Physics on a circle and geometry

Balazs Szendroi

University of Oxford
Classical physics deals with *particles* moving in *physical space* under the influence of *forces*.
Classical physics

Classical physics deals with *particles* moving in *physical space* under the influence of *forces*.

*Choose a simple physical space:*

the one-dimensional *circle*
Classical physics on a circle

- Equation of motion: Newton's equation
  \[ \frac{d^2x}{dt^2} = 0 \]
- Energy of particle
  \[ E = \frac{1}{2} \cdot v^2 \]
  \( v \) a real constant
- Continuous spectrum!
Quantum physics on a circle

- Equation of motion: Schrödinger's equation

- Energy of particle
  \[ E = \frac{n^2}{R^2} \]
  
  \( n \): an integer number

  \( R \): radius of circle

- Discrete spectrum!
String theory

- Idea: replace *particles* by *strings*.

- Strings move in space just like particles
- Strings have internal motion patterns, giving rise to physically particles
String theory on a circle

- Geometric input: string can wind round the circle!
- Energy of states:
  \[ E = \frac{n^2}{R^2} + m^2 R^2 \]
  - \( n \): quantum number
  - \( m \): winding number (integers)
  - \( R \): radius of circle
String theory on a circle

● Energy levels:

\[ E = \frac{n^2}{R^2} + m^2 R^2 \]

\(n,m\) integers; \(R\): radius of circle

● Energy spectrum \textit{invariant} under the transformation

\((n, m, R) \iff (m, n, 1/R)\)
Duality in string theory on a circle

\((n, m, R) \longleftrightarrow (m, n, 1/R)\)

**small circle** \(\longleftrightarrow\) **large circle**

**winding state** \(\longleftrightarrow\) **particle state**
Duality in string theory on a circle

\[(n, m, R) \iff (m, n, 1/R)\]

small circle \iff large circle

winding state \iff particle state

Physics on a small circle is indistinguishable from physics on a large circle!
Duality in string theory

• This example can be generalized: there are many examples, where geometrically very different spaces give rise to identical physics in string theory.

• One much studied example: mirror symmetry

• A famous example because of a historically important maths/physics debate
Rational curves on a quintic: maths

• Problem in classical enumerative geometry:
  "find the number $n_d$ of rational curves of degree $d$ on a quintic threefold"

• Mathematics results:

  $n_1 = 2875$ (Kleiman, 1979)

  $n_2 = 609,250$ (Katz, 1986)

  $n_3 = 2,682,549,425$ (Ellingsrud et al 1990)
Rational curves on a quintic: phys

\[ Y_1 = 5 + 2875 \frac{1^3 q}{1-q} + 609250 \frac{2^3 q^2}{1-q^2} + 317206375 \frac{3^3 q^3}{1-q^3} + 242467530000 \frac{4^3 q^4}{1-q^4} \\
+ 229305888887625 \frac{5^3 q^5}{1-q^5} + 248249742118022000 \frac{6^3 q^6}{1-q^6} \\
+ 295091050570845659250 \frac{7^3 q^7}{1-q^7} + 375632160937476603550000 \frac{8^3 q^8}{1-q^8} \\
+ 503840510416985243645106250 \frac{9^3 q^9}{1-q^9} \\
+ 704288164978454686113488249750 \frac{10^3 q^{10}}{1-q^{10}} \\
+ 1017913203569692432490203659468875 \frac{11^3 q^{11}}{1-q^{11}} \\
+ 1512323901934139334751675234074638000 \frac{12^3 q^{12}}{1-q^{12}} \\
+ 22994885681362666648325160104772265542625 \frac{13^3 q^{13}}{1-q^{13}} \\
+ 3565959228158001564810294084668822024070250 \frac{14^3 q^{14}}{1-q^{14}} \\
+ 5624656824668483274179483938371579753751395250 \frac{15^3 q^{15}}{1-q^{15}} \\
+ 9004003639871055462831535610291411200360685606000 \frac{16^3 q^{16}}{1-q^{16}} + \ldots \]
Rational curves on a quintic: phys

• Candelas et al in 1991 claims:

\[ n_1 = 2875 \]

\[ n_2 = 609,250 \]

\[ n_3 = 2,682,549,425 \]

\[ n_3 = 317,206,375 \]

\[ n_4 = \ldots \text{ etc} \]
Rational curves on a quintic: verdict

Date: Wed, 31 Jul 91 11:06:34 MDT
From: Herb Clemens
To: candelas@yyy.edu
Subject: Physics wins!
String theory in geometry

- Such computations come from exploiting the power of duality in string theory
- These ideas had an enormous influence on the development of pure mathematics!
- Development of subjects such as Gromov-Witten theory, derived geometry, non-commutative algebraic geometry,...