-Newtonian fluid flows in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. In the latter case, computable upper and lower bounds on the error are derived in terms of a natural energy norm which are explicit in the local mesh size and local polynomial degree of the approximating finite element method. A series of numerical experiments illustrate the performance of the proposed *a posteriori* error indicators within an automatic *hp*–adaptive refinement algorithm.

1. Introduction

In this article we develop the *a priori* and *a posteriori* error analysis, with respect to a mesh-dependent energy norm, for *hp*–version discontinuous Galerkin finite element methods (DGFEMs) for the quasi-Newtonian fluid flow problem:

$$-\nabla \cdot \{\mu\left(\mathbf{x}, \left|\underline{e}\left(\mathbf{u}\right)\right|\right) \underline{e}\left(\mathbf{u}\right)\} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma. \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded polygonal Lipschitz domain with boundary $\Gamma = \partial\Omega$, $\mathbf{f} \in \mathrm{L}^2(\Omega)^d$ is a given source term, $\mathbf{u} = (u_1, \ldots, u_d)^\top$ is the velocity vector, p is the pressure, and $\underline{e}(\mathbf{u})$ is the symmetric $d \times d$ strain tensor defined by

$$e_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, \ldots, d.$$

Furthermore, $|\underline{e}(\mathbf{u})|$ is the Frobenius norm of $\underline{e}(\mathbf{u})$.

In recent years, there has been considerable interest in DGFEMs for the numerical solution of a wide range of partial differential equations (PDEs); for an extensive survey of this area of research, we refer the reader to [12], and the references cited therein. DGFEMs have several important advantages over well established finite volume methods. The concept of higher-order discretization is inherent to the DGFEM. The stencil is minimal in the sense that each element communicates only with its direct neighbours. In particular, in contrast to the increasing stencil size needed to increase the accuracy of classical finite volume methods, the stencil of DGFEMs is the same for any order of accuracy, which has important advantages for the implementation of boundary conditions and for the parallel efficiency of the method. Moreover, because of the simple communication at element interfaces, elements with so–called hanging nodes can be easily treated, a fact that simplifies local mesh refinement (*h*–refinement). Additionally, the communication at element interfaces is identical for any order of the method, which simplifies the use of methods with different polynomial orders p in adjacent elements. This allows for the variation of the degrees of polynomials over the computational domain (*p*–refinement), which in combination with *h*–refinement leads to so–called *hp*–adaptivity.

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In the present article, we formulate a class of *hp*–version interior penalty DGFEMs for the numerical approximation of the quasi-Newtonian problem (1.1)–(1.3). This article represents the continuation of the work initiated in [16] and [24], where the *a priori* and *a posteriori* error analysis, respectively, of DGFEMs was developed for quasilinear elliptic boundary-value problems, in the case of a single equation; here, we focus on quasilinear elliptic systems. In particular, in the first part of this article we establish the existence and uniqueness of both the analytical solution to (1.1)–(1.3) and of its DGFEM counterpart. The *a priori* error analysis of the underlying class of DGFEMs is then undertaken, with respect to the underlying natural energy norm. In the second part of this article we derive computable upper and lower bounds on the error, again measured in terms of the energy norm, which are explicit in the local mesh size and the local polynomial degree of the approximating finite element method. At the expense of a slight suboptimality with respect to the polynomial degree of the approximating finite element method, this upper bound holds on general 1-irregular meshes. In particular, this means that elements can be divided into smaller elements without the need of connecting the resulting hanging nodes. This feature clearly improves both the feasibility and the flexibility of an *hp*–adaptive process. In addition, we note that the use of irregular meshes is very natural and quite easily realizable in the context of DGFEM schemes because of the discontinuous character of the corresponding finite element spaces. The proof of the upper bound is based on employing a suitable DGFEM space decomposition, together with an *hp*–version projection operator. This general approach was pursued in the series of articles [24, 17, 21, 19, 25]. The proof of the local lower error bounds (efficiency) is based on the techniques presented in [27], subject to the treatment of the nonlinearity. On the basis of these *a posteriori* error indicators, we design and implement the corresponding *hp*–adaptive algorithm to ensure reliable and efficient control of the discretization error. Numerical experiments are presented, which demonstrate the performance of the proposed algorithm. For related work on *h*–version local DGFEMs for quasi-linear PDEs, we refer to the articles [10, 11, 15], for example.

The article is organized as follows. In Section 2, we state the weak formulation of (1.1)–(1.3) and prove its well-posedness. In Section 3 we formulate the interior penalty *hp*–DGFEM for the numerical approximation of the boundary-value problem (1.1)–(1.3), and show that the proposed scheme is also well-posed. Section 4 is devoted to the *a priori* error analysis of the underlying *hp*–DGFEM. In Section 5 we establish both the upper and lower *a posteriori* error bounds. Section 6 contains a series of numerical experiments, which illustrate our theoretical results; in particular, we demonstrate the performance of an *hp*–adaptive algorithm based on the *hp*–error indicators. Finally, in Section 7 we summarise the main results of this article and draw some conclusions.

2. Weak Formulation

In this section, we will present a weak formulation for (1.1)–(1.3) and prove its well-posedness.

2.1. **Notation.**

Throughout this paper, we use the following standard function spaces. For a bounded Lipschitz domain $D \subset \mathbb{R}^d, d \geq 1$, we write $\mathrm{H}^t(D)$ to denote the usual Sobolev space of real-valued functions of order $t \geq 0$ with norm $\|\cdot\|_{t,D}$. In the case when $t = 0$, we set $\mathrm{L}^2(D) = \mathrm{H}^0(D)$. We define $\mathrm{H}^1_0(D)$ to be the subspace of functions in $\mathrm{H}^1(D)$ with zero trace on ∂D. Additionally, we set $\mathrm{L}^2_0(D) := \{q \in \mathrm{L}^2(D) : \int_D q \, \mathrm{d}\mathbf{x} = 0\}$. For a function space $X(D)$, we let $X(D)^d$ and $X(D)^{d\times d}$ denote the spaces of vector and tensor fields, respectively, whose components belong to $X(D)$. These spaces are equipped with the usual product norms which, for simplicity, we denote in the same way as the norm in $X(D)$.

For the d-component vector-valued functions $\mathbf{v}, \mathbf{w}$ and $d\times d$ matrix-valued functions $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{d\times d}$, we define the operators

$$(\nabla \mathbf{v})_{ij} := \frac{\partial v_i}{\partial x_j}, \qquad (\nabla \cdot \underline{\sigma})_i := \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j},$$

$$(\mathbf{v} \otimes \mathbf{w})_{ij} := v_i w_j, \qquad \underline{\sigma} : \underline{\tau} := \sum_{i=1}^{d} \sigma_{ij} \tau_{ij}.$$

For matrix-valued functions the Frobenius norm can be written as $|\underline{\tau}|^2 = \underline{\tau} : \underline{\tau}$.

2.2. **Variational Form.** By introducing the forms

$$A(\mathbf{u}, \mathbf{v}) := \int_\Omega \mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x}, \qquad B(\mathbf{v}, q) := -\int_\Omega q\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},$$

a natural weak formulation of the quasi-Newtonian problem (1.1)–(1.3) is to find $(\mathbf{u}, p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ such that

$$A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \tag{2.1}$$

$$-B(\mathbf{u}, q) = 0 \tag{2.2}$$

for all $(\mathbf{v}, q) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$. We note that the bilinear form B satisfies the following inf-sup condition: there exists a constant $\kappa > 0$ such that

$$\inf_{0\neq q\in \mathrm{L}^2_0(\Omega)} \sup_{0\neq \mathbf{v}\in \mathrm{H}^1_0(\Omega)^d} \frac{B(\mathbf{v}, q)}{\|q\|_{0,\Omega}\,\|\underline{e}(\mathbf{v})\|_{0,\Omega}} \geq \kappa; \tag{2.3}$$

see, e.g., [9].

We shall assume throughout this article that the function μ satisfies the following structural hypothesis.

Assumption 1. *We assume that the nonlinearity μ satisfies the following conditions:*

(A1) $\mu \in \mathrm{C}(\overline{\Omega} \times [0, \infty))$.

(A2) There exist constants $m_\mu, M_\mu > 0$ such that

$$m_\mu(t - s) \leq \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leq M_\mu(t - s), \qquad t \geq s \geq 0, \quad \mathbf{x} \in \overline{\Omega}. \tag{2.4}$$

From [4, Lemma 2.1], we note that as μ satisfies (2.4), there exists positive constants C_1 and C_2, such that for all symmetric $\underline{\tau}, \underline{\omega} \in \mathbb{R}^{d\times d}$ and all $\mathbf{x} \in \overline{\Omega}$,

$$|\mu(\mathbf{x}, |\underline{\tau}|)\underline{\tau} - \mu(\mathbf{x}, |\underline{\omega}|)\underline{\omega}| \leq C_1|\underline{\tau} - \underline{\omega}|, \tag{2.5}$$

$$C_2|\underline{\tau} - \underline{\omega}|^2 \leq (\mu(\mathbf{x}, |\underline{\tau}|)\underline{\tau} - \mu(\mathbf{x}, |\underline{\omega}|)\underline{\omega}) : (\underline{\tau} - \underline{\omega}). \tag{2.6}$$

For ease of notation we shall suppress the dependence of μ on $\mathbf{x}$ and write $\mu(t)$ instead of $\mu(\mathbf{x}, t)$.

2.3. **Well-Posedness.** We will now show that the weak formulation (2.1)–(2.2) admits a unique solution in the given spaces. To this end, we first give the following general theorem.

Theorem 2.1. *Let X be a real Hilbert space. Furthermore, consider forms $a : X \times X \to \mathbb{R}$ and $l : X \to \mathbb{R}$ with*

(a) l is linear and continuous on X,

(b) the functional $v \mapsto a(w, v)$ is linear and continuous on X for any fixed $w \in X$,

(c) there exists a constant $L > 0$ such that

$$|a(u, w) - a(v, w)| \leq L\|u - v\|_X\|w\|_X$$

for any $u, v, w \in X$,

(d) there exists a constant $c > 0$ such that for any $u, w \in X$, there exists $v \in X$ with

$$a(u, v) - a(w, v) \geq c\|u - w\|_X, \qquad \|v\|_X \leq 1.$$

(e) for any $0 \neq v \in X$,

$$\sup_{u\in X} a(u, v) > 0.$$

Then, there exists a unique solution $u \in X$ of the variational equation

$$a(u, v) = l(v) \qquad \forall v \in X. \tag{2.7}$$

Proof. We proceed in several steps.

Step 1: Denoting by $(\cdot,\cdot)_X$ the inner product on X, upon application of Riesz' theorem and (b), we deduce that, for any $w \in X$, there is an element denoted by $\mathsf{A}w \in X$ such that

$$a(w,u) = (\mathsf{A}w,u)_X \qquad \forall u \in X.$$

This defines an operator $\mathsf{A}: X \to X$. Using (d), we note that for any $u,w \in X$, there is $v \in X$ with $\|v\|_X \leq 1$ and

$$c\|u-w\|_X \leq a(u,v) - a(w,v) = (\mathsf{A}u - \mathsf{A}w, v)_X. \tag{2.8}$$

Furthermore, by (c), we observe that

$$|(\mathsf{A}u - \mathsf{A}v, w)_X| = |a(u,w) - a(v,w)| \leq L\|u-v\|_X\|w\|_X,$$

for any $u,v,w \in X$. Therefore,

$$\|\mathsf{A}u - \mathsf{A}v\|_X = \sup_{\|w\|_X \leq 1} |(\mathsf{A}u - \mathsf{A}v, w)_X| \leq L\|u-v\|_X. \tag{2.9}$$

In addition, again by Riesz' theorem and by recalling (a), there exists $\ell \in X$ such that

$$l(u) = (\ell, u)_X \qquad \forall u \in X.$$

Hence, the variational formulation (2.7) corresponds to the operator equation

$$\mathsf{A}u = \ell, \qquad u \in X. \tag{2.10}$$

Step 2: Our goal is to prove that the image $\mathrm{Im}(\mathsf{A})$ of A is the entire space X. This implies that (2.10) has a solution for any $\ell \in X$.

Step 2a: We begin by showing that $\mathrm{Im}(\mathsf{A})$ is closed in X. To this end, let us consider a sequence $\{z_n\}_{n=1}^\infty \subset \mathrm{Im}(\mathsf{A})$ that converges to $\bar{z} \in X$. Evidently, this sequence is a Cauchy sequence. Furthermore, there is a sequence $\{w_n\}_{n=1}^\infty \subset X$ such that $z_n = \mathsf{A}w_n$ for any $n \in \mathbb{N}$. Hence, for any $m,n \in \mathbb{N}$, using (2.8), we have that

$$\|z_m - z_n\|_X = \|\mathsf{A}w_m - \mathsf{A}w_n\|_X = \sup_{\|v\|_X \leq 1} (\mathsf{A}w_m - \mathsf{A}w_n, v)_X \geq c\|w_m - w_n\|_X;$$

thereby, $\{w_n\}_{n=1}^\infty$ is a Cauchy sequence, with a limit $\bar{w} \in X$. Furthermore,

$$\bar{z} = \lim_{n\to\infty} \mathsf{A}w_n = \lim_{n\to\infty} (\mathsf{A}w_n - \mathsf{A}\bar{w}) + \mathsf{A}\bar{w},$$

and by (2.9), we have

$$\|\mathsf{A}w_n - \mathsf{A}\bar{w}\|_X \leq L\|w_n - \bar{w}\|_X \to 0$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} \mathsf{A}w_n = \mathsf{A}\bar{w}$, and hence, $\bar{z} = \mathsf{A}\bar{w} \in \mathrm{Im}(\mathsf{A})$.

Step 2b: Suppose now that $\mathrm{Im}(\mathsf{A}) \neq X$. Then, since $\mathrm{Im}(\mathsf{A})$ is closed, we apply the Hahn–Banach theorem to deduce that there exists $0 \neq \widetilde{w} \in X$ with $(v,\widetilde{w})_X = 0$ for all $v \in \mathrm{Im}(\mathsf{A})$. In particular,

$$0 = (\mathsf{A}u, \widetilde{w}) = a(u, \widetilde{w}) \qquad \forall u \in X,$$

which contradicts (e). Consequently, $\mathrm{Im}A = X$ and (2.10) has a solution $u \in X$.

Step 3: We complete the proof by demonstrating that the solution of (2.10) is unique. Suppose that there are two solutions $u_1, u_2 \in X$ of (2.10). Then, recalling (2.8) there exists $w \in X$ such that

$$0 = (\mathsf{A}u_1 - \mathsf{A}u_2, w)_X \geq c\|u_1 - u_2\|_X,$$

and therefore, $u_1 = u_2$. □

We will now apply the above result to (2.1)–(2.2). To this end, we define the form

$$\mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) := A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) - B(\mathbf{u},q)$$

on the space $(\mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)) \times (\mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega))$, and the norm

$$|\!|\!|(\mathbf{u},p)|\!|\!|^2 := \|\underline{e}(\mathbf{u})\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2.$$

Proposition 2.2. *There exist two constants $L, c > 0$ such that the following hold:*

(a) *Continuity: For any* $(\mathbf{u}, p)$, $(\mathbf{v}, q)$, $(\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$, *we have:*

$$|\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q))| \leq L |\!|\!| (\mathbf{u} - \mathbf{w}, p - r) |\!|\!| \, |\!|\!| (\mathbf{v}, q) |\!|\!|.$$

(b) *Inf-sup stability: For any* $(\mathbf{u}, p), (\mathbf{w}, r) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ *there exists* $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ *such that*

$$\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q)) \geq c |\!|\!| (\mathbf{u} - \mathbf{w}, p - r) |\!|\!|, \qquad |\!|\!| (\mathbf{v}, q) |\!|\!| \leq 1.$$

(c) *For any* $0 \neq (\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$,

$$\sup_{(\mathbf{u}, p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) > 0.$$

Proof. We prove (a)–(c) separately.

Proof of (a): Applying the triangle inequality, we have that

$$\begin{aligned} &|\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q))| \\ &\qquad \leq |A(\mathbf{u}, \mathbf{v}) - A(\mathbf{w}, \mathbf{v})| + |B(\mathbf{v}, p - r)| + |B(\mathbf{u} - \mathbf{w}, q)|. \end{aligned}$$

Then, recalling (2.5) leads to

$$\begin{aligned} |A(\mathbf{u}, \mathbf{v}) - A(\mathbf{w}, \mathbf{v})| &\leq \int_\Omega |\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})||\underline{e}(\mathbf{v})| \,\mathrm{d}\mathbf{x} \\ &\leq C_1 \int_\Omega |\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})||\underline{e}(\mathbf{v})| \,\mathrm{d}\mathbf{x} \\ &\leq C_1 \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega}. \end{aligned}$$

Furthermore,

$$|B(\mathbf{v}, p - r)| \leq \int_\Omega |p - r||\nabla \cdot \mathbf{v}| \,\mathrm{d}\mathbf{x} \leq \|p - r\|_{0,\Omega} \|\nabla \mathbf{v}\|_{0,\Omega}.$$

According to Korn's inequality, there exist a positive constant C_* such that $\|\mathbf{v}\|_{1,\Omega} \leq C_* \|\underline{e}(\mathbf{v})\|_{0,\Omega}$ for all $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$; thus we arrive at

$$|B(\mathbf{v}, p - r)| \leq C_* \|p - r\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega}.$$

Similarly,

$$|B(\mathbf{u} - \mathbf{w}, q)| \leq C_* \|q\|_{0,\Omega} \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega}.$$

Combining these estimates we obtain

$$\begin{aligned} &|\mathcal{A}((\mathbf{u}, p); (\mathbf{v}, q)) - \mathcal{A}((\mathbf{w}, r); (\mathbf{v}, q))| \\ &\qquad \leq C_1 \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w}))\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega} \\ &\qquad\quad + C_* \|p - r\|_{0,\Omega} \|\underline{e}(\mathbf{v})\|_{0,\Omega} + C_* \|q\|_{0,\Omega} \|\underline{e}(\mathbf{u}) - \underline{e}(\mathbf{w})\|_{0,\Omega}. \end{aligned}$$

Thence, using the Cauchy–Schwarz inequality, we deduce (a).

Proof of (b): Let $p - r \in \mathrm{L}_0^2(\Omega)$, then, from the inf-sup condition (2.3) there exists $\boldsymbol{\xi} \in \mathrm{H}_0^1(\Omega)^d$ such that

$$-\int_\Omega (p - r)\nabla \cdot \boldsymbol{\xi} \,\mathrm{d}\mathbf{x} \geq \kappa \|p - r\|_{0,\Omega}^2, \qquad \|\underline{e}(\boldsymbol{\xi})\|_{0,\Omega} \leq \|p - r\|_{0,\Omega}. \tag{2.11}$$

Now, we choose

$$\hat{\mathbf{v}} := \alpha(\mathbf{u} - \mathbf{w}) + \beta\boldsymbol{\xi}, \qquad \hat{q} := \alpha(p - r),$$

with

$$\alpha := C_2^{-1}(1 + C_1^2\kappa^{-2}), \qquad \beta := 2\kappa^{-1},$$

where C_1, C_2 are the constants from (2.5)–(2.6). Now, using (2.5), (2.6), (2.11) and the arithmetic-geometric mean inequality we deduce that

$$\begin{aligned}
&\mathcal{A}((\mathbf{u},p);(\hat{\mathbf{v}},\hat{q})) - \mathcal{A}((\mathbf{w},r);(\hat{\mathbf{v}},\hat{q}))\\
&= \int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\hat{\mathbf{v}})\,\mathrm{d}\mathbf{x}\\
&\quad - \int_\Omega (p-r)\nabla\cdot\hat{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \int_\Omega \hat{q}\nabla\cdot(\mathbf{u}-\mathbf{w})\,\mathrm{d}\mathbf{x}\\
&= \alpha\int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\mathbf{u}-\mathbf{w})\,\mathrm{d}\mathbf{x}\\
&\quad + \beta\int_\Omega \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})\} : \underline{e}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{x} - \beta\int_\Omega (p-r)\nabla\cdot\boldsymbol{\xi}\,\mathrm{d}\mathbf{x}\\
&\geq \alpha C_2\int_\Omega |\underline{e}(\mathbf{u}-\mathbf{w})|^2\,\mathrm{d}\mathbf{x} - \frac{1}{2}\kappa\beta\int_\Omega |\underline{e}(\boldsymbol{\xi})|^2\,\mathrm{d}\mathbf{x} + \beta\kappa\,\|p-r\|_{0,\Omega}^2\\
&\quad - \frac{1}{2}\kappa^{-1}\beta\int_\Omega |\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{w})|)\underline{e}(\mathbf{w})|^2\,\mathrm{d}\mathbf{x}\\
&\geq (\alpha C_2 - \frac{1}{2}\kappa^{-1}\beta C_1^2)\,\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + \frac{1}{2}\beta\kappa\,\|p-r\|_{0,\Omega}^2\\
&= |\!|\!|(\mathbf{u}-\mathbf{w},p-r)|\!|\!|^2.
\end{aligned}$$

Using the triangle inequality, we deduce that

$$\begin{aligned}
|\!|\!|(\hat{\mathbf{v}},\hat{q})|\!|\!|^2 &= \|\underline{e}(\hat{\mathbf{v}})\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2\\
&\leq 2\alpha^2\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + 2\beta^2\|\underline{e}(\boldsymbol{\xi})\|_{0,\Omega}^2 + \alpha^2\,\|p-r\|_{0,\Omega}^2\\
&\leq 2\alpha^2\|\underline{e}(\mathbf{u}-\mathbf{w})\|_{0,\Omega}^2 + (\alpha^2+2\beta^2)\,\|p-r\|_{0,\Omega}^2\\
&\leq \max(2\alpha^2,\alpha^2+2\beta^2)|\!|\!|(\mathbf{u}-\mathbf{w},p-r)|\!|\!|^2.
\end{aligned}$$

Setting $(\mathbf{v},q) = \max(2\alpha^2,\alpha^2+2\beta^2)^{-\frac{1}{2}}|\!|\!|(\mathbf{u}-\mathbf{w},p-r)|\!|\!|^{-1}(\hat{\mathbf{v}},\hat{q})$ completes the proof.

Proof of (c): Let $(\mathbf{v},q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)\setminus\{(\mathbf{0},0)\}$. Then, for $\mathbf{v}\neq 0$, we have that

$$\sup_{(\mathbf{u},p)\in \mathrm{H}_0^1(\Omega)^d\times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) \geq \mathcal{A}((\mathbf{v},q);(\mathbf{v},q)) = A(\mathbf{v},\mathbf{v}),$$

and noting (2.6), yields

$$A(\mathbf{v},\mathbf{v}) = \int_\Omega \mu(|\underline{e}(\mathbf{v})|)\underline{e}(\mathbf{v}) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \geq C_2\|\underline{e}(\mathbf{v})\|_{0,\Omega}^2 > 0.$$

If $\mathbf{v}=\mathbf{0}, q\neq 0$, we use the inf-sup condition (2.3) to find $\mathbf{v}_q \in \mathrm{H}_0^1(\Omega)^d$ such that

$$\sup_{(\mathbf{u},p)\in \mathrm{H}_0^1(\Omega)^d\times \mathrm{L}_0^2(\Omega)} \mathcal{A}((\mathbf{u},p);(\mathbf{0},q)) \geq \mathcal{A}(-(\mathbf{v}_q,0);(\mathbf{0},q)) = B(\mathbf{v}_q,q) \geq \kappa\|q\|_{0,\Omega} > 0.$$

This completes the proof. □

We are now ready to prove the following result.

Theorem 2.3. *There exists exactly one solution* $(\mathbf{u},p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ *to the weak formulation* (2.1)–(2.2).

Proof. We notice that (2.1)–(2.2) is equivalent to finding $(\mathbf{u},p) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$\mathcal{A}((\mathbf{u},p);(\mathbf{v},q)) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \qquad \forall(\mathbf{v},q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega).$$

Thence, noticing that by Korn's inequality the linear form

$$\mathbf{v} \mapsto \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

is continuous, and applying Theorem 2.1, in combination with the above Proposition 2.2, implies the well-posedness of (2.1)–(2.2). □

3. DGFEM Approximation of Non-Newtonian Flows

In this section we present the discretization of (1.1)–(1.3) based on employing the hp–version of a family of interior penalty (IP) DGFEMs, which includes the symmetric, non-symmetric, and incomplete IP schemes. To this end, we first introduce the necessary notation.

3.1. **Meshes, Spaces, and Trace Operators.** Let $\mathcal{T}_h$ be a subdivision of Ω into disjoint open-element domains K such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. We assume that the family of subdivisions $\{\mathcal{T}_h\}_{h>0}$ is shape regular ([7, pp. 61, 118 and Remark 2.2, p. 114]) and each $K \in \mathcal{T}_h$ is an affine image of a fixed master element $\hat{K}$; i.e., for each $K \in \mathcal{T}_h$, there exists an affine mapping $T_K : \hat{K} \to K$ such that $K = T_K(\hat{K})$, where $\hat{K}$ is the open cube $(-1,1)^3$ in $\mathbb{R}^3$ or the open square $(-1,1)^2$ in $\mathbb{R}^2$. By h_K we denote the element diameter of $K \in \mathcal{T}_h$, $h = \max_{K\in\mathcal{T}_h} h_K$, and $\mathbf{n}_K$ signifies the unit outward normal vector to K. We allow the meshes $\mathcal{T}_h$ to be *1-irregular*, i.e., each face of any one element $K \in \mathcal{T}_h$ contains at most one hanging node (which, for simplicity, we assume to be at the centre of the corresponding face) and each edge of each face contains at most one hanging node (yet again assumed to be at the centre of the edge). Here, we suppose that $\mathcal{T}_h$ is *regularly reducible* [28], i.e., there exists a shape-regular conforming mesh $\widetilde{\mathcal{T}_h}$ such that the closure of each element in $\mathcal{T}_h$ is a union of closures in $\widetilde{\mathcal{T}_h}$, and that there exists a constant $C > 0$, independent of mesh sizes, such that for any two elements $K \in \mathcal{T}_h$ and $\widetilde{K} \in \widetilde{\mathcal{T}_h}$ with $\widetilde{K} \subseteq K$, we have that $h_K/h_{\widetilde{K}} \le C$. Note that these assumptions imply that the family $\{\mathcal{T}_h\}_{h>0}$ is of *bounded local variation*, i.e., there exists a constant $\rho_1 \ge 1$, independent of element sizes, such that

$$\rho_1^{-1} \le h_K/h_{K'} \le \rho_1 \tag{3.1}$$

for any pair of elements $K, K' \in \mathcal{T}_h$ which share a common face $F = \partial K \cap \partial K'$. We store the element sizes in the vector $\mathbf{h} := \{h_K : K \in \mathcal{T}_h\}$.

For a non-negative integer k, we denote by $\mathcal{Q}_k(\hat{K})$ the set of all tensor-product polynomials on $\hat{K}$ of degree k in each coordinate direction. To each $K \in \mathcal{T}_h$, we assign a polynomial degree $k_K \ge 1$ (local approximation order) and store these in a vector $\mathbf{k} = \{k_K : K \in \mathcal{T}_h\}$. We suppose that $\mathbf{k}$ is also of bounded local variation, i.e., there exists a constant $\rho_2 \ge 1$, independent of the element sizes and $\mathbf{k}$, such that, for any pair of neighbouring elements $K, K' \in \mathcal{T}_h$,

$$\rho_2^{-1} \le k_K/k_{K'} \le \rho_2. \tag{3.2}$$

With this notation we introduce the finite element spaces

$$\mathbf{V}_h := \left\{\mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \circ T_K \in \mathcal{Q}_{k_K}(\hat{K})^d, K \in \mathcal{T}_h\right\},$$
$$Q_h := \left\{q \in \mathrm{L}^2_0(\Omega) : q|_K \circ T_K \in \mathcal{Q}_{k_K-1}(\hat{K}), K \in \mathcal{T}_h\right\}.$$

We define an interior face F of $\mathcal{T}_h$ as the intersection of two neighbouring elements $K, K' \in \mathcal{T}_h$, i.e., $F = \partial K \cap \partial K'$. Similarly, we define a boundary face $F \subset \Gamma$ as the entire face of an element K on the boundary. We denote by $\mathcal{F}_\mathcal{I}$ the set of all interior faces, $\mathcal{F}_\mathcal{B}$ the set of all boundary faces and $\mathcal{F} = \mathcal{F}_\mathcal{I} \cup \mathcal{F}_\mathcal{B}$ the set of all faces.

We shall now define suitable face operators that are required for the definition of the proceeding DGFEM. Let q, $\mathbf{v}$, and $\underline{\tau}$ be scalar-, vector- and matrix-valued functions, respectively, which are smooth inside each element $K \in \mathcal{T}_h$. Given two adjacent elements, $K^+, K^- \in \mathcal{T}_h$, which share a common face $F \in \mathcal{F}_\mathcal{I}$, i.e., $F = \partial K^+ \cap \partial K^-$, we write $q^\pm$, $\mathbf{v}^\pm$, and $\underline{\tau}^\pm$ to denote the traces of the functions q, $\mathbf{v}$, and $\underline{\tau}$, respectively, on the face F, taken from the interior of $K^\pm$, respectively. With this notation, the averages of q, $\mathbf{v}$, and $\underline{\tau}$ at $\mathbf{x} \in F$ are given by

$$\{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-), \qquad \{\!\{\mathbf{v}\}\!\} := \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \qquad \{\!\{\underline{\tau}\}\!\} := \frac{1}{2}(\underline{\tau}^+ + \underline{\tau}^-),$$

respectively. Similarly, the jumps of q, $\mathbf{v}$, and $\underline{\tau}$ at $\mathbf{x} \in F$ are given by

$$[\![q]\!] := q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}, \qquad [\![\mathbf{v}]\!] := \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-},$$
$$\underline{[\![\mathbf{v}]\!]} := \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \qquad [\![\underline{\tau}]\!] := \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\mathbf{n}_{K^-}.$$

On a boundary face $F \in \mathcal{F}_\mathcal{B}$, we set $\{\!\{q\}\!\} := q$, $\{\!\{\mathbf{v}\}\!\} := \mathbf{v}$, $\{\!\{\underline{\tau}\}\!\} := \underline{\tau}$, $[\![q]\!] := q\mathbf{n}$, $[\![\mathbf{v}]\!] := \mathbf{v} \cdot \mathbf{n}$, $\underline{[\![\mathbf{v}]\!]} := \mathbf{v} \otimes \mathbf{n}$ and $[\![\underline{\tau}]\!] := \underline{\tau}\mathbf{n}$, with $\mathbf{n}$ denoting the unit outward normal vector on the boundary Γ.

With this notation, we note the following elementary identities for any scalar–, vector–, and matrix–valued functions $q, \mathbf{v}$, and $\underline{\tau}$, respectively:

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n}_K \,\mathrm{d}s &= \sum_{F\in\mathcal{F}} \int_F [\![q]\!] \cdot \{\!\{\mathbf{v}\}\!\} \,\mathrm{d}s + \sum_{F\in\mathcal{F}_\mathcal{I}} \int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!] \,\mathrm{d}s,\\ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \tau : (\mathbf{v} \otimes \mathbf{n}_K) \,\mathrm{d}s &= \sum_{F\in\mathcal{F}} \int_F \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\} \,\mathrm{d}s + \sum_{F\in\mathcal{F}_\mathcal{I}} \int_F \{\!\{\mathbf{v}\}\!\} \cdot [\![\underline{\tau}]\!] \,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Here, $\mathbf{n}_K$ denotes the unit outward normal vector to the element $K \in \mathcal{T}_h$.

3.2. **DGFEM Discretization.** Given a partition $\mathcal{T}_h$ of Ω, together with the corresponding polynomial degree vector $\mathbf{k}$, the IP DGFEM formulation is defined as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}), \tag{3.4}$$
$$-B_h(\mathbf{u}_h, q) = 0 \tag{3.5}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, where

$$\begin{aligned}A_h(\mathbf{u}, \mathbf{v}) :=& \int_\Omega \mu\left(|\underline{e}_h(\mathbf{u})|\right) \underline{e}_h(\mathbf{u}) : \underline{e}_h(\mathbf{v}) \,\mathrm{d}\mathbf{x} - \sum_{F\in\mathcal{F}} \int_F \{\!\{\mu\left(|\underline{e}_h(\mathbf{u})|\right) \underline{e}_h(\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]} \,\mathrm{d}s\\ &+ \theta \sum_{F\in\mathcal{F}} \int_F \{\!\{\mu(h_F^{-1}|\underline{[\![\mathbf{u}]\!]}|)\underline{e}_h(\mathbf{v})\}\!\} : \underline{[\![\mathbf{u}]\!]} \,\mathrm{d}s + \sum_{F\in\mathcal{F}} \int_F \sigma\underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]} \,\mathrm{d}s,\\ B_h(\mathbf{v}, q) :=& -\int_\Omega q\nabla_h \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \sum_{F\in\mathcal{F}} \int_F \{\!\{q\}\!\}[\![\mathbf{v}]\!] \,\mathrm{d}s\end{aligned}$$

and

$$F_h(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}.$$

Here, $\underline{e}_h(\cdot)$ and ∇_h denote the element-wise strain tensor and gradient operator, respectively, and $\theta \in [-1, 1]$. The *interior penalty parameter* σ is defined as follows:

$$\sigma := \gamma \frac{k_F^2}{h_F}, \tag{3.6}$$

where $\gamma \geq 1$ is a constant, which must be chosen sufficiently large (independent of the local element sizes and the polynomial degree). For a face $F \in \mathcal{F}$, we define h_F as the diameter of the face and the face polynomial degree k_F as

$$k_F := \begin{cases} \max(k_K, k_{K'}), & \text{if } F = \partial K \cap \partial K' \in \mathcal{F}_\mathcal{I},\\ k_K, & \text{if } F = \partial K \cap \Gamma \in \mathcal{F}_\mathcal{B}.\end{cases}$$

Remark 3.1. We note that the formulation (3.4)–(3.5) corresponds to the symmetric interior penalty (SIP) method when $\theta = -1$, the non-symmetric interior penalty (NIP) method when $\theta = 1$ and the incomplete interior penalty method (IIP) when $\theta = 0$.

We introduce the energy norms $\|\cdot\|_{1,h}$ and $\|(\cdot,\cdot)\|_{\mathrm{DG}}$ by

$$\|\mathbf{v}\|_{1,h}^2 := \|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}} \int_F \sigma|\underline{[\![\mathbf{v}]\!]}|^2 \,\mathrm{d}s$$

and

$$\|(\mathbf{v},q)\|_{\mathrm{DG}}^2 := \|\mathbf{v}\|_{1,h}^2 + \|q\|_{0,\Omega}^2 . \tag{3.7}$$

Lemma 3.2. *There exists a constant $C_{\mathcal{K}} > 0$, independent of* $\mathbf{h}$ *and* $\mathbf{k}$*, such that*

$$\|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2 \le \|\nabla_h \mathbf{v}\|_{0,\Omega}^2 \le C_{\mathcal{K}} \left(\|\underline{e}_h(\mathbf{v})\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}} \int_F h_F^{-1} |\underline{[\![\mathbf{v}]\!]}|^2 \,\mathrm{d}s \right)$$

for all $\mathbf{v} \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$, where $\mathrm{H}^1(\Omega,\mathcal{T}_h) = \left\{ \mathbf{v} \in \mathrm{L}^2(\Omega)^d : \mathbf{v}|_K \in \mathrm{H}^1(K)^d, K \in \mathcal{T}_h \right\}$.

Proof. The proof of the first bound follows from elementary manipulations and application of the Cauchy–Schwarz inequality. Furthermore, the second estimate is a discrete Korn inequality for piecewise H^1 vector fields; see [8, Equation 1.19]. □

3.3. **Well-Posedness of the DGFEM Formulation.** In this section we will prove that the DGFEM formulation (3.4)–(3.5) admits a unique solution. To this end, let us assume that the bilinear form B_h satisfies the following discrete inf-sup condition:

$$\inf_{0\neq q\in Q_h} \sup_{\mathbf{0}\neq \mathbf{v}\in \mathbf{V}_h} \frac{B_h(\mathbf{v},q)}{\|\mathbf{v}\|_{1,h}\|q\|_{0,\Omega}} \ge c \left(\max_{K\in\mathcal{T}_h} k_K \right)^{-1} . \tag{3.8}$$

We note that this inf-sup condition holds

- for $k_K \ge 2$, $K \in \mathcal{T}_h$, or
- for $k \ge 1$ if $\mathcal{T}_h$ is conforming and $k_K = k$ for all $K \in \mathcal{T}_h$;

see Theorem 6.2 and Theorem 6.12, respectively, in [29].

Theorem 3.3. *Provided that the penalty parameter γ arising in* (3.6) *is chosen sufficiently large, there is exactly one solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of the hp-DGFEM* (3.4)–(3.5).

Proof. We set

$$\mathcal{A}_h((\mathbf{u},p);(\mathbf{v},q)) := A_h(\mathbf{u},\mathbf{v}) + B_h(\mathbf{v},p) - B_h(\mathbf{u},q),$$

which allows the DGFEM defined in (3.4)–(3.5) to be written in the following compact form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\mathcal{A}_h((\mathbf{u}_h,p_h);(\mathbf{v},q)) = F_h(\mathbf{v}) \tag{3.9}$$

for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$.

We will now check conditions (a)–(e) of Theorem 2.1 separately.

(a) The continuity of the linear form F_h follows from applying the Cauchy–Schwarz inequality together with Lemma 3.2.

(b) The linearity of $(\mathbf{v},q) \mapsto \mathcal{A}_h((\mathbf{u},p);(\mathbf{v},q))$, for fixed $(\mathbf{u},p) \in \mathbf{V}_h \times Q_h$, follows directly from the definition of $\mathcal{A}_h$. Furthermore, the continuity is shown by using the Cauchy–Schwarz inequality and invoking (2.5).

(c) The estimate (c) is established as in the proof of Proposition 2.2 (a) by applying (2.5), the Cauchy–Schwarz inequality, and the discrete Korn inequality from Lemma 3.2. In addition, the flux terms are treated similarly as in [16, Lemma 2.2]; see also [20, Lemma 4.1] or [33, Lemma 5.5].

(d) The inf-sup stability of $\mathcal{A}_h$ is proved along the lines of [33, Theorem 5.4], provided that $\gamma > 0$ is large enough, using the discrete inf-sup condition (3.8), where the nonlinear terms are estimated as in Proposition 2.2 (b) using (2.5)–(2.6). More precisely, in this way, it can be seen that, for any $(\mathbf{u},p), (\mathbf{w},r) \in \mathbf{V}_h \times Q_h$, there exists $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} \mathcal{A}_h((\mathbf{u},p);(\mathbf{v},q)) - \mathcal{A}_h((\mathbf{w},r);(\mathbf{v},q)) &\ge c \left(\max_{K\in\mathcal{T}_h} k_K \right)^{-2} \|(\mathbf{u}-\mathbf{w}, p-r)\|_{\mathrm{DG}}, \\ \|(\mathbf{v},q)\|_{\mathrm{DG}} &\le 1. \end{aligned} \tag{3.10}$$

(e) We proceed as in the proof of Proposition 2.2 (c), and consider $(\mathbf{0},0) \neq (\mathbf{v},q) \in \mathbf{V}_h \times Q_h$. Firstly, if $\mathbf{v} \neq \mathbf{0}$, then

$$\sup_{(\mathbf{u},p)\in\mathbf{V}_h\times Q_h} \mathcal{A}_h((\mathbf{u},p);(\mathbf{v},q)) \geq \mathcal{A}_h((\mathbf{v},q);(\mathbf{v},q)) = A_h(\mathbf{v},\mathbf{v}).$$

Now, referring to [16, Lemma 2.3] (with $w_1 = v, w_2 = 0$ and $\gamma > 0$ sufficiently large), there exists a constant $C > 0$ independent of the mesh size and the local polynomial degrees such that

$$A_h(\mathbf{v},\mathbf{v}) \geq C\|\mathbf{v}\|_{1,h}^2 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{3.11}$$

Hence,

$$\sup_{(\mathbf{u},p)\in\mathbf{V}_h\times Q_h} \mathcal{A}_h((\mathbf{u},p);(\mathbf{v},q)) > 0.$$

Secondly, if $\mathbf{v} = \mathbf{0}$, $q \neq 0$, then we apply the discrete inf-sup condition (3.8), similar as in the proof of Proposition 2.2 (c), to obtain that

$$\sup_{(\mathbf{u},p)\in\mathbf{V}_h\times Q_h} \mathcal{A}_h((\mathbf{u},p);(\mathbf{0},q)) > 0.$$

This completes the proof. □

4. *A Priori* Error Analysis

The goal of this section is to derive an *a priori* bound for the *hp*–DGFEM proposed in this paper. To this end, we state the following result.

Theorem 4.1. *Let the penalty parameter γ be sufficiently large, and the solution $(\mathbf{u},p)$ of (1.1)–(1.3) belong to $(\mathrm{C}^1(\Omega)\cap\mathrm{H}^2(\Omega))^d \times (\mathrm{C}^0(\Omega)\cap\mathrm{H}^1(\Omega))$, and $\mathbf{u}|_K \in \mathrm{H}^{s_K+1}(K)^d$, $p|_K \in \mathrm{H}^{s_K}(K)$, $s_K \geq 1$, $K \in \mathcal{T}_h$. Then, provided that the discrete inf-sup condition (3.8) is valid, the following estimate holds*

$$\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}}^2 \leq C \max_{K\in\mathcal{T}_h} k_K^4 \sum_{K\in\mathcal{T}_h} \left(\frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2 + \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2 \right),$$

where $(\mathbf{u}_h,p_h)$ is the DGFEM solution defined in (3.4)–(3.5), and the constant $C > 0$ is independent of the mesh size and the polynomial degrees.

Proof. Let us consider two interpolants $\Pi_{\mathbf{u}}$ and Π_p satisfying

$$\begin{aligned} \|\mathbf{u}-\Pi_{\mathbf{u}}\mathbf{u}\|_{1,h}^2 &\leq C \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K-1}} \|\mathbf{u}\|_{s_K+1,K}^2, \\ \sum_{K\in\mathcal{T}_h} \left(\|p-\Pi_p p\|_{0,K}^2 + h_K k_K^{-1} \|p-\Pi_p p\|_{0,\partial K}^2 \right) &\leq C \sum_{K\in\mathcal{T}_h} \frac{h_K^{2\min\{s_K,k_K\}}}{k_K^{2s_K}} \|p\|_{s_K,K}^2; \end{aligned} \tag{4.1}$$

see [16, Equation (3.2)], and [22], respectively. Thence, defining

$$\begin{aligned} \mathbf{u}-\mathbf{u}_h &= (\mathbf{u}-\Pi_{\mathbf{u}}\mathbf{u}) + (\Pi_{\mathbf{u}}\mathbf{u}-\mathbf{u}_h) =: \boldsymbol{\eta}_{\mathbf{u}} + \boldsymbol{\xi}_{\mathbf{u}}, \\ p-p_h &= (p-\Pi_p p) + (\Pi_p p - p_h) =: \eta_p + \xi_p, \end{aligned}$$

we have $(\boldsymbol{\xi}_{\mathbf{u}},\xi_p) \in \mathbf{V}_h \times Q_h$. Next, by the inf-sup stability (3.10) we find $(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p) \in \mathbf{V}_h \times Q_h$ with $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)\|_{\mathrm{DG}} \leq 1$ and

$$c\left(\max_{K\in\mathcal{T}_h} k_K\right)^{-2} \|(\boldsymbol{\xi}_{\mathbf{u}},\xi_p)\|_{\mathrm{DG}} \leq \mathcal{A}_h((\Pi_{\mathbf{u}}\mathbf{u},\Pi_p p);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)) - \mathcal{A}_h((\mathbf{u}_h,p_h);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)).$$

Then, thanks to our regularity assumptions, the DGFEM (3.4)–(3.5) is consistent, and thus,

$$\begin{aligned} c\left(\max_{K\in\mathcal{T}_h} k_K\right)^{-2} \|(\boldsymbol{\xi}_{\mathbf{u}},\xi_p)\|_{\mathrm{DG}} &\leq \mathcal{A}_h((\Pi_{\mathbf{u}}\mathbf{u},\Pi_p p);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)) - \mathcal{A}_h((\mathbf{u},p);(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\hat{\xi}_p)) \\ &\leq |A_h(\Pi_{\mathbf{u}}\mathbf{u},\hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u},\hat{\boldsymbol{\xi}}_{\mathbf{u}})| + |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}},\Pi_p p - p)| + |B_h(\Pi_{\mathbf{u}}\mathbf{u}-\mathbf{u},\hat{\xi}_p)| \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

For term T_1, we apply [16, Lemma 3.2] to obtain

$$T_1 = |A_h(\Pi_{\mathbf{u}}\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}}) - A_h(\mathbf{u}, \hat{\boldsymbol{\xi}}_{\mathbf{u}})| \leq C \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|^2_{s_K+1,K} \right)^{\frac{1}{2}} \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h}.$$

For term T_2, by applying the Cauchy–Schwarz inequality, we arrive at

$$T_2 = |B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leq \|\nabla \cdot \hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{0,\Omega} \|\Pi_p p - p\|_{0,\Omega} + \left(\sum_{F \in \mathcal{F}} \int_F \sigma^{-1} |\{\!\{\Pi_p p - p\}\!\}|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\hat{\boldsymbol{\xi}}_{\mathbf{u}})]\!]}|^2 \, \mathrm{d}s \right)^{\frac{1}{2}}.$$

By applying Korn's inequality and recalling (3.6) we have that

$$|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leq C \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \left(\|p - \Pi_p p\|^2_{0,K} + h_K k_K^{-2} \|p - \Pi_p p\|^2_{0,\partial K} \right) \right)^{\frac{1}{2}}.$$

Invoking (4.1) results in

$$|B_h(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \Pi_p p - p)| \leq C \|\hat{\boldsymbol{\xi}}_{\mathbf{u}}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K}} \|p\|^2_{s_K,K} \right)^{\frac{1}{2}}.$$

Similarly,

$$T_3 = |B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \leq C \|\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}\|_{1,h} \left(\sum_{K \in \mathcal{T}_h} \left(\|\hat{\xi}_p\|^2_{0,K} + h_K k_K^{-2} \|\hat{\xi}_p\|^2_{0,\partial K} \right) \right)^{\frac{1}{2}}.$$

By applying an inverse estimate to the boundary term, see, e.g. [31, Theorem 4.76], and scaling, and using (4.1), leads to

$$|B_h(\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}, \hat{\xi}_p)| \leq C \|\Pi_{\mathbf{u}}\mathbf{u} - \mathbf{u}\|_{1,h} \|\hat{\xi}_p\|_{0,\Omega} \leq C \|\hat{\xi}_p\|_{0,\Omega} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min\{s_K, k_K\}}}{k_K^{2s_K - 1}} \|\mathbf{u}\|^2_{s_K+1,K} \right)^{\frac{1}{2}}.$$

Finally, recalling that $\|(\hat{\boldsymbol{\xi}}_{\mathbf{u}}, \hat{\xi}_p)\|_{\mathrm{DG}} \leq 1$, noting that

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathrm{DG}} \leq \|(\boldsymbol{\eta}_{\mathbf{u}}, \eta_p)\|_{\mathrm{DG}} + \|(\boldsymbol{\xi}_{\mathbf{u}}, \xi_p)\|_{\mathrm{DG}},$$

and combining the bounds on T_1, T_2 and T_3 completes the proof. □

5. *A Posteriori* Error Analysis

In this section, we develop the *a posteriori* error analysis of the DGFEM defined by (3.4)–(3.5). We define, for an element $K \in \mathcal{T}_h$ and face $F \in \mathcal{F}_{\mathcal{I}}$, the data-oscillation terms

$$\mathcal{O}_K^{(1)} := h_K^2 k_K^{-2} \, \|(\mathbb{I} - \Pi_{\mathcal{T}_h})|_K (\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\})\|^2_{0,K}$$

and

$$\mathcal{O}_F^{(2)} := h_K k_K^{-1} \, \|(\mathbb{I} - \Pi_{\mathcal{F}})|_F ([\![\mu(|\underline{e}_h(\mathbf{u}_h))\underline{e}_h(\mathbf{u}_h)]\!])\|^2_{0,F}\,,$$

respectively, which depend on the right-hand side $\mathbf{f}$ in (1.1) and the numerical solution $\mathbf{u}_h$ from (3.4)–(3.5). Here, $\mathbb{I}$ represents a generic identity operator, $\Pi_{\mathcal{T}_h}$ is an element-wise L^2–projector onto the finite element space with polynomial degree vector $\{k_K - 1 : K \in \mathcal{T}_h\}$ and $\Pi_{\mathcal{F}}|_F$ is the L^2–projector onto $\mathcal{Q}_{k_F-1}(F)$.

5.1. **Upper Bounds.** We now state the following *a posteriori* upper bound for the DGFEM defined by (3.4)–(3.5).

Theorem 5.1. *Let* $(\mathbf{u}, p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ *be the analytical solution to the problem* (1.1)–(1.3) *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be its DGFEM approximation obtained from* (3.4)–(3.5)*. Then, the following hp–version* a posteriori *error bound holds:*

$$\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}} \leq C \left(\sum_{K\in\mathcal{T}_h} \eta_K^2 + \mathcal{O}(\mathbf{f}, \mathbf{u}_h) \right)^{\frac{1}{2}}, \tag{5.1}$$

where the local error indicators $\eta_K, K \in \mathcal{T}_h$*, are defined by*

$$\begin{aligned} \eta_K^2 := {} & h_K^2 k_K^{-2} \left\| \Pi_{\mathcal{T}_h}(\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u_h})|)\underline{e}(\mathbf{u}_h)\}) - \nabla p_h \right\|_{0,K}^2 + \|\nabla \cdot \mathbf{u}_h\|_{0,K}^2 \\ & + h_K k_K^{-1} \left\| [\![p_h]\!] - \Pi_{\mathcal{F}}([\![\mu(|\underline{e}_h(\mathbf{u_h})|)\underline{e}_h(\mathbf{u}_h)]\!]) \right\|_{0,\partial K\setminus\Gamma}^2 + \gamma^2 h_K^{-1} k_K^3 \left\| \underline{[\![\mathbf{u}_h]\!]} \right\|_{0,\partial K}^2 \end{aligned} \tag{5.2}$$

and

$$\mathcal{O}(\mathbf{f}, \mathbf{u}_h) := \sum_{K\in\mathcal{T}_h} \mathcal{O}_K^{(1)} + \sum_{F\in\mathcal{F}_{\mathcal{I}}} \mathcal{O}_F^{(2)}. \tag{5.3}$$

Here, the constant $C>0$ *is independent of* $\mathbf{h}$*, the polynomial degree vector* $\mathbf{k}$ *and the parameter* γ*, and only depends on the shape-regularity of the mesh and the constants* ρ_1 *and* ρ_2 *from* (3.1) *and* (3.2)*, respectively.*

The proof of this result will follow in Section 5.3.

Remark 5.2. We observe a slight suboptimality with respect to the polynomial degree in the last term of the local error indicator η_K in (5.2). This results from the use of a non-conforming interpolant in the proof of Theorem 5.1 to deal with the possible presence of hanging nodes in $\mathcal{T}_h$. For conforming meshes, i.e., meshes without hanging nodes, a conforming *hp*–version Clément interpolant, as constructed in [26], can be employed which results in an *a posteriori* error bound of the form (5.1) with the final term in the local error indicators (5.2) replaced by the improved expression

$$\gamma h_K^{-1} k_K^2 \left\| \underline{[\![\mathbf{u}_h]\!]} \right\|_{0,\partial K}^2,$$

cf. [19].

5.2. **Local Lower Bounds.** For simplicity we shall restrict ourselves to local lower bounds on conforming meshes $\mathcal{T}_h$; the extension to non-conforming 1-irregular regularly reducible meshes follows analogously, cf., for example, [24, Remark 3.9]. The following result can be proved along the lines of the analyses contained in [17, 24]; for details, see [13].

Theorem 5.3. *Let* K *and* K' *be any two neighbouring elements in* $\mathcal{T}_h$*,* $F = \partial K \cap \partial K'$ *and* $\omega_F = (\overline{K} \cup \overline{K}')^\circ$*. Then, for all* $\delta \in (0, \frac{1}{2}]$*, the following hp–version* a posteriori *local bounds on the error between the analytical solution* $(\mathbf{u}, p) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2_0(\Omega)$ *satisfying* (1.1)–(1.3) *and the numerical solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *obtained by* (3.4)–(3.5) *hold:*

(a)

$$\begin{aligned} & \|\Pi_{\mathcal{T}_h}|_K(\mathbf{f} + \nabla \cdot \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\}) - \nabla p\|_{0,K} \\ & \qquad \leq C h_K^{-1} k_K^2 \left(\|\underline{e}(\mathbf{u}-\mathbf{u}_h)\|_{0,K} + \|p-p_h\|_{0,K} + k_K^{\delta-\frac{1}{2}} \sqrt{\mathcal{O}_K^{(1)}} \right), \end{aligned}$$

(b)

$$\|\nabla \cdot \mathbf{u}_h\|_{0,K} \leq C \|\underline{e}(\mathbf{u}-\mathbf{u}_h)\|_{0,K},$$

(c)

$$\begin{aligned} & \|[\![p_h]\!] - \Pi_{\mathcal{F}}|_F([\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!])\|_{0,F} \\ & \quad \leq C h_K^{-\frac{1}{2}} k_K^{\delta+\frac{3}{2}} \left(\|\underline{e}(\mathbf{u}-\mathbf{u}_h)\|_{0,\omega_F} + \|p-p_h\|_{0,\omega_F} + k_K^{\delta-\frac{1}{2}} \sum_{\tau\in\{K,K'\}} \sqrt{\mathcal{O}_\tau^{(1)}} + k_K^{-\frac{1}{2}} \sqrt{\mathcal{O}_F^{(2)}} \right), \end{aligned}$$

(d)

$$\left\| \underline{[\![\mathbf{u}_h]\!]} \right\|_{0,F} \leq C\gamma^{-\frac{1}{2}} h_K^{\frac{1}{2}} k_K^{-1} \left\| \sigma^{\frac{1}{2}} \underline{[\![\mathbf{u} - \mathbf{u}_h]\!]} \right\|_{0,F}.$$

Here, the generic constant $C > 0$ depends on δ, but is independent of $\mathbf{h}$ *and* $\mathbf{k}$.

5.3. **Proof of Theorem 5.1.** The proof of Theorem 5.1 is based on the techniques developed in [17, 24], cf. also [25].

5.3.1. *DGFEM Decomposition.* In order to admit 1-irregular meshes, we consider a subdivision $\mathcal{T}_h$ which is regularly reducible, i.e., $\mathcal{T}_h$ may be refined to create a conforming mesh $\widetilde{\mathcal{T}_h}$ as outlined in Section 3.1, cf. [28, 24]. We denote by $\widetilde{\mathbf{V}}_h$ and $\widetilde{Q}_h$ the corresponding DGFEM finite element spaces with polynomial degree vector $\widetilde{\mathbf{k}}$ defined by $\widetilde{k}_{\widetilde{K}} := k_K$ for any $\widetilde{K} \in \widetilde{\mathcal{T}_h}$ with $\widetilde{K} \subseteq K$, for some $K \in \mathcal{T}_h$. We note that $\mathbf{V}_h \subseteq \widetilde{\mathbf{V}}_h$, $Q_h \subseteq \widetilde{Q}_h$ and thanks to the assumptions in Section 3.1, the energy norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{\widetilde{1,h}}$ corresponding to the spaces $\mathbf{V}_h$ and $\widetilde{\mathbf{V}}_h$, respectively, are equivalent on $\mathbf{V}_h$; in particular, there exist constants $N_1, N_2 > 0$, independent of $\mathbf{h}$ and $\mathbf{k}$, such that

$$N_1 \sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s \leq \sum_{\widetilde{F} \in \widetilde{\mathcal{F}}} \int_{\widetilde{F}} \widetilde{\sigma} |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s \leq N_2 \sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}]\!]}|^2 \,\mathrm{d}s, \tag{5.4}$$

cf. [28, 24]. Here, $\widetilde{\mathcal{F}}$ denotes the set of all faces in the mesh $\widetilde{\mathcal{T}_h}$ and $\widetilde{\sigma}$ is the discontinuous penalization parameter on $\widetilde{\mathbf{V}}_h$ which is defined analogously to σ on $\mathbf{V}_h$.

An important step in the proof is to decompose the DGFEM space $\widetilde{\mathbf{V}}_h$ into two orthogonal subspaces: a conforming part $\widetilde{\mathbf{V}}_h^c = \widetilde{\mathbf{V}}_h \cap \mathrm{H}_0^1(\Omega)^d$ and a non-conforming part $\widetilde{\mathbf{V}}_h^{\perp}$, which is defined as the orthogonal complement of $\widetilde{\mathbf{V}}_h^c$ with respect to the energy inner product $(\cdot,\cdot)_{\widetilde{1,h}}$ (inducing the norm $\|\cdot\|_{\widetilde{1,h}}$), i.e.,

$$\widetilde{\mathbf{V}}_h = \widetilde{\mathbf{V}}_h^c \oplus_{\|\cdot\|_{\widetilde{1,h}}} \widetilde{\mathbf{V}}_h^{\perp}.$$

Based on this setting the DGFEM solution $\mathbf{u}_h$ may be split accordingly,

$$\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^{\perp}, \tag{5.5}$$

where $\mathbf{u}_h^c \in \widetilde{\mathbf{V}}_h^c$ and $\mathbf{u}_h^{\perp} \in \widetilde{\mathbf{V}}_h^{\perp}$. Furthermore, we define the error in the velocity vector as

$$\mathbf{e}_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h, \tag{5.6}$$

and error in the pressure as

$$e_p := p - p_h, \tag{5.7}$$

and let

$$\mathbf{e}_{\mathbf{u}}^c := \mathbf{u} - \mathbf{u}_h^c \in \mathrm{H}_0^1(\Omega)^d. \tag{5.8}$$

5.3.2. *Auxiliary Results.* In order to prove Theorem 5.1, we require the following auxiliary results.

Proposition 5.4. *Under the foregoing assumptions on the subdivision $\widetilde{\mathcal{T}_h}$, the following bound holds over the space $\widetilde{\mathbf{V}}_h^{\perp}$:*

$$\widetilde{C}\, \|\mathbf{v}\|_{\widetilde{1,h}}^2 \leq \sum_{\widetilde{F} \in \widetilde{\mathcal{F}}} \int_{\widetilde{F}} \widetilde{\sigma} |\underline{[\![\mathbf{v}]\!]}|^2 \,\mathrm{d}s \qquad \forall \mathbf{v} \in \widetilde{\mathbf{V}}_h^{\perp},$$

where the constant $\widetilde{C} > 0$ depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) *and* (3.2)*, respectively.*

Proof. The proof follows, for the case when $d = 2$, by first applying Lemma 3.2 and then extending [21, Proposition 4.1] and [24, Proposition 3.5] to vector–valued functions. The case when $d = 3$ can be similarly derived from [34, Theorem 4.1]. □

Corollary 5.5. *With $\mathbf{u}_h^{\perp}$ defined by (5.5), the following bound holds*

$$\left\|\mathbf{u}_h^{\perp}\right\|_{\widetilde{1,h}} \leq D \left(\sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h]\!]}|^2 \, \mathrm{d}s \right)^{\frac{1}{2}},$$

where the constant $D > 0$ is independent of γ, $\mathbf{h}$ and $\mathbf{k}$, and depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

Proof. Due to the fact that Proposition 5.4 holds we can simply extend the proof from [24, Corollary 3.6]. □

We now state the following approximation result:

Lemma 5.6. *For any $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$ there exists $\mathbf{v}_h \in \mathbf{V}_h$ such that*

$$\sum_{K \in \mathcal{T}_h} \left(\frac{k_K^2}{h_K} \left\|\mathbf{v} - \mathbf{v}_h\right\|_{0,K}^2 + \left\|\underline{e}(\mathbf{v} - \mathbf{v}_h)\right\|_{0,K}^2 + \frac{k_K}{h_K} \left\|\mathbf{v} - \mathbf{v}_h\right\|_{0,\partial K}^2 \right) \leq C_I \left\|\underline{e}_h(\mathbf{v})\right\|_{0,\Omega}^2,$$

with an interpolation constant $C_I > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$, which depends only on the shape regularity of the mesh and the constants ρ_1 and ρ_2 from (3.1) and (3.2), respectively.

Proof. This follows from applying [24, Lemma 3.7] componentwise to the vector field $\boldsymbol{v}$. □

5.3.3. *Proof of Theorem 5.1.* We now complete the proof of Theorem 5.1. To this end we recall the compact formulation (3.9) as well as the definition of the error, defined in (5.6) and (5.7). Then, by (5.4), Corollary 5.5 and the fact that $\gamma \geq 1$ and $k_K \geq 1$, we have that

$$\begin{aligned}
\left\|(\mathbf{e}_{\mathbf{u}}, e_p)\right\|_{\mathrm{DG}} &\leq \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} + \left\|\mathbf{u}_h^{\perp}\right\|_{1,h} \\
&= \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} + \left(\sum_{\widetilde{K} \in \widetilde{\mathcal{T}_h}} \left\|\underline{e}(\mathbf{u}_h^{\perp})\right\|_{0,\widetilde{K}}^2 + \sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h^{\perp}]\!]}|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} + \max(1, N_1^{-\frac{1}{2}}) \left\|\mathbf{u}_h^{\perp}\right\|_{\widetilde{1,h}} \\
&\leq \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} + \max(1, N_1^{-\frac{1}{2}}) D \left(\sum_{F \in \mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h]\!]}|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} + \max(1, N_1^{-\frac{1}{2}}) D \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.9}$$

To bound the term $\left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}}$, we invoke the result from Proposition 2.2 (b) which gives a function $(\mathbf{v}, q) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}_0^2(\Omega)$ such that

$$c \left\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\right\|_{\mathrm{DG}} \leq \mathcal{A}_h(\mathbf{u}, p, \mathbf{v}, q) - \mathcal{A}_h(\mathbf{u}_h^c, p_h, \mathbf{v}, q), \qquad \left\|(\mathbf{v}, q)\right\|_{\mathrm{DG}} \leq 1. \tag{5.10}$$

Notice here that, since $\boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d$, we have that $\underline{[\![\boldsymbol{v}]\!]} = \underline{0}$ on $\mathcal{F}$. Therefore, from (5.5), we deduce that

$$
\begin{aligned}
c\,\|(\mathbf{e}_{\mathbf{u}}^c, e_p)\|_{\mathrm{DG}} &\le \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h^c)|)\underline{e}(\mathbf{u}_h^c)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} (p-p_h)\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} q\nabla\cdot(\mathbf{u}-\mathbf{u}_h^c)\,\mathrm{d}\mathbf{x} \\
&= \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) - \mu(|\underline{e}(\mathbf{u}_h^c)|)\underline{e}(\mathbf{u}_h^c)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} (p-p_h)\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} q\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} q\nabla\cdot\mathbf{u}_h^\perp\,\mathrm{d}\mathbf{x} \\
&\equiv T_1 + T_2,
\end{aligned}
\tag{5.11}
$$

where

$$
\begin{aligned}
T_1 &= \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} \{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u}) - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} (p-p_h)\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} q\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\mathbf{x}, \\
T_2 &= \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} \{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) - \mu(|\underline{e}(\mathbf{u}_h^c)|)\underline{e}(\mathbf{u}_h^c)\} : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} q\nabla\cdot\mathbf{u}_h^\perp\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We start by bounding T_1. To this end, employing integration by parts and equations (1.1) and (1.2), we get

$$
\begin{aligned}
T_1 &= \sum_{K\in\mathcal{T}_h} \int_K (-\nabla\cdot\{\mu(|\underline{e}(\mathbf{u})|)\underline{e}(\mathbf{u})\} + \nabla p)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K p_h\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{K\in\mathcal{T}_h} \int_K q\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\mathbf{x} \\
&= \sum_{K\in\mathcal{T}_h} \int_K \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v})\,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K p_h\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We let $\mathbf{v}_h \in \mathbf{V}_h$ be the elementwise interpolant of $\mathbf{v}$, which satisfies Lemma 5.6. Then, noting from (3.9) that $\mathcal{A}_h(\mathbf{u}_h, p_h, \mathbf{v}_h, 0) - F_h(\mathbf{v}_h) = 0$, for all $\mathbf{v}_h \in \mathbf{V}_h$, gives

$$\begin{aligned}
T_1 = &\sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h}\int_K \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : \underline{e}(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&- \sum_{F\in\mathcal{F}}\int_F \Big(\{\!\{\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)\}\!\} : \underline{[\![\mathbf{v}_h]\!]} - \theta\{\!\{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\}\!\} : \underline{[\![\mathbf{u}_h]\!]}\Big)\,\mathrm{d}s \\
&+ \sum_{K\in\mathcal{T}_h}\int_K p_h\nabla\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathcal{F}}\int_F \{\!\{p_h\}\!\}[\![\mathbf{v}_h]\!]\,\mathrm{d}s \\
&- \sum_{K\in\mathcal{T}_h}\int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathcal{F}}\int_F \sigma\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
T_1 = &\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f}+\nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&+ \sum_{K\in\mathcal{T}_h}\int_{\partial K} (p_h(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}_K - \mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) : (\mathbf{v}-\mathbf{v}_h)\otimes\mathbf{n}_K)\,\mathrm{d}s \\
&- \sum_{F\in\mathcal{F}}\int_F \Big(\{\!\{\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)\}\!\} : \underline{[\![\mathbf{v}_h]\!]} - \theta\{\!\{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\}\!\} : \underline{[\![\mathbf{u}_h]\!]}\Big)\,\mathrm{d}s \\
&+ \sum_{F\in\mathcal{F}}\int_F \{\!\{p_h\}\!\}[\![\mathbf{v}_h]\!]\,\mathrm{d}s - \sum_{K\in\mathcal{T}_h}\int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} + \sum_{F\in\mathcal{F}}\int_F \sigma\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s.
\end{aligned}$$

Since $\mathbf{v} \in \mathrm{H}_0^1(\Omega)^d$, we have that $\underline{[\![\mathbf{v}]\!]} = \underline{0}$, which implies that $|\underline{[\![\mathbf{v}_h]\!]}| = |\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}|$ on $\mathcal{F}$. Thereby, using this result, together with the application of (3.3), gives

$$\begin{aligned}
T_1 = &\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f}+\nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \\
&+ \sum_{F\in\mathcal{F}_\mathcal{I}}\int_F ([\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!])\cdot\{\!\{\mathbf{v}-\mathbf{v}_h\}\!\}\,\mathrm{d}s - \sum_{K\in\mathcal{T}_h}\int_K q\nabla\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} \\
&+ \theta\sum_{F\in\mathcal{F}}\int_F \{\!\{\mu(h^{-1}|\underline{[\![\mathbf{u}_h]\!]}|)\underline{e}_h(\mathbf{v}_h)\}\!\} : \underline{[\![\mathbf{u}_h]\!]}\,\mathrm{d}s + \sum_{F\in\mathcal{F}}\int_F \sigma\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h-\mathbf{v}]\!]}\,\mathrm{d}s \\
\le &\sum_{K\in\mathcal{T}_h} \|\mathbf{f}+\nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\|_{0,K}\,\|\mathbf{v}-\mathbf{v}_h\|_{0,K} \\
&+ \sum_{K\in\mathcal{T}_h} \|q\|_{0,K}\,\|\nabla\cdot\mathbf{u}_h\|_{0,K} \\
&+ C\sum_{K\in\mathcal{T}_h} \|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\Gamma}\,\|\mathbf{v}-\mathbf{v}_h\|_{0,\partial K\setminus\Gamma} \\
&+ M_\mu|\theta|\left(\sum_{F\in\mathcal{F}}\int_F h_F^{-1}k_F^2|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{\frac12}\left(\sum_{F\in\mathcal{F}}\int_F h_F k_F^{-2}|\{\!\{|\underline{e}_h(\mathbf{v}_h)|\}\!\}|^2\,\mathrm{d}s\right)^{\frac12} \\
&+ \left(\sum_{F\in\mathcal{F}}\int_F \sigma k_F|\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s\right)^{\frac12}\left(\sum_{F\in\mathcal{F}}\int_F \sigma k_F^{-1}|\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}|^2\,\mathrm{d}s\right)^{\frac12}.
\end{aligned}$$

Exploiting the trace inequalities in [31, Theorem 4.76] and [29, Lemma 7.1], and noting that $k_F \geq 1$, we get

$$\begin{aligned}
T_1 \leq & \sum_{K\in\mathcal{T}_h} h_K k_K^{-1} \left\|\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\right\|_{0,K} h_K^{-1} k_K \left\|\mathbf{v}-\mathbf{v}_h\right\|_{0,K} \\
& + \sum_{K\in\mathcal{T}_h} \|q\|_{0,K} \|\nabla\cdot\mathbf{u}_h\|_{0,K} \\
& + C \sum_{K\in\mathcal{T}_h} h_K^{1/2} k_K^{-1/2} \left\|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\right\|_{0,\partial K\setminus\Gamma} h_K^{-1/2} k_K^{1/2} \left\|\mathbf{v}-\mathbf{v}_h\right\|_{0,\partial K\setminus\Gamma} \\
& + C|\theta| \left(\sum_{F\in\mathcal{F}} \int_F h_F^{-1} k_F^2 |\underline{[\![\mathbf{u}_h]\!]}|^2 \,\mathrm{d}s\right)^{\frac12} \left(\sum_{K\in\mathcal{T}_h} \|\underline{e}(\mathbf{v}_h)\|_{0,K}^2\right)^{\frac12} \\
& + C\gamma^{1/2} \left(\sum_{F\in\mathcal{F}} \int_F \sigma k_F |\underline{[\![\mathbf{u}_h]\!]}|^2 \,\mathrm{d}s\right)^{\frac12} \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} k_K \left\|\mathbf{v}-\mathbf{v}_h\right\|_{0,\partial K}^2\right)^{\frac12} \\
\leq & C \Bigg\{ \sum_{K\in\mathcal{T}_h} \Big(h_K^2 k_K^{-2} \left\|\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\right\|_{0,K}^2 + \|\nabla\cdot\mathbf{u}_h\|_{0,K}^2 \\
& \quad + h_K k_K^{-1} \left\|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u}_h)|)\underline{e}_h(\mathbf{u}_h)]\!]\right\|_{0,\partial K\setminus\Gamma}^2 \Big) + \gamma \sum_{F\in\mathcal{F}} \int_F \sigma k_F |\underline{[\![\mathbf{u_h}]\!]}|^2 \,\mathrm{d}s \Bigg\}^{\frac12} \\
& \times \left\{ \sum_{K\in\mathcal{T}_h} \left(h_K^{-2} k_K^2 \left\|\mathbf{v}-\mathbf{v}_h\right\|_{0,K}^2 + h_K^{-1} k_K \left\|\mathbf{v}-\mathbf{v}_h\right\|_{0,\partial K}^2 + |\theta| \, \|\underline{e}(\mathbf{v}_h)\|_{0,K}^2 + \|q\|_{0,K}^2 \right) \right\}^{\frac12} .
\end{aligned}$$

For $K \in \mathcal{T}_h$, we write

$$\begin{aligned}
\widetilde{\eta}_K^2 = & h_K^2 k_K^{-2} \left\|\mathbf{f} + \nabla\cdot\{\mu(|\underline{e}(\mathbf{u_h})|)\underline{e}(\mathbf{u}_h)\} - \nabla p_h\right\|_{0,K}^2 + \|\nabla\cdot\mathbf{u}_h\|_{0,K}^2 \\
& + h_K k_K^{-1} \left\|[\![p_h]\!] - [\![\mu(|\underline{e}_h(\mathbf{u_h})|)\underline{e}_h(\mathbf{u}_h)]\!]\right\|_{0,\partial K\setminus\Gamma}^2 + \gamma^2 h_K^{-1} k_K^3 \left\|\underline{[\![\mathbf{u}_h]\!]}\right\|_{0,\partial K}^2 .
\end{aligned}$$

Then, noting that $\gamma \geq 1 \geq |\theta| \geq 0$, $\|\underline{e}(\mathbf{v}_h)\|_{0,K}^2 \leq \|\underline{e}(\mathbf{v}-\mathbf{v}_h)\|_{0,K}^2 + \|\underline{e}(\mathbf{v})\|_{0,K}^2$, applying Lemma 5.6 and (5.10) gives

$$\begin{aligned}
T_1 & \leq C \left(\sum_{K\in\mathcal{T}_h} \widetilde{\eta}_K^2\right)^{\frac12} \left(\sum_{K\in\mathcal{T}_h} \left\{ \|\underline{e}(\mathbf{v})\|_{0,K}^2 + \|q\|_{0,K}^2 \right\}\right)^{\frac12} \\
& \leq C \left(\sum_{K\in\mathcal{T}_h} \widetilde{\eta}_K^2\right)^{\frac12} \|(\mathbf{v},q)\|_{\mathrm{DG}} \leq C \left(\sum_{K\in\mathcal{T}_h} \widetilde{\eta}_K^2\right)^{\frac12} .
\end{aligned}$$

By application of the triangle inequality we deduce the following bound for T_1:

$$T_1 \leq C \left(\sum_{K\in\mathcal{T}_h} \eta_K^2 + \mathcal{O}(\mathbf{f},\mathbf{u}_h)\right)^{\frac12} . \tag{5.12}$$

We now consider the T_2 term. By using the bound (2.5) and the trace inequality, we get that

$$
\begin{aligned}
T_2 &\le \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} |\mu(|\underline{e}(\mathbf{u}_h)|)\underline{e}(\mathbf{u}_h) - \mu(|\underline{e}(\mathbf{u}_h^c)|)\underline{e}(\mathbf{u}_h^c)||\underline{e}(\mathbf{v})|\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} |q||\nabla\cdot\mathbf{u}_h^{\perp}|\,\mathrm{d}\mathbf{x} \\
&\le C_1 \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} |\underline{e}(\mathbf{u}_h^{\perp})||\underline{e}(\mathbf{v})|\,\mathrm{d}\mathbf{x} + \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \int_{\widetilde{K}} |q||\nabla\cdot\mathbf{u}_h^{\perp}|\,\mathrm{d}\mathbf{x} \\
&\le C_1 \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \left(\left\|\underline{e}(\mathbf{u}_h^{\perp})\right\|_{0,\widetilde{K}} \left\|\underline{e}(\mathbf{v})\right\|_{0,\widetilde{K}} + \|q\|_{0,\widetilde{K}} \left\|\nabla\cdot\mathbf{u}_h^{\perp}\right\|_{0,\widetilde{K}} \right) \\
&\le C \left\{ \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \left(\left\|\underline{e}(\mathbf{u}_h^{\perp})\right\|_{0,\widetilde{K}}^2 + \left\|\nabla\cdot\mathbf{u}_h^{\perp}\right\|_{0,\widetilde{K}}^2 \right) \right\}^{\frac{1}{2}} \left\{ \sum_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \left(\|\underline{e}(\mathbf{v})\|_{0,\widetilde{K}}^2 + \|q\|_{0,\widetilde{K}}^2 \right) \right\}^{\frac{1}{2}}.
\end{aligned}
$$

We note that because of Lemma 3.2 we have that

$$
\sum_{\tilde{K}\in\widetilde{\mathcal{T}_h}} \left\|\nabla\cdot\mathbf{u}_h^{\perp}\right\|_{0,\widetilde{K}}^2 \le d \sum_{\tilde{K}\in\widetilde{\mathcal{T}_h}} \left\|\nabla\mathbf{u}_h^{\perp}\right\|_{0,\widetilde{K}}^2 \le dC_{\mathcal{K}} \left\|\mathbf{u}_h^{\perp}\right\|_{\widetilde{1,h}}^2.
$$

Therefore, applying Corollary 5.5, gives

$$
T_2 \le C\left((1+dC_{\mathcal{K}}) \left\|\mathbf{u}_h^{\perp}\right\|_{\widetilde{1,h}}^2 \right)^{\frac{1}{2}} \|(\mathbf{v},q)\|_{\mathrm{DG}} \le C \left(\sum_{F\in\mathcal{F}} \int_F \sigma |\underline{[\![\mathbf{u}_h]\!]}|^2\,\mathrm{d}s \right)^{\frac{1}{2}} \|(\mathbf{v},q)\|_{\mathrm{DG}}.
$$

Recalling (5.10), we deduce that

$$
T_2 \le C \left(\sum_{K\in\mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}}. \tag{5.13}
$$

Substituting (5.11), (5.12) and (5.13) into (5.9) completes the proof.

6. Numerical Experiments

In this section we present a series of numerical experiments to numerically verify the *a priori* error estimate derived in Section 4, as well as to demonstrate the performance of the *a posteriori* error bound derived in Theorem 5.1 within an automatic hp–adaptive refinement procedure based on 1-irregular quadrilateral elements for $\Omega \subset \mathbb{R}^2$. Throughout this section the DGFEM solution (u_h, p_h) defined by (3.4)–(3.5) is computed with $\theta = -1$, i.e., we employ the SIP DGFEM. Additionally, we set the constant γ appearing in the interior penalty parameter σ defined by (3.6) equal to 10. The resulting system of nonlinear equations is solved based on employing a damped Newton method; for each inner (linear) iteration, we employ the Multifrontal Massively Parallel Solver (MUMPS), see [1, 2, 3].

The hp–adaptive meshes are constructed by first marking the elements for refinement/derefinement according to the size of the local error indicators η_K; this is achieved via a fixed fraction strategy where the refinement and derefinement fractions are set to 25% and 5%, respectively. We employ the hp–adaptive strategy developed by [23] to decide whether h– or p–refinement/derefinement should be performed on an element $K \in \mathcal{T}_h$ marked for refinement/derefinement. We note here that we start with a polynomial degree of $k_K = 3$ for all $K \in \mathcal{T}_h$.

The purpose of these experiments is to demonstrate that the *a posteriori* error indicator in Theorem 5.1 converges to zero at the same asymptotic rate as the actual error in the DGFEM energy norm $\|(\cdot,\cdot)\|_{\mathrm{DG}}$, on a sequence of non-uniform hp–adaptively refined meshes. We also demonstrate that the hp–adaptive strategy converges at a higher rate than an h–adaptive refinement strategy, which uses the same 25% and 5% refinement/derefinement fixed fraction strategy, but only undertakes mesh subdivision for a fixed (uniform) polynomial degree distribution. As in [5, 24] we set the constant C arising in Theorem 5.1 equal to one for simplicity; in general this constant must be determined numerically from the underlying problem to ensure the reliability of the error estimator, cf. [14]. We are then able to check that the effectivity indices, defined as the

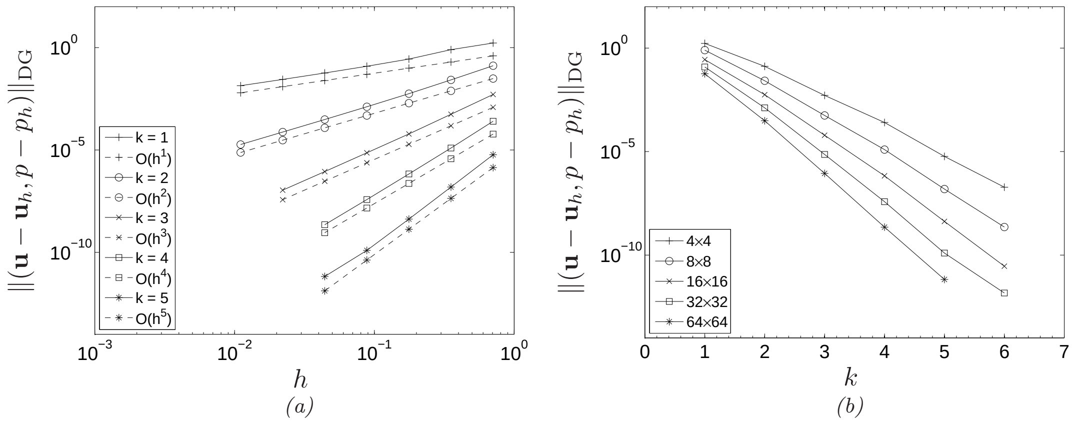


Figure 1. Example 1. Convergence of the DGFEM with (a) h–refinement; (b) p–refinement.

ratio of the *a posteriori* error bound and the DGFEM energy norm of the true error, is roughly constant. We also ignore in all our experiments the data-oscillation terms arising in Theorem 5.1.

6.1. **Example 1: Smooth solution.** In this first example, we let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and consider the nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = 2 + \frac{1}{1+|e(\mathbf{u})|^2}.$$

In addition, we select $\mathbf{f}$ so that the analytical solution to (1.1)–(1.3) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} -\mathrm{e}^x(y\cos(y)+\sin(y)) \\ \mathrm{e}^x y \sin(y) \end{pmatrix},$$
$$p(x,y) = 2\mathrm{e}^x \sin(y) - (2(1-e)(\cos(1)-1))/3.$$

Here, we investigate the convergence of the DGFEM (3.4)–(3.5) on a sequence of hierarchically and uniformly refined square meshes for different (fixed) values of the polynomial degree k. To this end, in Figure 1(a) we present a comparison of the DGFEM energy norm $\|(\cdot,\cdot)\|_{\mathrm{DG}}$ with the mesh function h for k ranging between 1 and 5. Here, we clearly see that $\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|_{\mathrm{DG}}$ converges like $\mathcal{O}(h^k)$ as h tends to zero for each (fixed) k, which is in complete agreement with Theorem 4.1. Secondly, we investigate the convergence of the DGFEM with p–enrichment for fixed h. Since the analytical solution to this problem is a real analytic function, we expect to observe exponential rates of convergence. Indeed, Figure 1(b) clearly illustrates this behaviour: on the linear–log scale, the convergence plots for each mesh become straight lines as the degree of the approximating polynomial is increased.

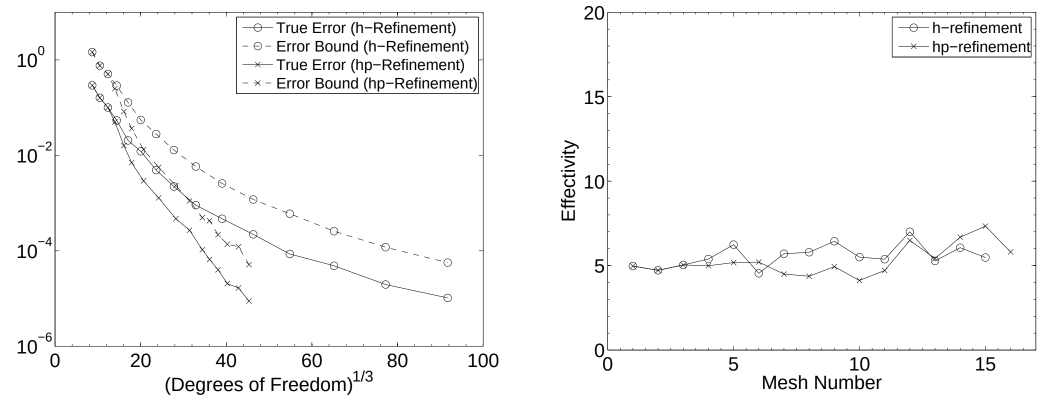
6.2. **Example 2: Cavity problem.** In this example we consider the cavity-like problem from [6, Section 6.1] using the Carreau law nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = k_\infty + (k_0 - k_\infty)(1+\lambda|\underline{e}(\mathbf{u})|^2)^{(\theta-2)/2},$$

with $k_\infty = 1, k_0 = 2, \lambda = 1$ and $\theta = 1.2$. We let Ω be the unit square $(0,1)^2 \subset \mathbb{R}^2$ and select the forcing function $\mathbf{f}$ so that the analytical solution to (1.1)–(1.3) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} \left(1-\cos\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^\theta-1}\right)\right)\sin(2\pi y) \\ -\theta\mathrm{e}^{\theta x}\sin\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^\theta-1}\right)\frac{1-\cos(2\pi y)}{\mathrm{e}^\theta-1} \end{pmatrix},$$

$$p(x,y) = 2\pi\theta\mathrm{e}^{\theta x}\sin\left(2\frac{\pi\left(\mathrm{e}^{\theta x}-1\right)}{\mathrm{e}^\theta-1}\right)\frac{\sin(2\pi y)}{\mathrm{e}^\theta-1}.$$



(a) *(b)*

Figure 2. Example 2. (a) Comparison of the error in the DGFEM norm employing both h– and hp–refinement, with respect to the number of degrees of freedom; (b) Effectivity index using both h– and hp–refinement.

In this example, we now turn our attention to the performance of the proposed hp–adaptive refinement algorithm. To this end, in Figure 2(a) we present a comparison of the actual error measured in the DGFEM norm and the *a posteriori* error bound versus the third root of the number of degrees of freedom on a linear-log scale for the sequence of meshes generated by both the h– and hp–adaptive algorithm; in each case the initial value of the polynomial degree k is set equal to 3. We remark that the choice of the third root of the number of degrees of freedom is based on the *a priori* analysis performed in [30] for the linear Stokes problem, cf. [18]. We observe that the error bound over-estimates the true error by roughly a consistent factor; this is confirmed in Figure 2(b), where the effectivity indices for the sequence of meshes which, although slightly oscillatory, all lie in roughly the range 4–7. From Figure 2(a) we can also see that the DGFEM norm of the error converges to zero at an exponential rate when hp–adaptivity is employed. Consequently, we observe the superiority of the grid adaptation algorithm based on employing hp–refinement in comparison to a standard h–version method; on the final mesh the DGFEM norm of the discretization error is over an order of magnitude smaller when the former algorithm is employed, in comparison to the latter, for a fixed number of degrees of freedom.

In Figures 3(a) and (b) we show the meshes generated after 10 mesh refinements using the h– and hp–adaptive mesh refinement strategies, respectively. Figure 3(c) displays the analytical solution to this example for comparison to the meshes; as noted in [6] the flow exhibits a counter-clockwise vortex around the point $((1/\theta)\log((\mathrm{e}^{\theta}+1)/2), 1/2)$, though the analytical solution is relatively smooth. We can see that the h–adaptive refinement strategy performs nearly uniform h–refinement as we would expect for such a smooth analytical solution, with more refinement around the vortex centre and the hill and valley on the right side of the vortex. With the hp–refinement strategy, we note that mostly p–refinement has occurred, which is as expected for a smooth analytical solution, with the main p–refinement occurring around the vortex centre and more h–refinement occurring around the centre of the hills and valleys in the pressure function; further h–refinement has also occurred in the 'tighter' hill and valley on the right caused by the off-centre vortex.

6.3. **Example 3: Singular solution.** For this example we consider a nonlinear version of the singular solution from [32, p. 113], see also [17], using the nonlinearity

$$\mu(|\underline{e}(\mathbf{u})|) = 1 + \mathrm{e}^{-|\underline{e}(\mathbf{u})|}.$$

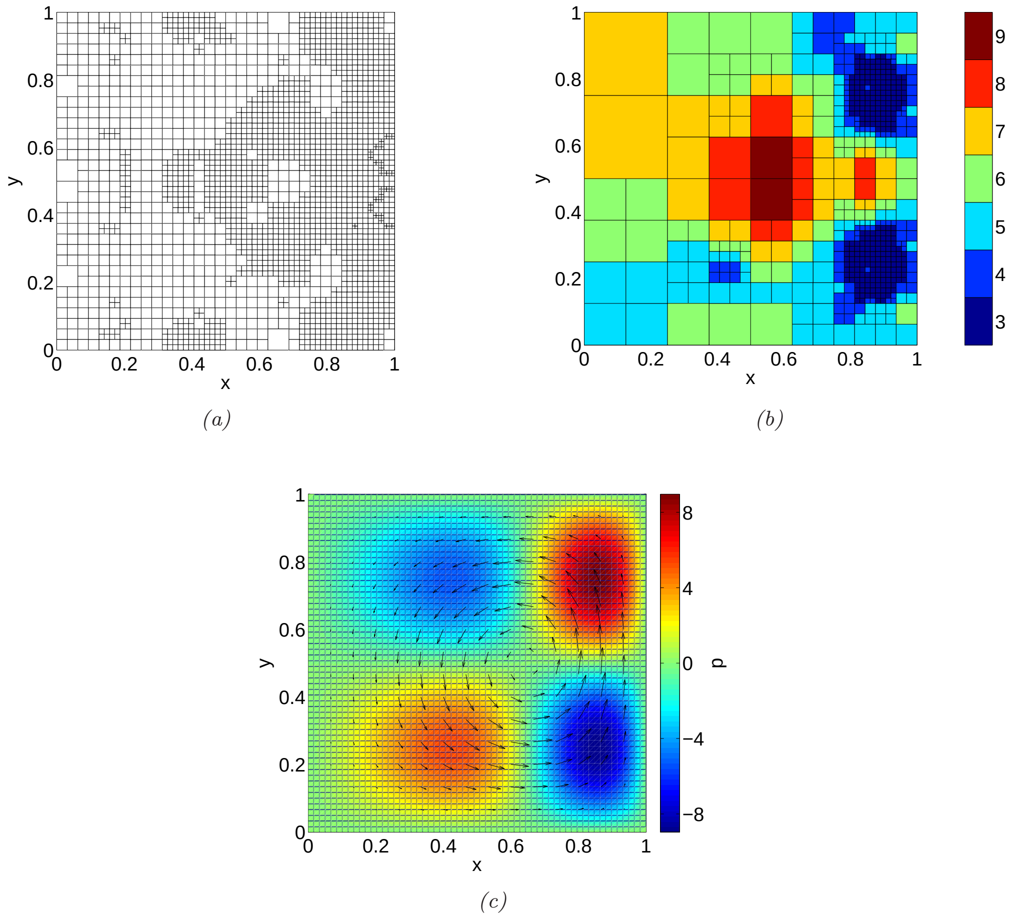


Figure 3. Example 2. Finite element mesh after 10 adaptive refinements: (a) h–adaptivity; (b) hp–adaptivity; (c) Analytical solution.

We let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$ and select $\mathbf{f}$ so that the analytical solution to (1.1)–(1.3), where (r,φ) denotes the system of polar coordinates, is given by

$$\mathbf{u}(x,y) = r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi) \\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi) \end{pmatrix},$$

$$p(x,y) = -r^{\lambda-1}\frac{(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)}{(1-\lambda)},$$

where

$$\Psi(\varphi) = \frac{\sin((1+\lambda)\varphi)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\varphi) - \frac{\sin((1-\lambda)\varphi)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\varphi),$$

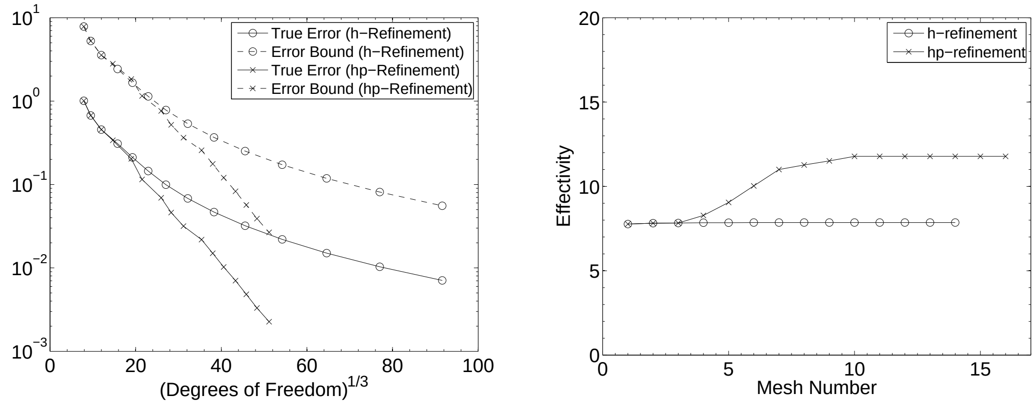
and $\omega = \frac{3\pi}{2}$. Here, the exponent λ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda\sin(\omega) = 0;$$

thereby, $\lambda \approx 0.54448373678246$.

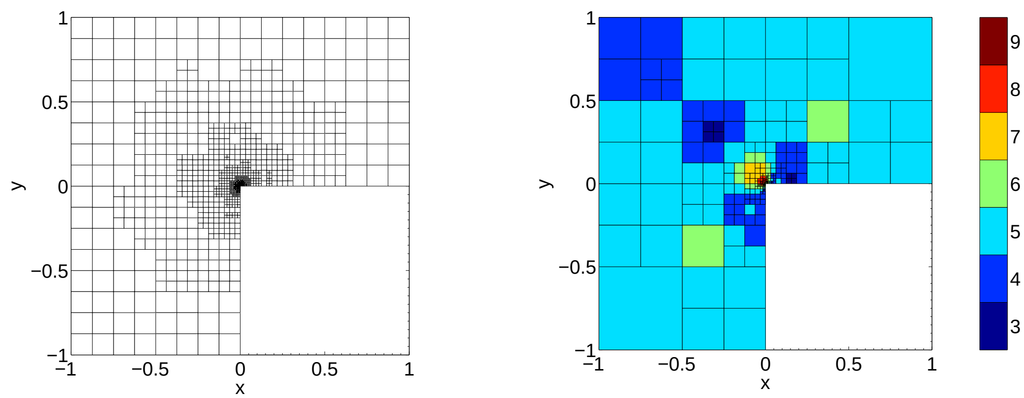
We note that $(\mathbf{u}, p)$ is analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but both $\nabla\mathbf{u}$ and p are singular at the origin; indeed, $\mathbf{u} \notin \mathrm{H}^2(\Omega)^2$ and $p \notin \mathrm{H}^1(\Omega)$.

Figure 4(a) presents the comparison of the actual error in the DGFEM norm and the *a posteriori* error bound versus the third root of the number of degrees of freedom on a linear-log scale for



(a) (b)

Figure 4. Example 3. (a) Comparison of the error in the DGFEM norm employing both h– and hp–refinement, with respect to the number of degrees of freedom; (b) Effectivity index using both h– and hp–refinement.



(a) (b)

Figure 5. Example 3. Finite element mesh after 8 adaptive refinements: (a) h–adaptivity; (b) hp–adaptivity.

the sequence of meshes generated by both the h– and hp–adaptive algorithm. We again observe that the error bound over-estimates the true error by a roughly consistent factor, although the hp–refinement has some initial increase before stabilizing at a higher value than for h–refinement; this is confirmed again by the effectivity indices for the sequence of meshes, cf. Figure 4(b). From Figure 4(a) we can also see that yet again the error in the DGFEM norm converges to zero at an exponential rate when the hp–adaptive algorithm is employed, leading to a greater reduction in the error for a given number of degrees of freedom when compared with the corresponding quantity computed using h–refinement.

Figures 5(a) and (b) show the meshes generated after 8 mesh refinements using the h– and hp–adaptive mesh refinement strategies, respectively. We can see that both refinement strategies perform mostly h–refinement in the region of the singularity at the origin. However, the hp–adaptive strategy is able to perform less h–refinement around the origin as it only performs enough to isolate the singularity; then it performs mostly uniform p–refinement, with a larger p–refinement to the immediate top-left of the singularity.

7. Concluding Remarks

In this article, we have studied the numerical approximation of a quasi-Newtonian flow problem of strongly monotone type by means of *hp*-interior penalty discontinuous Galerkin methods. We have established well-posedness for both the given PDE system as well as for the proposed *hp*-DGFEM. In addition, *a priori* and *a posteriori* error bounds in the discontinuous Galerkin energy norm (3.7) have been derived. In the latter case, both global upper and local lower residual-based *a posteriori* error bounds have been given. The proof of the upper bound is based on employing a suitable DGFEM space decomposition, together with an *hp*–version projection operator. At the expense of a slight suboptimality with respect to the polynomial degree of the approximating finite element method, this upper bound holds on general 1-irregular meshes. The numerical experiments undertaken in this article demonstrate the theoretical results. In particular, we have shown that the *a posteriori* upper bound converges to zero at the same asymptotic rate as the true error measured in the DGFEM energy norm on sequences of *hp*–adaptively refined meshes.

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References

[1] P. R. Amestoy, I. S. Duff, J. Koster, and J.-Y. L'Excellent. A fully asynchronous multifrontal solver using distributed dynamic scheduling. *SIAM Journal on Matrix Analysis and Applications*, 23(1):15–41, 2001.

[2] P. R. Amestoy, I. S. Duff, and J.-Y. L'Excellent. Multifrontal parallel distributed symmetricand unsymmetric solvers. *Comput. Methods Appl. Mech. Eng.*, 184:501–520, 2000.

[3] P. R. Amestoy, A. Guermouche, J.-Y. L'Excellent, and S. Pralet. Hybrid scheduling for the parallel solution of linear systems. *Parallel Computing*, 32(2):136–156, 2006.

[4] J. W. Barrett and W. B. Liu. Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow. *Numer. Math.*, 68:437–456, 1994.

[5] R. Becker, P. Hansbo, and M. G. Larson. Energy norm a posteriori error estimation for discontinuous Galerkin methods. *Comput. Methods Appl. Mech. Engrg.*, 192(5–6):723–733, 2003.

[6] S. Berrone and E. Süli. Two-sided a posteriori error bounds for incompressible quasi-Newtonian flows. *IMA J. Numer. Anal.*, 28:382–421, 2008.

[7] D. Braess. *Finite elements. Theory, fast solvers, and applications in solid mechanics*. Cambridge University Press, Cambridge, second edition, 2001.

[8] S. C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Math. Comp.*, 73(73):1067–1087, 2004.

[9] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer-Verlag New York, Inc., New York, NY, USA, 1991.

[10] R. Bustinza and G. Gatica. A local discontinuous Galerkin method for nonlinear diffusion problems with mixed boundary conditions. *SIAM J. Sci. Comput.*, 26(1):152–177, 2004.

[11] R. Bustinza, G. Gatica, and B. Cockburn. An a posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems. *J. Sci. Comput.*, 22(1):147–185, 2005.

[12] B. Cockburn, G. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin Methods. Theory, Computation and Applications*, volume 11 of *Lect. Notes Comput. Sci. Engrg.* Springer, 2000.

[13] S. Congreve. *Discontinuous Galerkin Finite Element Methods for Non-Newtonian Flows*. PhD thesis, University of Nottingham, in preparation.

[14] K. Eriksson, D. J. Estep, P. Hansbo, and C. Johnson. Introduction to adaptive methods for differential equations. *Acta Numerica*, 4:105–158, 1995.

[15] G. G. M. González and S. Meddahi. A low-order mixed finite element method for a class of quasi-Newtonian Stokes flows. Part I: A-priori error analysis. *Comput. Methods Appl. Mech. Engrg.*, 193(9-11):881–892, 2004.

[16] P. Houston, J. Robson, and E. Süli. Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems. I. The scalar case. *IMA J. Numer. Anal.*, 25(4):726–749, 2005.

[17] P. Houston, D. Schötzau, and T. P. Wihler. *hp*-adaptive discontinuous Galerkin finite element methods for the Stokes problem. In P. Neittaanmäki, T. Rossi, S. Korotov, J. Périaux, and D. Knörzer, editors, *Proc. of the European Congress on Computational Methods in Applied Sciences and Engineering*, volume II, 2004.

[18] P. Houston, D. Schötzau, and T. P. Wihler. Mixed *hp*-discontinuous Galerkin finite element methods for the Stokes problem in polygons. Technical Report 2004/02, University of Leicester, School of Mathematics & Computer Science, 2004.

[19] P. Houston, D. Schötzau, and T. P. Wihler. Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Stokes problem. *J. Sci. Comput.*, 22(1):347–370, 2005.

[20] P. Houston, D. Schötzau, and T. P. Wihler. An *hp*-adaptive mixed discontinuous Galerkin FEM for nearly incompressible linear elasticity. *Comput. Methods Appl. Mech. Engrg.*, 195(25-28):3224–3246, 2006.

[21] P. Houston, D. Schötzau, and T. P. Wihler. Energy norm a posteriori error estimation of *hp*-adaptive discontinuous Galerkin methods for elliptic problems. *Math. Model. Methods Appl. Sci.*, 17(1):33–62, 2007.

[22] P. Houston, C. Schwab, and E. Süli. Discontinuous *hp*-finite element methods for advection–diffusion–reaction problems. *SIAM J. Numer. Anal.*, 39:2133–2163, 2002.

[23] P. Houston and E. Süli. A note on the design of *hp*-adaptive finite element methods for elliptic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 194(2–5):229–243, 2005.

[24] P. Houston, E. Süli, and T. P. Wihler. A posteriori error analysis of *hp*-version discontinuous Galerkin finite-element methods for second-order quasi-linear PDEs. *IMA J. Numer. Anal.*, 28(2):245–273, 2008.

[25] O. Karakashian and F. Pascal. A posteriori error estimation for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 41(6):2374–2399, 2003.

[26] J. M. Melenk. *hp*-interpolation of nonsmooth functions and an application to *hp*-a posteriori error estimation. *SIAM J. Numer. Anal.*, 43(1):127–155, 2005.

[27] J. M. Melenk and B. I. Wohlmuth. On residual-based a posteriori error estimation in *hp*-FEM. *Adv. Comp. Math.*, 15:311–331, 2001.

[28] C. Ortner and E. Süli. Discontinuous Galerkin finite element approximation of nonlinear second-order elliptic and hyperbolic systems. *SIAM J. Numer. Anal.*, 45(4):1370–1397, 2007.

[29] D. Schötzau, C. Schwab, and A. Toselli. Mixed *hp*-DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, 40(6):2171–2194, 2002.

[30] D. Schötzau and T. P. Wihler. Exponential convergence of mixed *hp*-DGFEM for Stokes flow in polygons. Technical Report 2002-19, ETH Zurich, Seminar für Angewandte Mathematik, 2002.

[31] C. Schwab. *p- and hp-FEM — Theory and Applications in Solid and Fluid Mechanics*. Oxford University Press, Oxford, 1998.

[32] R. Verfürth. *A review of a posteriori error estimation and adaptive mesh-refinement techniques*. Teubner, 1996.

[33] T. P. Wihler and M. Wirz. Mixed *hp*-discontinuous Galerkin FEM for linear elasticity in three dimensions. Technical Report 2011-01, Mathematics Institute, University of Bern, 2011.

[34] L. Zhu, S. Giani, P. Houston, and D. Schötzau. Energy norm a posteriori error estimation for *hp*-adaptive discontinuous Galerkin methods for elliptic problems in three dimensions. *Math. Model. Methods Appl. Sci.*, 21(2):267–306, 2011.

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