Touching representations by comparable boxes

Jane Tan (Oxford)

with

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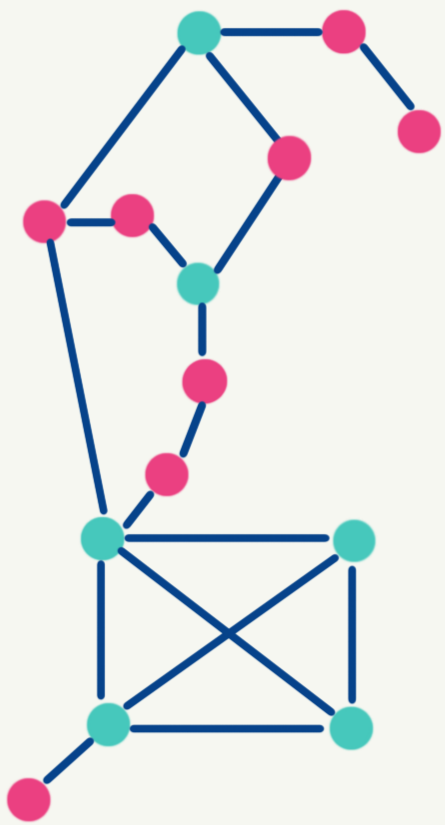

A $\boxed{\text{geometric}}$ representation of a graph G consists of a collection $\mathcal{C}$ of

$$\boxed{\text{objects in } \mathbb{R}^d}$$

one per vertex $f : V(G) \hookrightarrow\!\!\!\!\rightarrow \mathcal{C}$

$$v \longmapsto f(v)$$

such that there is an edge $uv \in E(G)$

$\Leftrightarrow$ $f(u)$ and $f(v)$ $\boxed{\text{interact in a specific geometric way}}$.

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A coin representation of a graph G consists of a collection $\mathcal{C}$ of

interior-disjoint balls (disks) in $\mathbb{R}^2$

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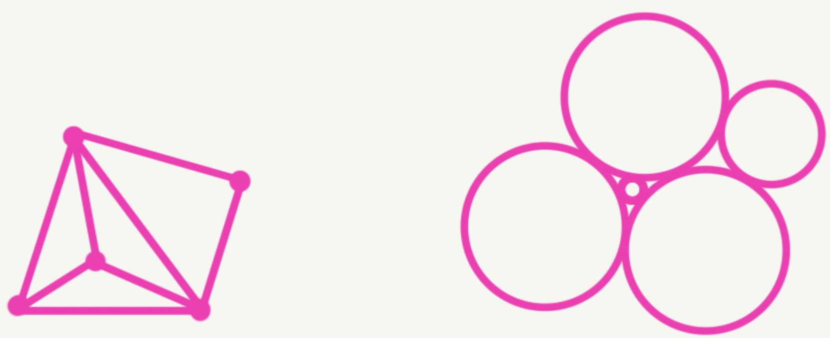

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Thm (Koebe 1936): A graph is planar iff it has a coin representation.

A $\boxed{\text{penny}}$ representation of a graph G consists of a collection $\mathcal{C}$ of

$\boxed{\text{unit disks in } \mathbb{R}^2}$

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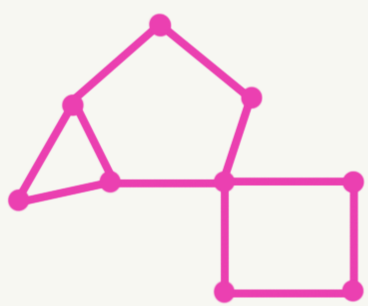
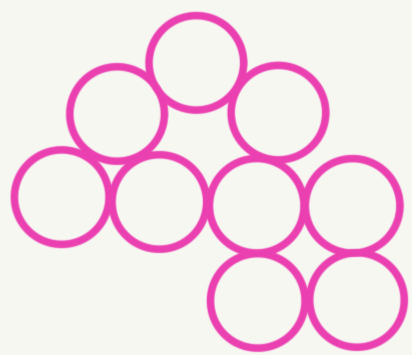




A $\boxed{\text{circle}}$ representation of a graph G consists of a collection $\mathcal{C}$ of

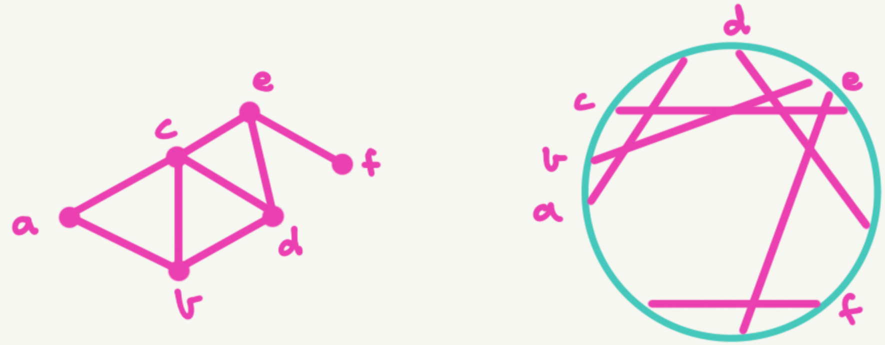
$\boxed{\text{chords on a circle}}$

one per vertex $f : V(G) \longleftrightarrow\!\!\!\twoheadrightarrow \mathcal{C}$

$v \longmapsto f(v)$

such that there is an edge $uv \in E(G)$

$\Leftrightarrow$ $f(u)$ and $f(v)$ $\boxed{\text{intersect}}$.



An interval representation of a graph G consists of a collection $\mathcal{C}$ of

intervals in $\mathbb{R}$

one per vertex $f : V(G) \twoheadleftarrow\!\!\!\rightarrow \mathcal{C}$

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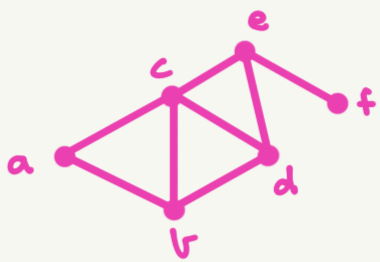
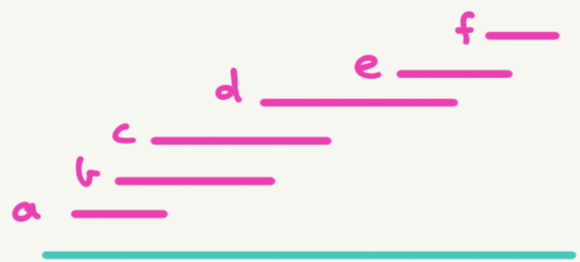
such that there is an edge $uv \in E(G)$

$\Leftrightarrow$ $f(u)$ and $f(v)$ intersect.





A $\boxed{\text{box}}$ representation of a graph G consists of a collection $\mathcal{C}$ of

$\boxed{\text{axis-aligned boxes in } \mathbb{R}^d}$

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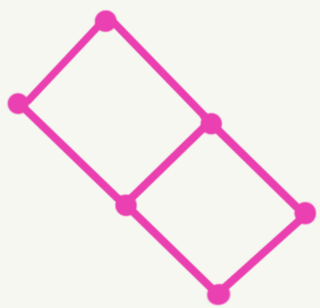

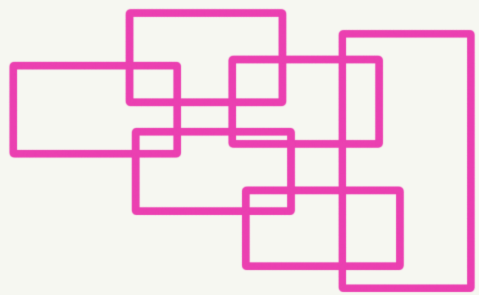

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Def^{n} : The boxicity of a graph G is the minimum d s.t G has a representation by boxes in $\mathbb{R}^d$.

A $\boxed{\text{touching box}}$ representation of a graph G consists of a collection $\mathcal{C}$ of

$\boxed{\text{interior-disjoint axis-aligned boxes in } \mathbb{R}^d}$

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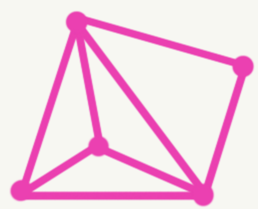

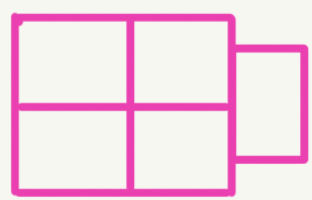

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Thm (Thomassen 1986): Every planar graph has a touching representation by boxes in $\mathbb{R}^3$

A cube representation of a graph G consists of a collection $\mathcal{C}$ of

interior disjoint cubes in $\mathbb{R}^3$

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Thm (Felsner & Francis 2011): Every planar graph has a cube representation.

What makes an interesting* representation ?

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- [ ] can represent sparse graphs

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  A representation of a graph is c-thin if every point in the space is contained in $\leq c$ objects.

- [ ] The class of representable graphs has some nice propertie

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Touching representation by comparable boxes

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Axis-aligned boxes in $\mathbb{R}^d$
i.e. Cartesian product of closed intervals with non-zero length.

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intersecting interior-disjoint
objects correspond to edges
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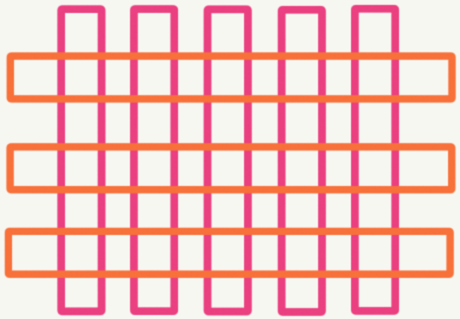

$K_{m,n}$ represented by boxes in $\mathbb{R}^3$

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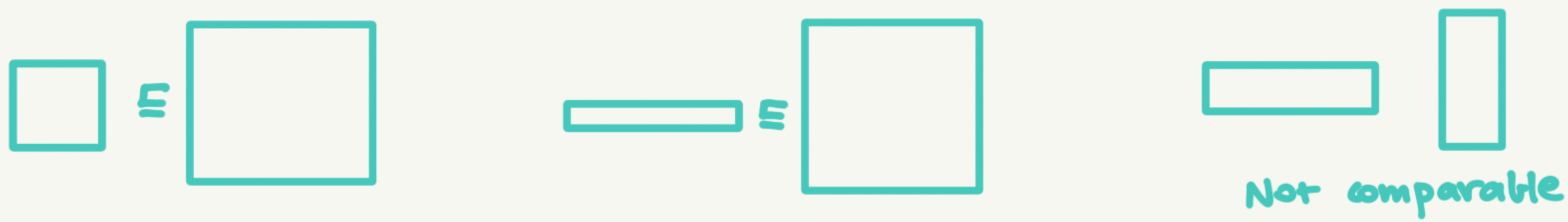
Touching representation by comparable boxes

If B_1 is a subset of a translate of B_2, then $B_1 \sqsubseteq B_2$
We say B_1 and B_2 are comparable if $B_1 \sqsubseteq B_2$ or $B_2 \sqsubseteq B_1$.



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Not comparable

$\dim_{cb}(G)$

Def^{n} : The comparable box dimension of a graph G is the smallest d s.t. G has a touching representation by comparable boxes in $\mathbb{R}^d$.

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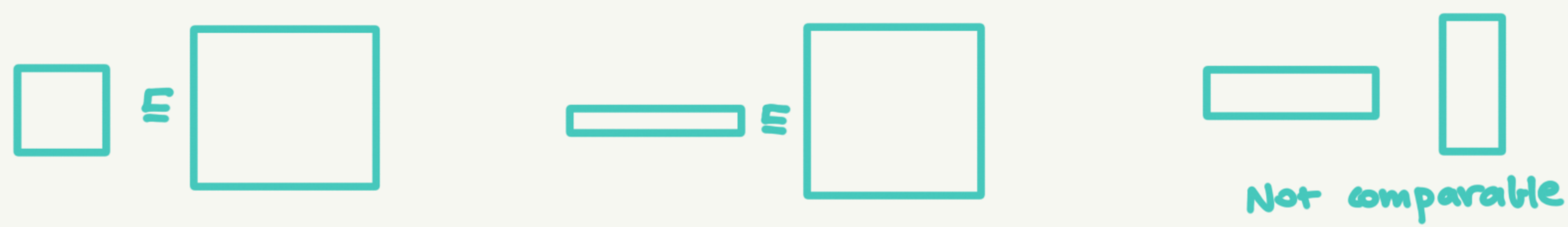

Def$^{\underline{n}}$: The comparable box dimension of a graph G $\xrightarrow{} \dim_{cb}(G)$ is the smallest d s.t. G has a touching representation by comparable boxes in $\mathbb{R}^d$.

For a class of graphs $\mathcal{C}$ we let $\dim_{cb}(\mathcal{C}) = \sup\{\dim_{cb}(G) : G \in \mathcal{C}\}$
(can be ∞)

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Properties of classes with bounded comparable box dimension

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(Dvořák, McCarty, Norin 2020)

- Polynomial strong colouring numbers

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  $\Downarrow$

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(Dvořák, Gonçalves, Lahiri, T., Ueckerdt 2021)

- Fractionally treewidth-fragile

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Thm (from Thomassen or Felsner & Francis)
If $\mathcal{C}$ is the class of planar graphs, then

$$\dim_{cb}(\mathcal{C}) = 3 .$$

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The comparable box dimension of every proper minor-closed class of graphs is finite.

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Our main result

Thm (Dvořák, Gonçalves, Lahiri, T., Ueckerdt 2021)

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$\underline{\dim}_{cb}$ vs some basic parameters

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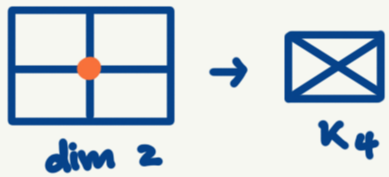

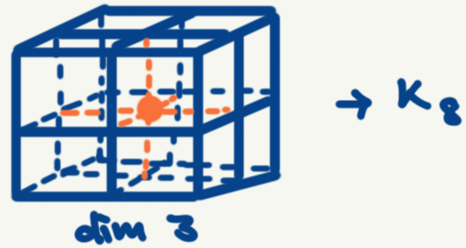

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Let f be a representation of G by comparable boxes in $\mathbb{R}^d$, $d = \dim_{cb}(G)$

Key idea: colour greedily in decreasing order of box volume.

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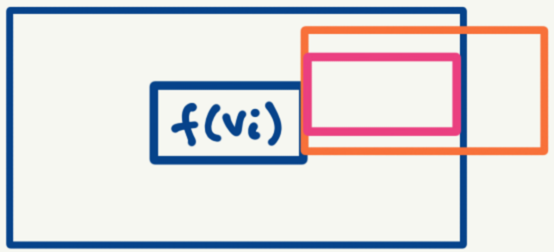

$B = 3 \times f(v_i)$

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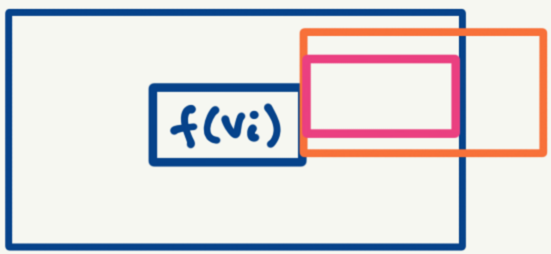

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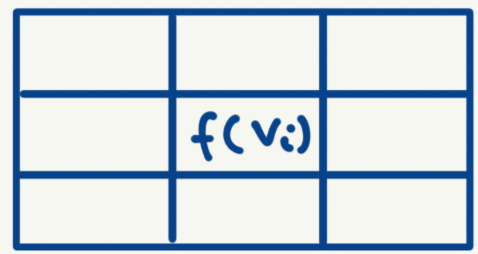

worst case

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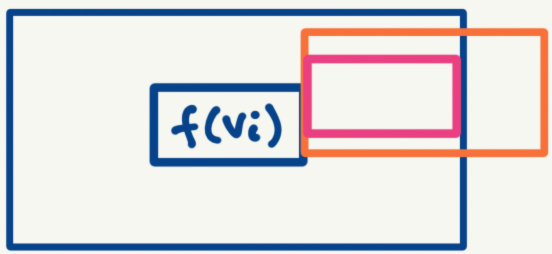

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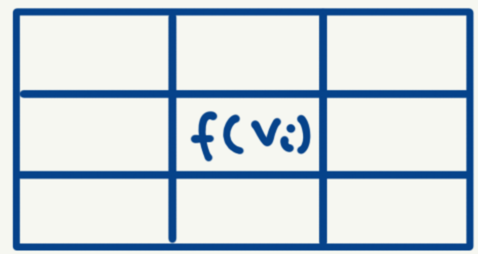

worst case

So # vertices v_j with $j < i$ s.t. $f(v_j)$ touches $f(v_i)$

(so $f(v_i) \leq f(v_j)$)

$$\leq \frac{\mathrm{vol}(B)}{\mathrm{vol}(f(v_i))} - 1 = 3^{d} - 1.$$

Hence at most 3^{d} colours are needed.

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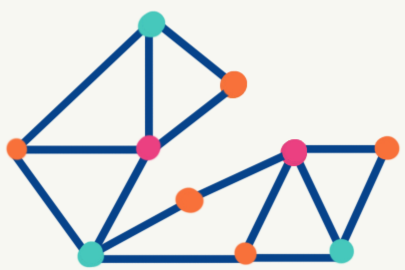
$$\chi_s(G) \leq 2 \cdot 9^{\dim_{cb}(G)}$$

Def$^{\underline{n}}$: A proper star colouring of a graph G is a proper colouring of G s.t. every path on 4 vertices has at least 3 colours.



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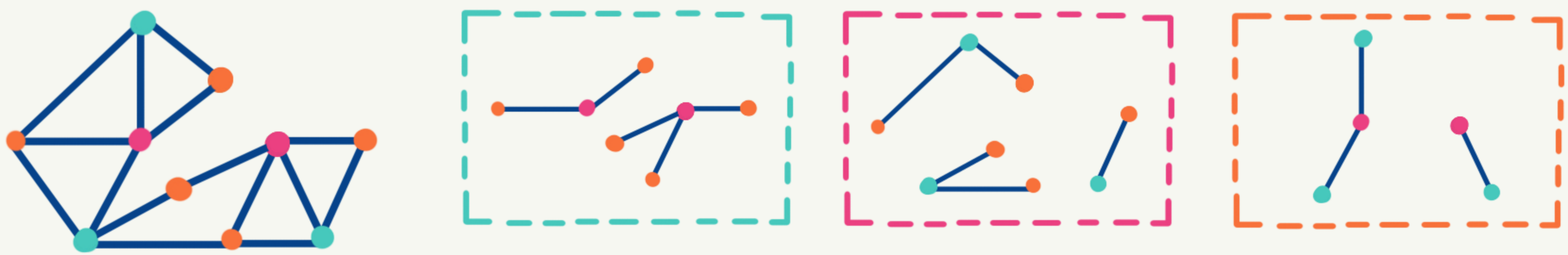

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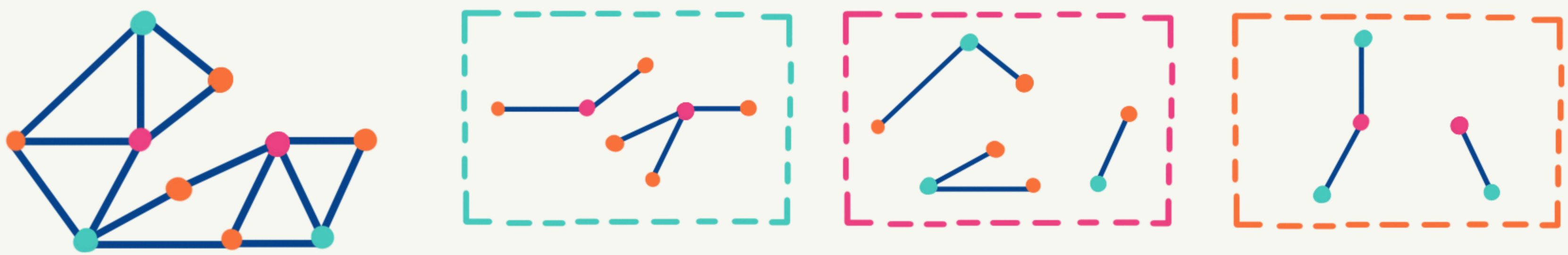

$\chi_s(G)$ = fewest # colours needed in a proper star colouring of G.

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③ Treewidth $\qquad \dim_{cb}(G) \le tw(G) + 1$

$\dim_{cb}$ vs some basic operations

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① Vertex addition

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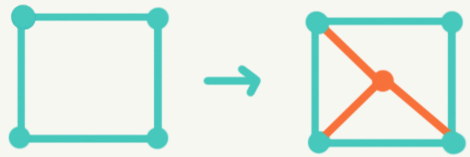

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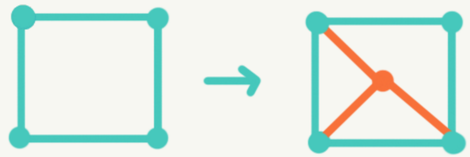

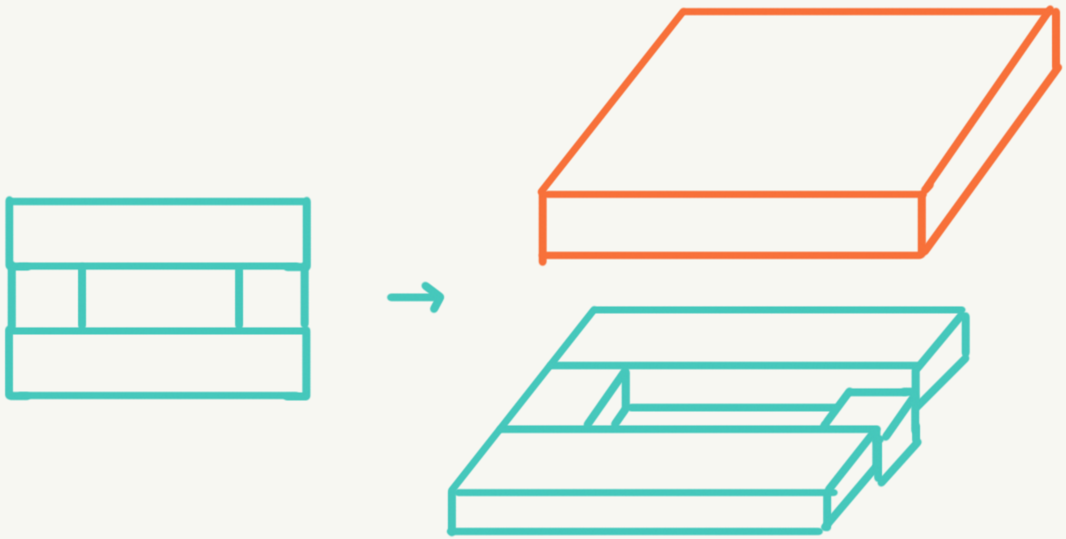

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② Induced subgraphs

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For any G and $H \subseteq G$ induced, $\dim_{cb}(H) \leq \dim_{cb}(G)$.

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③ Subgraphs

Aside on subgraphs

WARNING: When $H \subseteq G$, it is possible that $\dim_{cr}(H) > \dim_{cr}(G)$

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An exponential increase is possible

$$\dim_{cr}(K_{2d}) = d$$

$$\dim_{cr}(K_{2d} - \text{p.m.}) = 2^{d-1}$$

Aside on subgraphs

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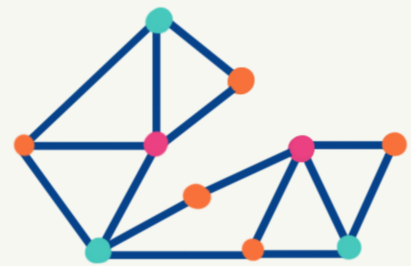

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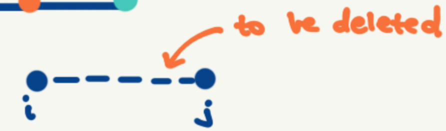

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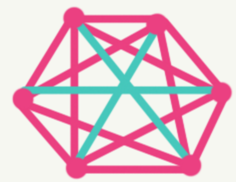

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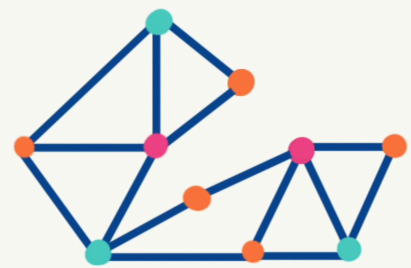

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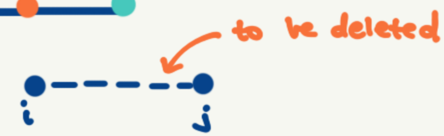

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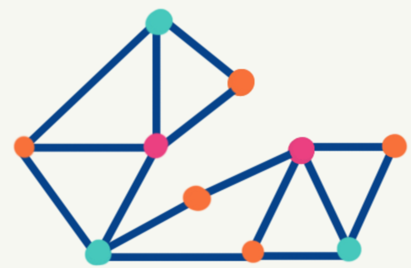

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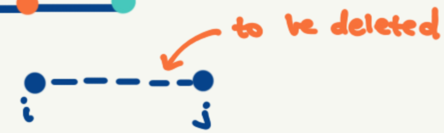

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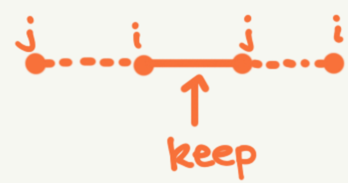

$\dim_{cb}$ vs some basic operations

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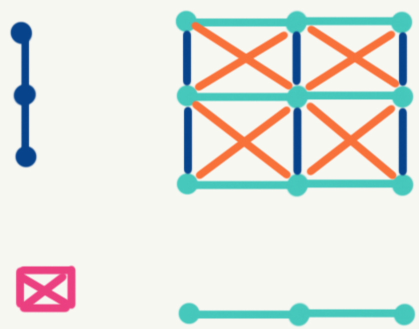

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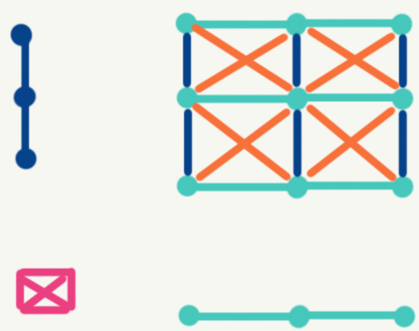

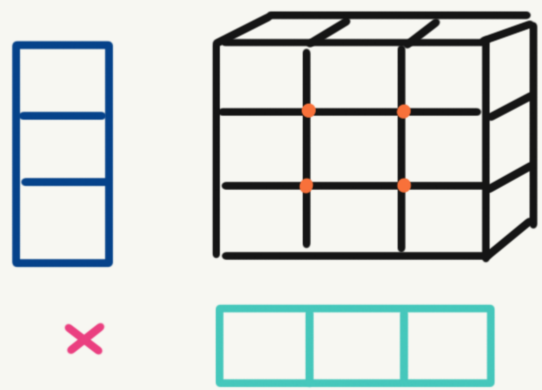

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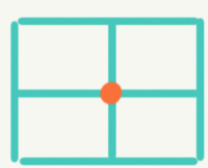

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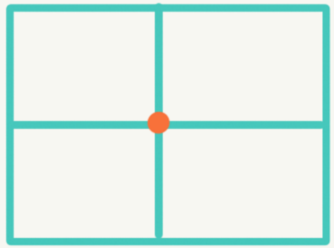

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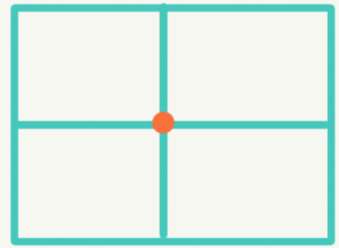

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A touching representation h of G in $\mathbb{R}^d$ is clique-sum extendable if :

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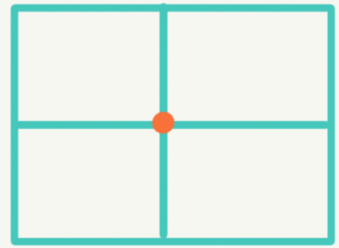

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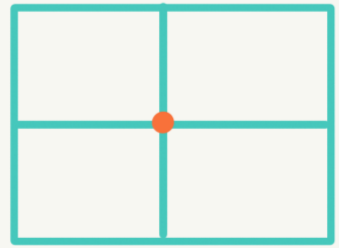

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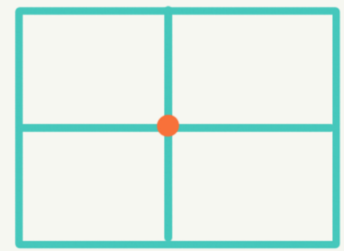

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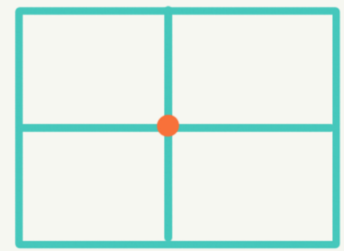

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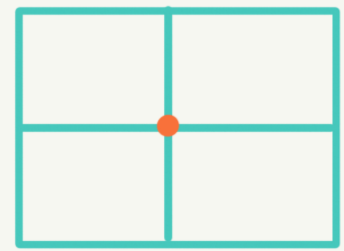

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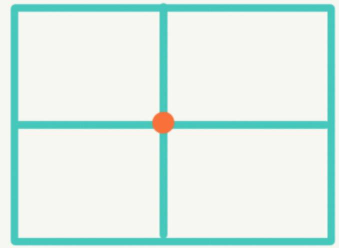

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- $\exists\, \varepsilon$ s.t $h^{\varepsilon}(C)$ $\big([p(C)[i], p(C)[i] + \varepsilon]\big)$ intersects a box $h(v) \Leftrightarrow v \in V(C)$ and their intersection is a facet of $h^{\varepsilon}(C)$ incident to $p(C)$.

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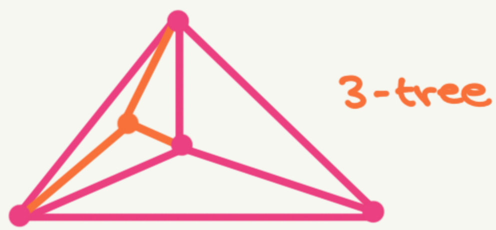

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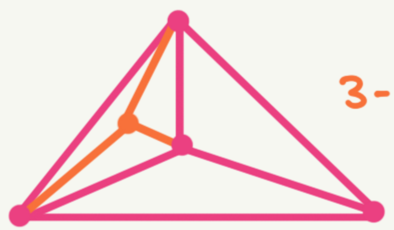

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