Orientations for gauge-theoretic moduli problems

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Plan of talk:

1. Gauge-theoretic moduli problem
   – general picture, motivation
2. Anti-self-dual instanton moduli space
   – Atiyah–Hitchin–Singer complex, Kuranishi model
3. Orientations
   – orientability and orientations of gauge-theoretic moduli spaces
- \( X \): smooth manifold of real dimension \( n \)
- \( P \to X \): principal \( G \)-bundle over \( X \), \( G \): Lie group

The gauge group \( \mathcal{G}_P := \text{Aut}(P) \) acts on \( \mathcal{A}_P \), the space of all connections on \( P \), by \( u(A) := A - (d_A u)u^{-1} \), where \( u \in \mathcal{G}_P \) and \( A \in \mathcal{A}_P \), \( \mathcal{G}_P \) is identified with \( \Gamma(P \times \text{Ad } G) \) and \( d_A \) is the covariant derivative on \( \Gamma(P \times \text{Ad } G) \) induced by \( A \). We denote the quotient \( \mathcal{A}_P / \mathcal{G}_P \) by \( \mathcal{B}_P \).

Gauge-theoretic equations (e.g. anti-self-dual instanton equations) assign a vector bundle \( \mathcal{E} \) over \( \mathcal{B}_P \) and a section \( s \) of \( \mathcal{E} \).

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow \\
\mathcal{M}_P \subset \mathcal{B}_P.
\end{array}
\]

We call \( \mathcal{M}_P := s^{-1}(0) \) a gauge-theoretic moduli space.
Gauge-theoretic moduli problem

**Problem**: construct a (virtual) fundamental cycle out of $\mathcal{M}_P$.

**Applications**: *intersection theory* on the (virtual) fundamental cycles produces deformation invariants such as Donaldson invariants, Gromov–Witten, Seiberg–Witten, Donaldson–Thomas ones and so on. Furthermore, the generating functions of these invariants typically have non-trivial properties such as modularity, which could indicate the origins of these theories perhaps.

**Issues**: smoothness, orientability, compactness of $\mathcal{M}_P$.

- For smoothness: use Freed–Uhlenbeck perturbation, virtual techniques by Behrend–Fantechi et al., or invoke derived stacks.
- For compactness: take up Uhlenbeck, Gieseker compactifications for vector bundles/sheaves, or stable map compactification for pseudo-holomorphic curves.
Anti-self-dual instantons

- $X$: closed, oriented, smooth four-manifold
- $P \to X$: principal $G$-bundle over $X$, $G$: Lie group

Fix a Riemannian metric $g$ on $X$, and consider the Hodge star operator $*_{g}$ on $\Lambda_{X}^{2} := (\Lambda^{2}T^{*}X)$. This satisfies $*_{g}^{2} = 1$, so $\Lambda_{X}^{2}$ decomposes as $\Lambda_{X}^{2} = \Lambda_{X}^{+} \oplus \Lambda_{X}^{-}$.

**Definition**: A connection on $P$ is said to be an *anti-self-dual instanton*, or *ASD instanton* for short, if the curvature $F_{A}$ of $A$ satisfies $F_{A}^{+} := \pi_{+}(F_{A}) = 0$, where $\pi_{+} : \Gamma(g_{P} \otimes \Lambda_{X}^{2}) \to \Gamma(g_{P} \otimes \Lambda_{X}^{+})$ is the projection and $g_{P}$ is the adjoint bundle of $P$.

Consider $\mathcal{M}_{P, g}^{ASD} := \{ A \in \mathcal{A}_{P} : F_{A}^{+} = 0 \}/G_{P}$, the *anti-self-dual instanton moduli space*. (The corresponding $(\mathcal{E}, s)$ in the earlier slide is given by $\mathcal{E} := \mathcal{A}_{P} \times_{g_{P}} \Omega_{X}^{+}(g_{P}) \to B_{P}$ and $s := F_{A}^{+}$.)
Atiyah–Hitchin–Singer complex: the infinitesimal deformation of an anti-self-dual instanton \( A \) is described by the following elliptic complex:

\[
0 \to \Gamma(g_P \otimes \Lambda_X^0) \xrightarrow{d_A} \Gamma(g_P \otimes \Lambda_X^1) \xrightarrow{d_A^+} \Gamma(g_P \otimes \Lambda_X^+) \to 0,
\]

where \( d_A^+ := \pi_+ \circ d_A \).

We write its cohomology by \( \mathbb{H}_A^i \) for \( i = 0, 1, 2 \).

Denote by \( \Gamma_A := \{ u \in G_P : u(A) = A \} \) the stabilizer group of \( G_P \) at \( [A] \in B_P \).

**Definition:** A connection \( A \) of \( P \) is called *irreducible* if \( \Gamma_A \) coincides with the centre of \( G \) and *reducible* otherwise.
Theorem (Atiyah–Hitchin–Singer) Let $A$ be an anti-self-dual instanton. Then there exists an open neighbourhood $U$ of 0 in $\mathbb{H}_A^1$ and a differentiable map $\kappa : U \to \mathbb{H}_A^2$ with $\kappa(0) = 0$ and the first derivative of $\kappa$ vanishing at 0, which is $\Gamma_A$-equivariant if $A$ is reducible, such that the moduli space $\mathcal{M}_{P,g}^{ASD}$ around $[A]$ is locally modeled on $\kappa^{-1}(0)/\Gamma_A$

Remark: One needs an appropriate Sobolev space setting to prove the above, for example in order to use an implicit function theorem in the infinite-dimensional setting.

Note that $\mathbb{H}_A^0 = 0$, if $A$ is irreducible. Also, if $\mathbb{H}_A^2 = 0$, then $\mathbb{H}_A^1$ is the tangent space at $[A] \in \mathcal{M}_{P,g}^{ASD}$ for $A$ irreducible, so $\mathbb{H}_A^2$ is the obstruction space to deforming the equivalence classes $[\tilde{A}]$ of irreducible connections in $\mathcal{M}_{P,g}^{ASD}$. 
In the analytic setting, if \( G = SU(2) \) or \( SO(3) \), then \( H^2_\mathbb{A} = 0 \) for a generic choice of Riemannian metrics \( g \) on \( X \). In addition, if \( b^+_X \), the dimension of maximal positive subspace for the intersection form on \( H_2(X, \mathbb{Z}) \), is positive, then there are no reducible connections other than the trivial one again for a generic metric \( g \). Hence we obtain:

**Theorem** (Atiyah–Hitchin–Singer, Freed–Uhlenbeck, Donaldson–Kronheimer) Let \( X \) be a closed, oriented, simply-connected, smooth four-manifold, and let \( P \to X \) be a principal \( G \)-bundle over \( X \). Take the structure group \( G \) of \( P \) to be \( SU(2) \) or \( SO(3) \), and assume that \( b^+_X > 0 \). Then \( \mathcal{M}^{ASD}_{P,g} \) is a smooth manifold of the expected dimension for a generic choice of metrics \( g \) on the underlying four-manifold.

Remark: One can use *topological stacks* and *derived manifolds* when the above assumptions fail.
Orientations

Finite-dimensional model: let $X$ be a smooth $n$-manifold.

a) $X$ is orientable if the determinant bundle $L := \Lambda^n TX$ of $TX$ is trivial.

b) An orientation is a choice of trivialization of $L$.

ASD instantons case: consider the family of elliptic operators parametrized by $\mathcal{A}_P$ given by:

$$\delta_A := (d_A^*, d_A^+) : \Gamma(g_P \otimes \Lambda^1_X) \to \Gamma(g_P \otimes (\Lambda^0 \oplus \Lambda^+_X)).$$

By the Fredholm property of elliptic operators, it has a well-defined determinant:

$$\mathcal{L}^A := \det(\text{ind } \delta_A) := \det(\ker \delta_A) \otimes \det(\text{coker } \delta_A)^*.$$

This defines a line bundle on $\mathcal{A}_P$, which descends to $\mathcal{L} \to \mathcal{B}_P$. 
If there are no reducible connections (so $\mathbb{H}^0_A = 0$ and $\mathbb{H}^2_A = 0$ for all $[A] \in \mathcal{M}^{ASD}_{P,g}$, then $\iota^*(\mathcal{L})$ is isomorphic to the determinant line bundle of the tangent bundle of $\mathcal{M}^{ASD}_{P,g}$, where $\iota : \mathcal{M}^{ASD}_{P,g} \hookrightarrow \mathcal{B}_P$ is the natural inclusion.

**Theorem** (i) Donaldson, ii) Donaldson–Kronheimer) Let $X$ be a closed, oriented, smooth 4-manifold, and let $P \to X$ be a principal $G$-bundle over $X$. Assume that either i) the structure group $G$ of $P$ is $U(m)$ or $SU(m)$; or ii) $X$ is simply-connected and $G$ is a simply-connected, simple Lie group. Then

a) $\mathcal{L} \to \mathcal{B}_P$ is trivial, hence the smooth part of $\mathcal{M}^{ASD}_{P,g}$ is orientable; and

b) a canonical orientation can be determined by choosing an orientation on $H^1(X)$ and $H^+(X)$. 

In general, suppose we are given $E_0, E_1 \to X$ real vector bundles of the same rank over a compact manifold $X$, and $D : \Gamma(E_0) \to \Gamma(E_1)$, a linear elliptic operator. We write $E_{\bullet} := (E_0, E_1, D)$.

Let $A \in \mathcal{A}_P$. This induces a connection on $g_P \to X$. Then consider the elliptic linear operator twisted by $A$:

$$D^A : \Gamma(g_P \otimes E_0) \to \Gamma(g_P \otimes E_1).$$

As $D^A$ is elliptic on a compact manifold, we have that

$$\det(D^A) = \det(\ker D^A) \otimes \det(\text{coker } D^A)^*$$

is a one-dimensional vector space.

This defines a line bundle on $\mathcal{A}_P$, which descends to a line bundle $L^E_{P\bullet} \to \mathcal{B}_P$, the determinant line bundle of $\mathcal{B}_P$.

We call $O^E_{P\bullet} := (L^E_{P\bullet} \setminus 0(\mathcal{B}_P))/(0, \infty)$ the orientation bundle of $\mathcal{B}_P$, where $0(\mathcal{B}_P)$ is the zero section. This is a principal $\mathbb{Z}_2$-bundle.
Definition:

a) \((\mathcal{B}_P, E_{\bullet})\) is orientable if \(O_P^{E_{\bullet}}\) is isomorphic to the trivial principal \(\mathbb{Z}_2\)-bundle \(\mathcal{B}_P \times \mathbb{Z}_2\).

b) An orientation on \((\mathcal{B}_P, E_{\bullet})\) is an isomorphism \(\mathcal{B}_P \times \mathbb{Z}_2 \xrightarrow{\sim} O_P^{E_{\bullet}}\) of principal \(\mathbb{Z}_2\)-bundles.

Problems:

a) Under what condition on \(X, G, P, E_{\bullet}\), is \((\mathcal{B}_P, E_{\bullet})\) orientable?

b) If it is orientable, can we construct a natural orientation, i.e. is there a way of choosing an orientation which is independent of extra data (e.g. Riemannian metrics) on \(X\)?

Methods for orientations

- Excision theorems for the orientation bundles.
- Orientations from complex structures on $E_\bullet = (E_0, E_1, D)$ or $G$, namely, if $E_0, E_1 \to X$ are complex vector bundles, and the symbol of $D$ is complex linear, then we have a canonical trivialization $O^{E_\bullet}_P \xrightarrow{\mathbb{R}} B_P \times \mathbb{Z}_2$ coming from the complex structures.
- Relating orientations of moduli spaces for Lie subgroups $H \subset G$ such as $U(m_1) \times U(m_2) \subset U(m_1 + m_2)$, $U(m) \hookrightarrow SU(m + 1)$, $U(m) \hookrightarrow Sp(m)$ and so on.
- Stabilization, e.g. consider the direct limit $B_{P \oplus \mathbb{C}^\infty} := \varinjlim_{k \to \infty} B_{P \oplus \mathbb{C}^k}$ for a principal $U(m)$-bundle via gluing maps $B_{P \oplus \mathbb{C}^k} \to B_{P \oplus \mathbb{C}^{k+1}}$, and the direct limit of principal $\mathbb{Z}_2$-bundles $O^{E_\bullet}_{P \oplus \mathbb{C}^\infty} \to B_{P \oplus \mathbb{C}^\infty}$.
- etc. see [Joyce–T–Upmeier].
For the anti-self-dual instanton moduli spaces, we obtain:

**Theorem** (Joyce–T–Upmeier)

Let $X$ be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal $G$-bundle over $X$, where $G$ is a connected Lie group.

1) Choose an orientation on $H^0(X) \oplus H^1(X) \oplus H^+(X)$ and on $\mathfrak{g}$, and a $Spin^c$-structure $\mathfrak{s}$ on $X$. Then we can construct a canonical orientation on $M_{ASD}^P$ for all principal $G$-bundles $P \to X$.

2) If $G$ is simply-connected, or if $G = U(m)$, then the orientation on $M_{ASD}^P$ in the above 1) is independent of the choice of $Spin^c$-structure $\mathfrak{s}$.

Part 1) is new, both the orientability of $B_P$ and the use of $Spin^c$-structures in constructing canonical orientations.
Structure of proof:

- Find a CW complex $Y \subset X$ of dimension 2, and a trivialization $P|_{X \setminus Y} \to (X \setminus Y) \times G$. Then choose an open neighbourhood $U$ of $Y$ in $X$ such that $U$ retract onto $Y$, an open subset $V \subset X$ with $\bar{V} \subset X \setminus Y$ and $U \cup V = X$, and connection $\hat{A}$ on $P$, which is trivial over $V \subset X \setminus Y$.

- $X$ does not necessarily have an almost complex structure, but by using a $Spin^c$-structure, one can introduce an almost complex structure on the above $U$.

- Applying an excision theorem to these $U, V$ etc., one obtains a choice for orientations at $[\hat{A}]$, using the almost complex structure on $E_\bullet|_U$. Then take paths from $[\hat{A}]$ to other connections $[A]$ in the space of connections on $P$ in order to make choices at $[A]$. Prove that this orientation is independent of choices in the construction.

For the orientation problems for $G_2$-instantons, $Spin(7)$-instantons and $DT4$ moduli spaces, see

- D. Joyce and M. Upmeier, *Canonical orientations for moduli spaces of $G_2$-instantons with gauge group SU$(m)$ or U$(m)$*, arXiv:1811.02405, 2018.
I. Kapustin–Witten equations on closed four-manifolds

Let $X$ be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal $G$-bundle over $X$, where $G$ is a connected Lie group. For $(A, a) \in \mathcal{A}_P \times \Gamma(g_P \otimes \Lambda^1_X)$, we consider the following equations:

\[
d^* A a = 0, \quad d^- A a = 0, \quad \text{and} \quad F^+_A + \pi_+([a, a]) = 0,
\]

where $d^-_A := \pi_-(d_A)$, and $\pi_- : \Gamma(g_P \otimes \Lambda^2_X) \to \Gamma(g_P \otimes \Lambda^-)$ is the projection.

Remark: Kapustin and Witten introduced a one parameter family of equations. The above is obtained by specifying the parameter, and corresponds to Simpson’s equations (i.e. (poly-)stable Higgs bundles) when the underlying manifold is a Kähler surface.
The gauge group $G_P = \text{Aut}(P)$ acts on pairs $(A, \alpha) \in \mathcal{A}_P \times \Gamma(g_P \otimes \Lambda^1_X)$. We say that $(A, \alpha)$ is irreducible if the stabilizer group in $G_P$ is trivial.

Denote by $\mathcal{M}_P^{KW}$ the moduli space of gauge equivalence classes of irreducible solutions $(A, \alpha)$ to the Kapustin–Witten equations.

The elliptic operator $E_\bullet = (E_0, E_1, D)$ in this case is

$$D : \Gamma(\Lambda^1_X \oplus \Lambda^1_X) \rightarrow \Gamma(\Lambda^0_X \oplus \Lambda^+_X \oplus \Lambda^-_X \oplus \Lambda^0_X),$$

where

$$D := \begin{pmatrix} d^* & d^+ & 0 & 0 \\ 0 & 0 & d^- & d^* \end{pmatrix}^T.$$

So $E_\bullet$ is the direct sum $E_\bullet = E_\bullet^+ \oplus E_\bullet^-$, where $E_\bullet^+$ is as in the anti-self-dual instantons case, and $E_\bullet^-$ is $E_\bullet^+$ for the opposite orientation on $X$. Thus $O_P^{E_\bullet} \cong O_P^{E_\bullet^+} \otimes_{\mathbb{Z}_2} O_P^{E_\bullet^-}$. 


The orientation bundle of $\mathcal{M}_P^{KW}$ is the pull-back of $O_P^{E^\bullet} \to \mathcal{B}_P$ under the forgetful map $\mathcal{M}_P^{KW} \to \mathcal{B}_P$. Hence we obtain:

**Theorem** (Joyce–T–Upmeier)

Let $X$ be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal $G$-bundle over $X$, where $G$ is a connected Lie group.

1) Choose an orientation on $H^2(X)$ and on $g$, and a $Spin^c$-structure $\mathfrak{s}$ on $X$. Then we can construct a canonical orientation on $\mathcal{M}_P^{KW}$ for all principal $G$-bundles $P \to X$, as a derived manifold.

2) If $G$ is simply-connected, or if $G = U(m)$, then the orientation on $\mathcal{M}_P^{KW}$ in the above 1) is independent of the choice of $Spin^c$-structure $\mathfrak{s}$. 
II. Vafa–Witten equations on closed four-manifolds

Let $X$ be a closed, oriented, smooth four-manifold, and let $P \rightarrow X$ be a principal $G$-bundle over $X$, where $G$ is a connected Lie group.

For $(A, B, C) \in \mathcal{A}_P \times \Gamma(g_P \otimes \Lambda^+_X) \times \Gamma(g_P \otimes \Lambda^0_X)$, we consider the following equations:

\[
d^*_A B + d_A C = 0, \quad \text{and} \quad F_A^+ + [B, C] + [B.B] = 0,
\]

where $[B.B] \in \Gamma(g_P \otimes \Lambda^+_X)$.

The gauge group $\mathcal{G}_P = \text{Aut}(P)$ acts on pairs $(A, B, C) \in \mathcal{A}_P \times \Gamma(g_P \otimes \Lambda^+_X) \times \Gamma(g_P \otimes \Lambda^0_X)$. We say that $(A, B, C)$ is irreducible if the stabiliser group in $\mathcal{G}_P$ is trivial.

Denote by $\mathcal{M}^{VW}_P$ the moduli space of gauge equivalence classes of irreducible solutions $(A, B, C)$ to the Vafa–Witten equations.
The elliptic operator $E_\bullet = (E_0, E_1, D)$ in this case is

$$D : \Gamma(\Lambda^0_X \oplus \Lambda^+_X \oplus \Lambda^1_X) \to \Gamma(\Lambda^1_X \oplus \Lambda^0_X \oplus \Lambda^+_X),$$

where

$$D := \begin{pmatrix} d & d^* & 0 \\ 0 & 0 & d^* \\ 0 & 0 & d^+ \end{pmatrix}.$$

So, $E_\bullet = E^+_\bullet \oplus (E^+_\bullet)^\ast$. Thus, the orientation bundle $O^E_\bullet$ becomes $O^E_\bullet \cong O^{E^+}_\bullet \otimes_{\mathbb{Z}_2} O^{(E^+)^\ast}_\bullet \cong O^{E^+}_\bullet \otimes_{\mathbb{Z}_2} (O^{E^+}_\bullet)^\ast \cong \mathcal{B}_P \times \mathbb{Z}_2$, namely it is canonically trivial. As in the case of the Kapustin–Witten equations, the orientation bundle of $\mathcal{M}_P^{VW}$ is the pull-back of $O^E_\bullet \to \mathcal{B}_P$ under the forgetful map $\mathcal{M}_P^{VW} \to \mathcal{B}_P$. Hence we obtain:

**Theorem (Joyce–T–Upmeier)**

The Vafa–Witten moduli spaces $\mathcal{M}_P^{VW}$ have canonical orientations for all $G$ and principal $G$-bundles $P \to X$, as a derived manifold.