Numerical Solution of Differential Equations: Problem Sheet 1 (of 4)

1. Verify that the following functions satisfy a Lipschitz condition with respect to y, uniformly in x, on the respective intervals:

   (a) $f(x,y) = 2yx^{-4}, \qquad x \in [1,\infty), \quad y \in \mathbb{R};$

   (b) $f(x,y) = \mathrm{e}^{-x^2}\tan^{-1} y, \qquad x \in [1,\infty), \quad y \in \mathbb{R};$

   (c) $f(x,y) = 2y(1+y^2)^{-1}(1+\mathrm{e}^{-|x|}), \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}.$

2. Suppose that m is a fixed positive integer. Show that the initial-value problem

$$y' = y^{2m/(2m+1)}, \qquad y(0) = 0,$$

   has infinitely many continuously differentiable solutions. Why does this not contradict Picard's Theorem?

3. Van der Pol's equation

$$y'' - \varepsilon(1-y^2)y' + y = 0$$

   subject to the initial conditions $y(a) = A_1$ and $y'(a) = A_2$, where A_1 and A_2 are given real numbers, and $\varepsilon > 0$ a parameter, models electrical circuits connected with electronic oscillators. Rewrite the equation as a coupled system of two first-order differential equations with appropriate initial conditions. Formulate Euler's method for this system, when $\varepsilon = 1$, $a = 0$, $A_1 = 1/2$ and $A_2 = 1/2$, on the interval $[0,1]$ using a mesh of uniform spacing h. Compute the Euler approximations to $y(x)$ and $y'(x)$ at the point $x = h$.

4. Consider the scalar initial-value problem $y' = y\sin(x^2)$, $y(0) = 1$.

   (a) Compute the approximation of $y(0.1)$ obtained using one step of: (i) the explicit Euler method; (ii) the implicit Euler method; and (iii) the implicit midpoint rule.

   (b) Complete the MATLAB-script `elegantoscillatorycurve.m`, which, for each of the methods mentioned under (a), plots the numerical approximation of $y(x)$ for $x = 0.1, 0.2, \ldots, 8$, in steps of $h = 0.1$.

5. Consider the initial-value problem

$$y' = \log\log(4+y^2), \qquad x \in [0,1], \qquad y(0) = 1,$$

   and the sequence $(y_n)_{n=0}^{N}$, $N \geq 1$, generated by the explicit Euler method

$$y_{n+1} = y_n + h\log\log(4+y_n^2), \qquad n = 0,\ldots,N-1, \qquad y_0 = 1,$$

   using the mesh points $x_n = nh$, $n = 0,\ldots,N$, with spacing $h = 1/N$. Here log denotes the logarithm with base e.

   (a) Let T_n denote the consistency error of Euler's method at $x = x_n$ for this initial value problem. Show that $|T_n| \leq h/(4\mathrm{e})$.

b) Verify that

$$|y(x_{n+1}) - y_{n+1}| \leq (1 + hL)|y(x_n) - y_n| + h|T_n|, \qquad n = 0, \ldots, N-1,$$

where $L = 1/(2\log 4)$.

c) Find a positive integer N_0, as small as possible, such that

$$\max_{0 \leq n \leq N} |y(x_n) - y_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

6. The explicit Euler method, the implicit Euler method, and the implicit midpoint rule are Runge–Kutta methods. Write down the formulae for their stages when considered as Runge–Kutta methods.