Guide to the Serre Spectral Sequence in 10 easy steps!

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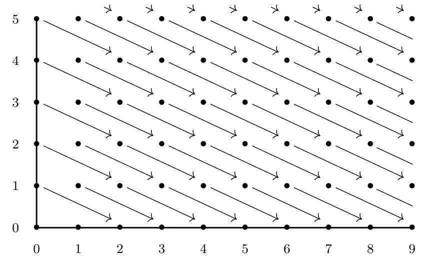
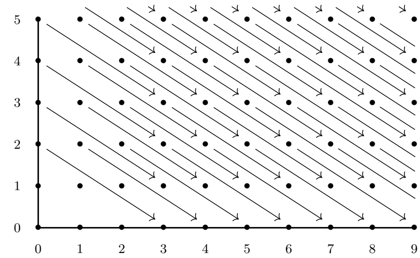
These notes are intended to serve as a practical and painless introduction to the Serre spectral sequence, based on a lecture by André Henriques. Through 10 step-by-step examples, we'll see that a lot can be deduced from just the cohomology of spheres and some well-known fibrations. There will be no proofs or even precise statements made; the aim is simply to get a feel for the calculations involved and pick up some basic properties of the spectral sequence along the way. To this end, instead of a definition, we begin by describing what a spectral sequence looks like.

The data of a spectral sequence can be naturally organised into *pages* E_r for $r \geq 1$. Each page consists of a 2-dimensional lattice of objects, as well as maps of a specified bidegree between any two objects for which the degrees match up. These form a collection of chain complexes. The spectral sequence that we consider is *first-quadrant*. This means that if we view our groups as being located at points of the integer lattice on the Cartesian plane, everything below the horizontal axis or right of the vertical axis is zero. It will also be *cohomologically graded*, meaning differentials d_r on E_r have bidegree $(r, 1-r)$. The group at position (s,t) on the rth page is denoted $E_r^{s,t}$. Differentials on the rth page are therefore of the form $E_r^{s,t} \to E_r^{s+r,t+1-r}$.

Here is a generic picture of E_2 and E_3 with dots representing groups and arrows representing differentials.



E_2



E_3

By convention, we have not drawn in any differentials that are necessarily the zero map. This includes, for instance, all differentials going out of rows 0, 1, and 2 of the E_3 page which have codomain located below the horizontal axis. In our examples, the nonzero groups and differentials are usually sufficiently sparse to stack the pages into one picture.

Let us now specialise to the (cohomological) Serre spectral sequence, which relates the cohomology of the total space of a fibration to the cohomology of the fiber and base spaces. One can compactly describe the essential features of the spectral sequence via a signature of the form

$$E_2^{s,t} = H^s(B, H^t(F)) \implies H^{s+t}(X)$$

where $F \to X \to B$ is the input fibration. This provides a formula for objects on the E_2 page of the spectral sequence, and tells us that they 'converge' in some sense to the cohomology of the total space.

It is useful to note that when there is no torsion[1], the formula simplifies to

$$E_2^{s,t} = H^s(B) \otimes H^t(F). \tag{$*$}$$

We will focus on integral cohomology, so these entries, and indeed the entries on each page, are abelian groups. It is worth noting that there is also a homological version of the spectral sequence, although we make no further mention of it.

In the steps that follow, we feed a series of fibrations into the Serre spectral sequence and see what we can extract. These have been chosen so that the starting point is minimal, the only assumed knowledge being $H^*(S^n)$.

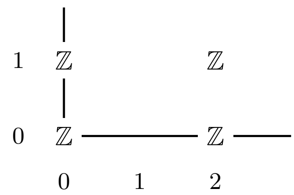
Step 1: $S^1 \to S^3 \to S^2$. (Working backwards to the Serre SS)

Consider the quotient map from $\mathbb{C}^2 - \{0\}$ to the complex projective line $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ given by identifying (z, w) with $(\lambda z, \lambda w)$ for any nonzero complex number λ, that is, $\lambda \in \mathbb{C}^\times$. If we view S^3 as the unit sphere in $\mathbb{C}^2$, and $\mathbb{C}P^1$ as S^2, then this restricts to a map $S^3 \to S^2$ for which the fiber over each point is the space of complex numbers of unit norm, and hence can be identified with S^1. This describes a fibration $S^1 \to S^3 \to S^2$ known as the *Hopf fibration.*

We already know the cohomology groups of each space involved in this case, so the idea in this first step is to work backwards from the answer to determine the associated Serre spectral sequence. From the formula given in ($*$), we see that $H^*(B)$ is 'plotted' on the horizontal axis, and $H^*(F)$ on the vertical axis. For the Hopf fibration, this gives $\mathbb{Z}$s at positions $(0,0)$, $(0,1)$, and $(2,0)$. We then fill in $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ at $(2,1)$. Due to all of the 0s on the axes, every other position will just contain 0. The picture so far is as follows, where we adopt the

[1]Specifically, the condition is that we require at least one of $H^*(B)$ and $H^*(F)$ to be free and finitely generated.

convention of suppressing 0 groups, so any blank entry should be interpreted as 0, and as before do not draw in any differentials that are necessarily trivial.



There is no direct expression for groups on E_3 or higher. Instead, we 'turn the page' to see these groups by taking homology at every position. In order for this to make sense, note that all of the maps really are differentials satisfying $d_r^2 = 0$. Here (and henceforth), we use d_r to denote all differentials on the the E_r page. We can specify which one we mean by the position of its domain or codomain when necessary, but do so only if it is not clear from context.

Property 1. $E_{r+1}^{s,t} = \ker d_r / \operatorname{im} d_r$ *at* $E_r^{s,t}$. *More precisely,*

$$E_{r+1}^{s,t} = \ker d_r : E_r^{s,t} \to E_r^{s+r,t+1-r} / \operatorname{im} d_r : E_r^{s-r,t-1+r} \to E_r^{s,t}.$$

Observe that if the differentials going in and out of a particular entry, say $E_r^{s,t}$, are both zero, then $E_r^{s,t} = E_{r+1}^{s,t}$. In particular, for any fixed position (s,t), since the differentials at (s,t) become longer on successive pages, there exists an ℓ such that $E_\ell^{s,t} = E_r^{s,t}$ for any $r > \ell$. Taking $\ell = \max\{s,t\} + 1$ would do. Let the collection of stabilised groups be the entries $E_\infty^{s,t}$ on the so-called E_∞ page. The Serre spectral *converges* to the cohomology of the total space X in the sense that $H^n(X)$ comes from the $E_\infty^{s,t}$ for which $s + t = n$. In general, we say that the *total degree* of $E_r^{s,t}$ is $s + t$.

Applying the above to our example, note that all differentials d_r are 0 for $r \geq 3$. This is because 0 groups remain the same when taking a quotient, so the four $\mathbb{Z}$s that we have found indicate the only possible positions of nonzero groups in all higher pages. Then, it is easy to check that all differentials going in and out of any non-zero group land outside of the first quadrant. It follows that $E_3 \cong E_\infty$.

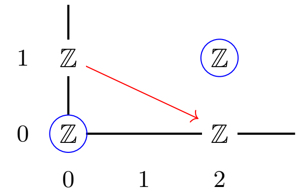
Since $H^1(S^3) = H^2(S^3) = 0$, we deduce from convergence that each $E_\infty^{s,t} = E_3^{s,t}$ for which $s + t = 1$ or 2 must also be 0. This means that we need to kill the $\mathbb{Z}$s at $(0,1)$ and $(2,0)$. Fortunately, we have just the right differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ between these groups. This is in fact the only possible (nonzero) differential on E_2 as the domain or codomain vanish for all others. From

$$0 = E_\infty^{0,1} = E_3^{0,1} = \ker d_2 / \operatorname{im} d_2,$$

it follows that d_2 must be injective to make $\ker(d_2) = 0$. Similarly, for

$$0 = E_\infty^{2,0} = E_3^{2,0} = \ker d_2 / \operatorname{im} d_2 = \mathbb{Z} / \operatorname{im} d_2$$

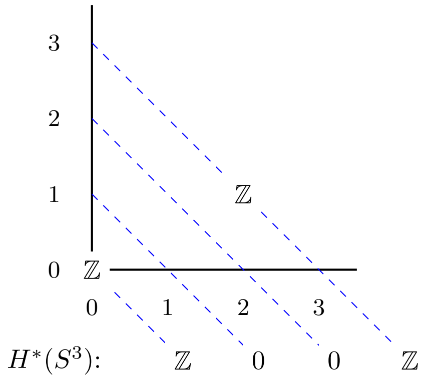
we need $\operatorname{im} d_2 = \mathbb{Z}$, so d_2 is also surjective. Hence, we find that d_2 is an isomorphism. In the next picture, this isomorphism is shown in red, so the circled groups are the only ones that survive to infinity.



Turning to the $E_3 = E_\infty$ page, it is now possible to read off the cohomology of the total space S^3 by assembling along the diagonals. In this case, we have

$$H^n(S^3) = \bigoplus_{s+t=n} E_\infty^{s,t}$$

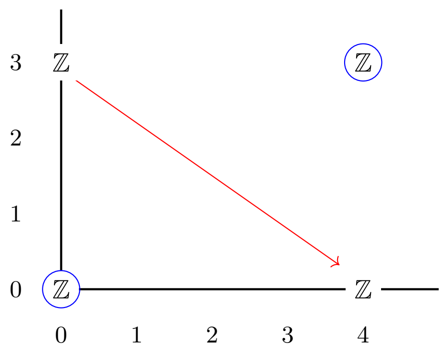
which gives a $\mathbb{Z}$ in dimension 0 from $E_\infty^{0,0}$, and $\mathbb{Z}$ in dimension 3 from $E_\infty^{2,1}$ as expected. Since $E_3 = E_\infty$, we say that the spectral sequences *collapses* at E_3.



Step 2: $S^3 \to S^7 \to S^4$ (Later pages)

With the basic set-up in hand, we now essentially repeat Step 1 only this time working over the quaternions $\mathbb{H}$. Specifically, if we view S^7 as living in $\mathbb{H}^2$ and identify S^4 with the quaternionic projective line $\mathbb{H}P^1 = \mathbb{H} \cup \{\infty\}$, then one can analogously define a mapping $S^7 \to S^4$ with fiber S^3, or the unit quaternions. Thus, there is also a quaternionic Hopf fibration $S^3 \to S^7 \to S^4$.

As before, we start by filling in the groups on the E_2 page using the formula $(*)$. On the horizontal axis, which represents $H^*(S^4)$, we have a $\mathbb{Z}$ in positions $(0,0)$ and $(0,4)$, and all other entries are 0. We also have a $\mathbb{Z}$ on the vertical axis at $(3,0)$ which comes from $H^3(S^3)$. Due to all of the 0s on the axes, the only other nontrivial entry is $\mathbb{Z} \otimes \mathbb{Z}$ at (4,3). Given the cohomology of S_7, we know that the $\mathbb{Z}$s at $(0,0)$ and $(4,3)$ will survive to E_∞ whilst the other two have to be killed at some point. All of the groups are plotted in the following picture.

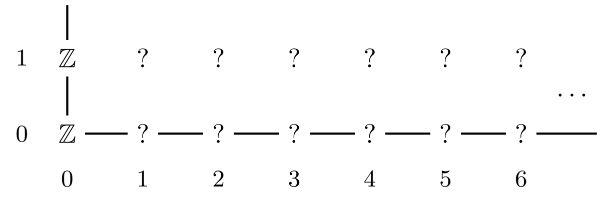


Unlike the previous example, every d_2 differential here is 0; they have bidegree $(2, -1)$ which, given that the interesting columns are separated by 3 empty ones, forces at least one of the domain and codomain to be 0. This means that all $\mathbb{Z}$s survive to the E_3 page. The differentials d_3 have bidegree $(3, -2)$, and by the same reasoning they must all be 0. Now we have four $\mathbb{Z}$s on the E_4 page, and there is exactly one possible differential $d_4 : E_4^{0,3} \to E_4^{4,0}$, drawn in red above. Note that all d_r for $r > 4$ are also necessarily 0 since they are too long. This d_4 is therefore the only opportunity to kill the groups at $(0, 3)$ and $(3, 0)$, and so must be an isomorphism. We conclude that $E_5 = E_\infty$, and we have completely worked out what the spectral sequence looks like for this fibration.

Step 3: $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (First computation)

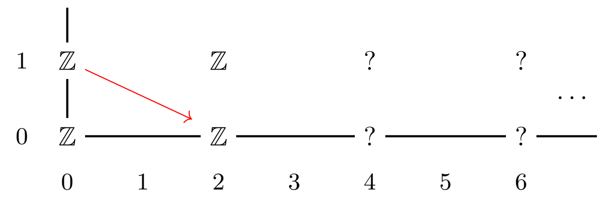
Generalizing the Hopf fibration, consider again the quotient map from $\mathbb{C}^{n+1}$ without the origin to n-dimensional complex projective space $\mathbb{C}P^n$ which identifies (z, w) with $(\lambda z, \lambda w)$ for any $\lambda \in \mathbb{C}^\times$. We may view S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, so the restriction of the quotient produces a map $S^{2n+1} \to \mathbb{C}P^n$ for which the fiber over a point is $S^1 \subset \mathbb{C}^\times$. This gives rise to a fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. This will be our input for a first example that actually computes some cohomology that is not already assumed, although we will be running the spectral sequence “in reverse”. That is, since we know the cohomology of the total space and the fiber in $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, our goal will be to deduce the cohomology of the base space. To start with, we plot the cohomology of S^1 on the vertical axis. It follows immediately that only the bottom two rows of the E_2 page can contain nonzero groups.

From $H^*(S^{2n+1})$, the E_∞ page will have a $\mathbb{Z}$ in degree 0 and one in degree $2n + 1$. This means $E_2^{1,0} = 0$, as there is no space for non-zero differentials in or out, and hence $E_2^{1,1} = 0$ as well. Similarly, $E_2^{3,0} = 0$ because a d_2 differential landing in $E_2^{3,0}$ would have to originate from $E_2^{1,1}$, and all higher differentials

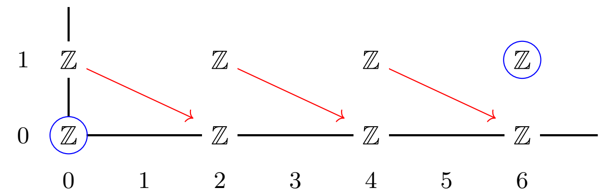


point outside of the first quadrant. Continuing this reasoning, we find that there can only be non-zero groups in positions with even horizontal coordinate.

In addition, we know that the $\mathbb{Z}$ at $(0,1)$ does not survive to E_∞. Note that any d_r for $r \geq 3$ going out of $(0,1)$ is zero since it will point below the horizontal axis, so the only opportunity to kill this $\mathbb{Z}$ is if $d_2 : E_2^{0,1} \to E_2^{2,0}$ is injective. We claim that it is also surjective. If not, $E_3^{2,0}$ would be non-zero and survive to E_∞, but this contradicts $H^2(S^{2n+1}) = 0$. Thus, this d_2 is an isomorphism and we have a $\mathbb{Z}$ at $(2,0)$. We can then also fill in $\mathbb{Z}$ at $(2,1)$.



Now, the $\mathbb{Z}$ at $(2,1)$ must be killed since $H^3(S^{2n+1}) = 0$, and the only way for this to occur without creating some other non-zero surviving class is for $d_2 : E_2^{2,1} \to E_2^{4,0}$ to be an isomorphism. It follows that there are $\mathbb{Z}$s at $(4,0)$ and $(4,1)$. Playing the same game, we find that on the E_2 page there is a $\mathbb{Z}$ at $(x,0)$ and $(x,1)$ as well as isomorphisms $d_2 : E_2^{x,1} \to E_2^{x+2,0}$ for each even $x \leq 2n$.



The argument stops once we have a $\mathbb{Z}$ at $(2n,1)$ since we do actually want $H^{2n+1}(S^{2n+1}) = \mathbb{Z}$. At this point, there are two possible positions with the correct total degree, namely $(2n,1)$ and $(2n+1,0)$. To rule out the latter being nonzero, it suffices to input the fact that $\mathbb{C}P^n$ is $2n$-dimensional. Thus, we conclude that the Z at $(2n,1)$ survives to E_∞ and we now have a complete picture of the E_2 page. The cohomology groups of the base space are on the horizontal axis, from which we read off that they are $\mathbb{Z}$ in even dimensions up to $2n$, and 0 otherwise.

Step 4: $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (Ring structure I)

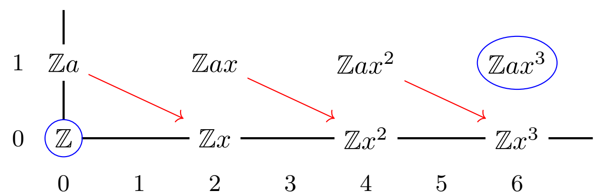
It turns out that one can also deduce the ring structure of $H^*(\mathbb{C}P^n)$ from the spectral sequence. Returning to the previous calculation, let a and x be generators for $E_2^{0,1}$ and $E_2^{2,0}$ respectively. Recall that each nonzero d_2 is an isomorphism in this picture, so $d_2(a) = x$. We also have that xa generates the $\mathbb{Z}$ at $(2,1)$. What can we say about $d_2(xa)$?

The feature that we need to proceed is that there is a multiplicative structure on each page of the spectral sequence which is induced by the multiplication on the previous page, and on E_2 coincides (up to a sign) with the cup product. Moreover, differentials are derivations.

Property 2. *Each page $E_r^{*,*}$ is a ring, and each d_r satisfies the Leibniz rule*

$$d_r(ab) = d_r(a)b + (-1)^{\deg(a)} a d_r(b).$$

Using Property 2, we compute $d_2(xa) = d_2(a)x + (-1)^{0+1} a d_2(x) = x^2 - a \cdot 0 = x^2$ where we have used the fact that $d_2(x) = 0$ since the codomain is $E^{4,-1} = 0$. That gives the generators of $\mathbb{Z}$ at $(4,0)$. Then the $\mathbb{Z}$ at $(4,1)$ is generated by ax^2, so we compute $d(ax^2) = d_2(a)x^2 - ad_2(x^2) = x^3 - 0 = x^3$ which generates $\mathbb{Z}$ at $(6,0)$ and so on. Once this terminates at degree $2n$ on the horizontal axis, we see that the groups are generated by powers of x, giving the structure of a truncated polynomial ring. That is, $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$ with $|x| = 2$.



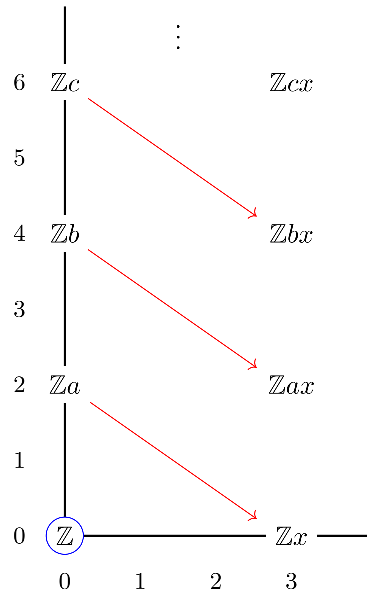
Step 5: $\Omega S^3 \to PS^3 \to S^3$ (Ring structure II)

Given a based space $(X, *)$, define $P(X) := \{\gamma : [0,1] \to X \mid \gamma(0) = *\}$ to be its *path space*, and $\Omega(X) := \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = *\}$ to be its loop space. Consider the map $PX \to X$ which sends each path γ to its endpoint. Then the fiber over $*$ consists of those paths for which $\gamma(0) = \gamma(1) = *$, which are just loops based at $*$. This gives rise to a path space fibration $\Omega(X) \to PX \to X$. Note that the path space is always contractible, which can be seen by gradually truncating paths.

Let us specialise to $X = S^3$. We will determine the cohomology ring of ΩS^3. This time, all we know initially is that there are $\mathbb{Z}$s at $(0,0)$ and $(3,0)$ coming from the cohomology of the base space, and for degree reasons there can only be non-zero differentials on E_3. Since PS^3 is contractible, the E_∞ page will be empty except for the $\mathbb{Z}$ at (0,0). Thus, we need to kill the $\mathbb{Z}$ at $(3,0)$. The only way to do this is for $d_3 : E_3^{0,2} \to E_3^{3,0}$ to be surjective. In fact, it must also be injective, otherwise some nonzero group would survive to E_∞ at $(0,2)$, but this would contradict $H^2(PS^3) = 0$. Next, we can fill in a Z at (3,2), but this again must be killed by an isomorphism $d_3 : E_3^{0,4} \to E_3^{3,2}$.

Repeating the previous sequence of deductions upward indefinitely, we find that the E_3 page has $\mathbb{Z}$ at $(0,y)$ and $(3,y)$ for each even coordinate y. There is no space for d_2 differentials since $H^*(S^3)$ forces a gap of size 2 between the interesting columns, so these groups must also live on E_2 in the same positions. In particular, the cohomology groups of ΩS^3 consists of a $\mathbb{Z}$ in every even dimension, as seen on the vertical axis.

We now turn to ring structure. Let a, b, and x be generators for the $\mathbb{Z}$s at $(0,2)$, $(0,4)$, and $(3,0)$ respectively. Then ax generates $E_2^{3,2}$, and we can fill in the following picture.



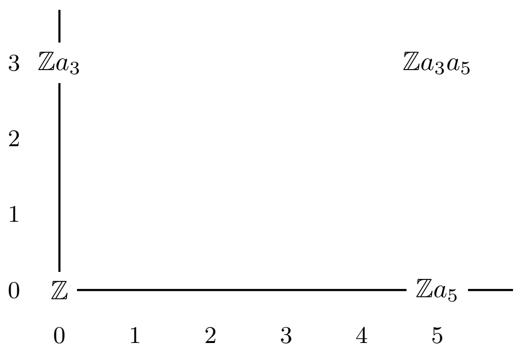
Inspired by the previous example, we might wonder how b relates to a^2. By definition we have $d_3(a) = x$, so $d_3(a^2) = d_3(a)a + ad_3(a) = 2ax$ by the Leibniz rule. Since d_3 is an isomorphism and $d_3(b) = ax$ as well, this implies that $a^2 = 2b$. Next, let c be the generator of $E_2^{0,6}$, so $d_3(c) = bx$ by definition. At the same time,

$$d_3(ab) = d_3(a)b + ad_3(b) = xb + a^2x = xb + 2bx = 3bx.$$

Again using the fact that d_3 is an isomorphism, we obtain $3c = ab = a\frac{a^2}{2}$. Thus, $a^3 = 6c$. One can continue these computations to find that $a^n = n! \cdot g$ where g generates $E_2^{0,2n}$. This describes a divided power ring, which we now know to be the structure of $H^*(\Omega S^3)$.

Step 6: $SU(n-1) \to SU(n) \to S^{2n-1}$ for $n = 3, 4$
(Collapse for degree reasons)

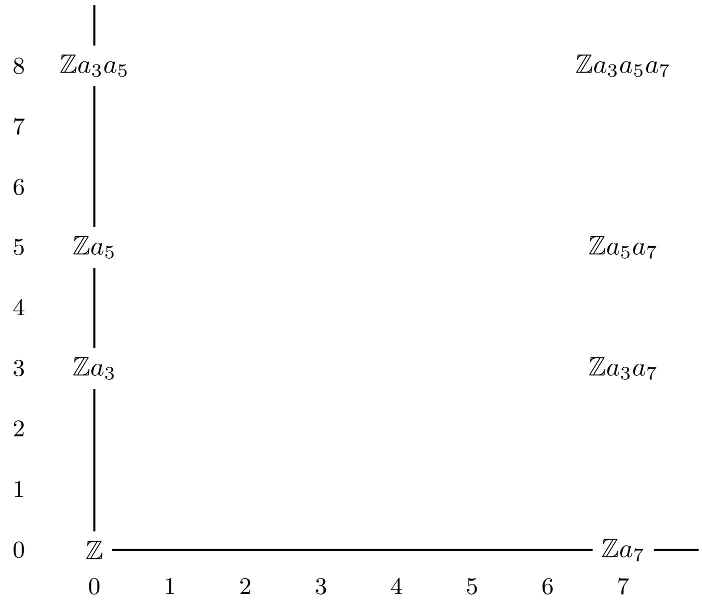
Here is an example in which we are actually computing the cohomology of the total space. The special unitary group $SU(n)$ acts on $\mathbb{C}^n$, and hence on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. This action is transitive and the stabiliser of any given point is a copy of $SU(n-1)$. It follows that there is a fibration $0 \to SU(n-1) \to SU(n) \to S^{2n-1} \to 0$. If we set $n = 2$, then as $SU(1)$ is trivial, we see that $SU(2) \cong S^3$. Setting $n = 3$ then gives a fibration $S^3 \to SU(3) \to S^5$. Given that we know the cohomology of the fiber and base space, we can fill in all of the groups on E_2 by plotting $H^*(S^3)$ and $H^*(S^5)$ along the vertical and horiztonal axes, after which we get one more $\mathbb{Z}$ at (5,3).



The interesting columns are 0 and 5, which immediately limits us to d_5 differentials. However, these have bidegree $(5, -4)$, and there are no two nonzero groups that are 4 apart vertically. In short, we observe that there is no space at all for nonzero differentials. In such situations, we say that the spectral sequence "collapses for degree reasons". For the current calculation, this means that $E_2 = E_\infty$, and hence $SU(3)$ has nontrivial cohomology groups in degrees 0, 3, 5, and 8.

The ring structure is also easy to extract. Let a_3 and a_5 be the generators of $E_2^{0,3}$ and $E_2^{5,0}$ respectively, so a_3a_5 generates $H^8(SU(3))$. Then $H^*(SU(3))$ has the structure of the exterior algebra $\Lambda(a_3, a_5)$ on two generators with $|a_3| = 3$ and $|a_5| = 5$.

The same reasoning allows us to quickly obtain $H^*(SU(4))$ as well, which we'll do using the fibration $SU(3) \to SU(4) \to S^7$ which is the $n = 4$ case. The now known cohomology of $SU(3)$ together with the cohomology of S^7 determine all of the groups on the E_2 page. Again, there is no space for differentials; the horizontal distance of 7 restricts us to d_7's, but there are no two nonzero groups in $H^*(SU(3))$ that differ in degree by 6. The spectral sequence therefore collapses for degree reasons, and we read off from $E_2 = E_\infty$ that $H^*(SU(4)) = \Lambda(a_3, a_5, a_7)$ with $|a_i| = i$.

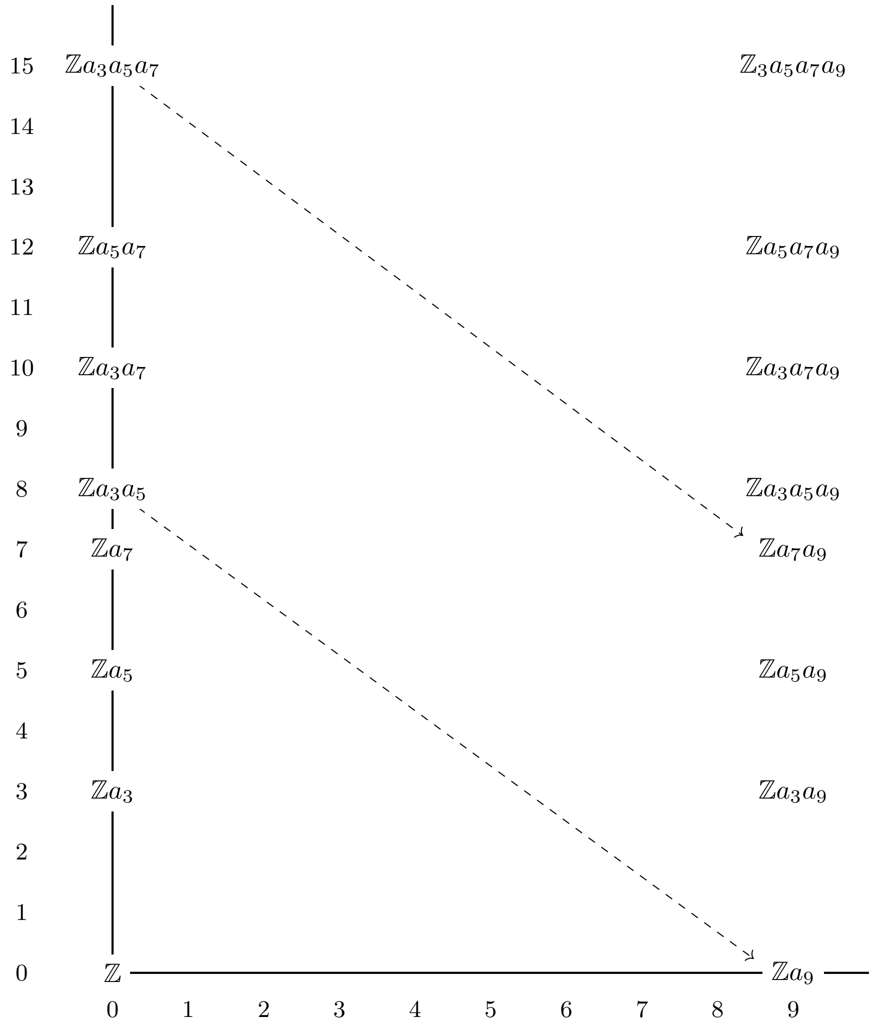


Step 7: $SU(n-1) \to SU(n) \to S^{2n-1}$ for $n \geq 5$
(Deducing collapse from ring structure)

Next in line is $H^*(SU(5))$, which we compute from the fibrations discussed in the preceding step with $n = 5$, namely $SU(4) \to SU(5) \to S^9$, together with $H^*(SU(4))$ from the last calculation and $H^*(S^9)$. The groups on the E_2 page, together with their generators, are shown in the next picture. Most differentials are necessarily zero for degree reasons, but there are two d_9's that need to be determined by a further argument. This is a situation in which we need to use the ring structure to even work out all of the cohomology groups.

Consider $d_9 : E_9^{0,8} \to E_9^{9,0}$, where the domain is generated by a_3a_5. Note that $d_9(a_3) = 0$ and $d_9(a_5) = 0$. By the Leinbniz rule we have $d_9(a_3a_5) = d_9(a_3)a_5 + a_3d_9(a_5) = 0$. Thus, this d_9 is also 0. As $d_9(a_7) = 0$ as well, a similar calculation shows that $d_9 : E_9^{0,15} \to E_9^{9,7}$ is 0. That rules out all potential differentials, so the spectral sequences collapses at E_2 (although not for degree reasons) and we conclude that $H^*(SU(5)) = \Lambda(a_3, a_5, a_7, a_9)$ with $|a_i| = i$.

We can now prove by induction that $H^*(SU(n)) = \Lambda(a_3, a_5, \ldots, a_{2n-1})$. The induction step is essentially a repeat of the previous argument, but starting instead with the fibration $SU(n) \to SU(n+1) \to S^{2n+1}$ where $H^*(SU(n))$ is known by the hypothesis. This allows us to fill in all of the groups on the E_2 page. The only possible differentials live on E_{2n+1} and have bidegree $(2n+1, -2n)$. It follows that $d_{2n+1}(a_i) = 0$ for any $i \leq 2n-1$, but this interval includes all of the generators of the exterior algebra $\Lambda(a_3, a_5, \ldots, a_{2n-1}) = H^*(SU^n)$. Thus, all d_{2n+1} differentials are zero by the Leibniz rule. From this we conclude that $E_2 = E_\infty$, and $H^*(SU(n+1))$ is the claimed exterior algebra.



Step 8: $S^1 \to \mathbb{R}P^3 \to S^2$ (Torsion)

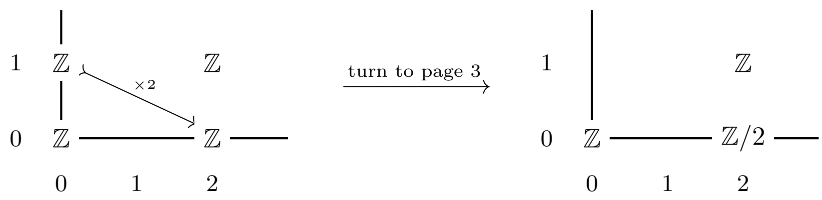
The Hopf fibration $\eta : S^3 \to S^2$ factors through real projective 3-space $\mathbb{R}P^3 = S^3/\{\pm 1\}$. This can be written as a commutative diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \xrightarrow{\eta} & S^2 \\ \downarrow q & & \downarrow q & & \| \\ S^1/\{\pm 1\} & \longrightarrow & S^3/\{\pm 1\} & \longrightarrow & S^2 \end{array}$$

where q denotes quotient maps, from which we see that there is a fibration $S^1 \to \mathbb{R}P^3 \to S^2$. This will allow us to compute $H^*(\mathbb{R}P^3)$. The E_2 page of the associated spectral sequence has $\mathbb{Z}$s at $(0,0)$, $(0,1)$, $(2,0)$ and $(2,1)$, and 0 elsewhere. For degree reasons, the only possible differential is $d_2 : E_2^{0,1} \to E_2^{2,0}$, so at this point we already know that $H^0(\mathbb{R}P^3) = H^3(\mathbb{R}P^3) = \mathbb{Z}$.

To fill in the remaining groups, we'll need to work out what d_2 is, and this requires some more input. Given that $\pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$ which is abelian, then $H_1(\mathbb{R}P^3) = \pi_1(\mathbb{R}P^3)^{ab} = \mathbb{Z}/2$. By Poincaré duality, we then have $H^2(\mathbb{R}P^3) = \mathbb{Z}/2$. Alternatively, this can also be determined using the universal coefficient theorem.

Once we know that we need a $\mathbb{Z}/2$ at some position with total degree 2, which can only be $(2,0)$, it follows that d_2 is multiplication by 2. On the next page, taking cohomology kills $E_2^{0,1}$ and leaves the required $\mathbb{Z}/2$ at (2,0). There is no more space for differentials, so the spectral sequence collapses at E_3. Thus, the cohomology groups of $\mathbb{R}P^3$ are $\mathbb{Z}$ 0 $\mathbb{Z}/2$ $\mathbb{Z}$.

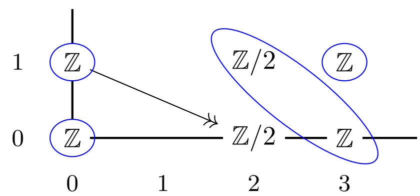


Step 9: $S^1 \to U(2) \to \mathbb{R}P^3$ (Additive extension problems)

The last two steps are cautionary tales of sorts. We have so far had an easy time reading off cohomology groups from E_∞ since there has only been at most one nonzero group in each degree. However, there is often a lot more to it.

Consider the mapping $S^1 = U(1) \to U(2)$ which sends $\lambda \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Note that $U(2)/U(1) = SU(2)/\{\pm 1\}$ as $SU(2)$ intersects $U(1) \leq U(2)$ in $\{\pm 1\}$. In addition, we may identify $SU(2)$ as S^3, so $SU(2)/\{\pm 1\} = \mathbb{R}P^3$. Thus, this fits into a fibration $S^1 \to U(2) \to \mathbb{R}P^3$. We would like to use this to compute the cohomology of $U(2)$.

As usual, we begin by plotting $H^*(S^1)$ and $H^*(\mathbb{R}P^3)$ and then filling in the remaining groups. Since $H^2(U(2)) = 0$, the $\mathbb{Z}/2$ at $(2,0)$ must die. The only chance to kill it is for $d_2 : E_2^{0,1} \to E^{2,0}$ to be surjective. There is no space for any other d_2 differentials, nor any higher differentials as everything outside of the bottom two rows are 0. Thus, $E_3 = E_\infty$. From this, we can read off the groups $H^n(U(2))$ for $n = 0, 1, 4$ directly. However, in degree 3 we have $\mathbb{Z}/2$ and $\mathbb{Z}$, and we are left wondering how to put them together.



The general problem of determining how groups of the same total degree in E_∞ assemble into the desired cohomology group is called an *additive extension problem*. This refers to the fact that there is an underlying group extension problem, which becomes apparent once we say a few words on convergence.

Property 3. *E_∞ is the associated graded with respect to some filtration of $H^*(X)$. That is, there is a filtration $0 = F^n_{-1} \subset F^n_0 \subset F^n_1 \subset \cdots \subset F^n_n = H^n(X)$ such that for $i = 0, 1, \ldots, n$ we have a short exact sequence*

$$0 \longrightarrow F^n_{i-1} \longrightarrow F^n_i \longrightarrow E_\infty^{n-i,i} \longrightarrow 0\ .$$

Starting from $i = 0$, the dream would be to work our way up solving the group extension problems and thus work out the filtration. There are nice cases such as when we are working over a field, or if we otherwise only have free modules and hence trivial extensions, in which we can just assemble by direct sum along the diagonals as we have been doing. In general though, we cannot solve such extension problems without some additional information.

In the present calculation, we can sidestep the problem using our earlier calculation of $H^*(SU(n))$. There is a split short exact sequence of Lie groups

$$0 \longrightarrow SU(n) \longrightarrow U(n) \underset{\det}{\overset{s}{\rightleftarrows}} U(1) \longrightarrow 0$$

with the section s given by $\lambda \mapsto M$ where $m_{11} = \lambda$, $m_{1i} = 0 = m_{i1}$, and the minor M_{11} is I_{n-1}. Notably, this is different from our earlier embedding $U(1) \to U(2)$. Hence, as a manifold we have $U(n) = SU(n) \times U(1)$. Now recalling that $H^*(U(1)) = H^*(S^1) = \Lambda(a_1)$ with $|a_1| = 1$ and $H^*(SU(n)) = \Lambda(a_3, a_5, \ldots, a_{2n-1})$ with $|a_i| = i$, it then follows from the Künneth formula that $H^*(U(n)) = \Lambda(a_1, a_3, a_5, \ldots, a_{2n-1})$ with $|a_i| = i$.

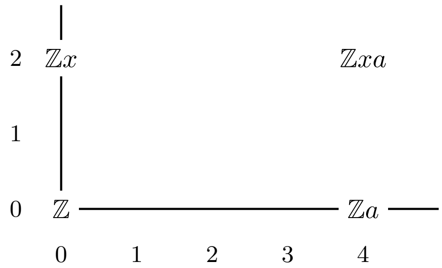
In particular, we obtain $H^*(U(2)) = \Lambda(a_1, a_3)$ with $|a_i| = i$ which was what we were originally trying to compute. Returning to the additive extension problem, we see that the $\mathbb{Z}/2$ and $\mathbb{Z}$ with total degree 3 on the E_2 page should assemble to give $H^3(U(2)) = \mathbb{Z}$. We can now work backwards from the answer to deduce that the filtration for $n = 3$ was $0 \subset 2\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Z} = H^3(U(2))$.

Step 10: $S^2 \to \mathbb{C}P^3 \to S^4$ (Multiplicative extension problems)

There are also situations in which we can determine the cohomology groups from the spectral sequence, but not the ring structure. Just as we factored the Hopf fibration through $\mathbb{R}P^3$ in Step 8, one can factor the quaternionic Hopf fibration through $\mathbb{C}P^3$ by taking quotients as in

$$\begin{array}{ccccc}
S^3 & \longrightarrow & S^7 & \xrightarrow{\ \eta\ } & S^4 \\
\big\downarrow q & & \big\downarrow q & & \big\| \\
S^3/S^1 & \longrightarrow & S^7/S^1 & \longrightarrow & S^4
\end{array}$$

to obtain a fibration $S^2 \to \mathbb{C}P^3 \to S^4$. We already know the cohomology of each space, but suppose we wish to recalculate $H^*(\mathbb{C}P^3)$ using our new fibration. Once we have filled in the groups on the E_2 page of the associated spectral sequence, we observe that the spectral sequence collapses at E_2 for degree reasons. What can we say about the ring structure? If we let $H^2(\mathbb{C}P^3)$ be generated by x and $H^4(\mathbb{C}P^3)$ by a, then xa generates $H^6(\mathbb{C}P^3)$ is generated by xa. After that though, we are stuck staring at the following picture.



In fact, we know from Step 3 that we should have $a = x^2$ since $H^*(\mathbb{C}P^3) = \mathbb{Z}[x]/x^4$ with $|x| = 2$, but there is no way to deduce this from the spectral sequence alone. The trouble again comes from there being more to the statement of convergence.

Property 4. *The ring structure on the E_∞ page is the associated graded of the ring structure on $H^*(X)$ with respect to some filtration.*

Note that by Property 3, a map $E_\infty^{m-i,i} \times E_\infty^{n-j,j} \to E_\infty^{m+n-i-j,i+j}$ corresponds to a map $F_i^m/F_{i-1}^m \times F_j^n/F_{j-1}^n \to F_{i+j}^{m+n}/F_{i+j-1}^{m+n}$ where $0 \subset F_0 \subset F_1 \subset \cdots \subset F_n \subset H^*(X)$ is some filtration. Property 4 then says that the multiplication on E_∞ is given by quotient maps induced by the multiplication $F_i^m \times F_j^n \to F_{i+j}^{m+n}$, which is in turn induced by the cup product in $H^*(X)$. Taking the associated graded in this way can kill some of the multiplications, so it is nontrivial to recover the graded ring structure of $H^*(X)$ from the E_∞ page. This is known as a *multiplicative extension problem.* Like additive extension problems, these generally require some additional information to be solved.