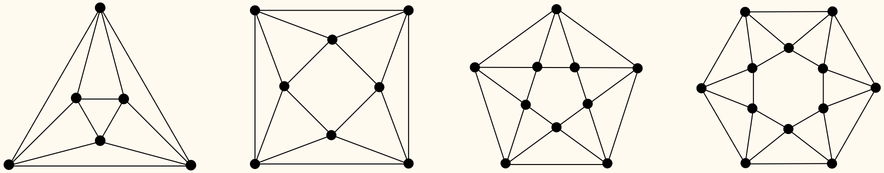
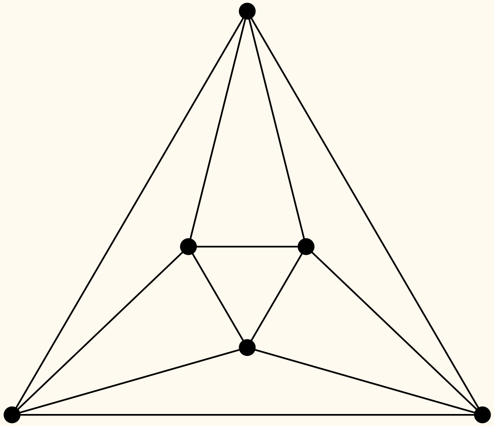
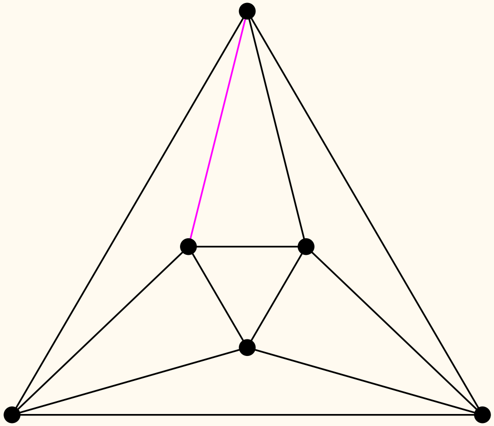
Eulerian circuits and path decompositions in quartic planar graphs

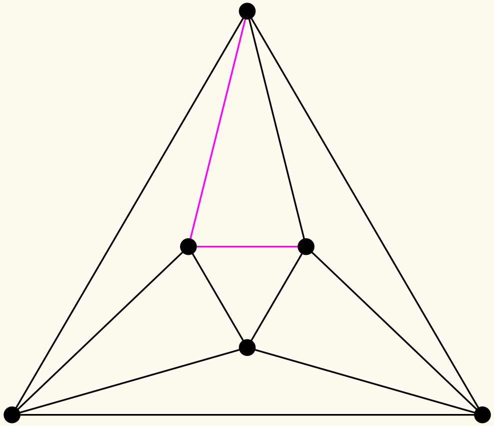
Jane Tan
University of Oxford

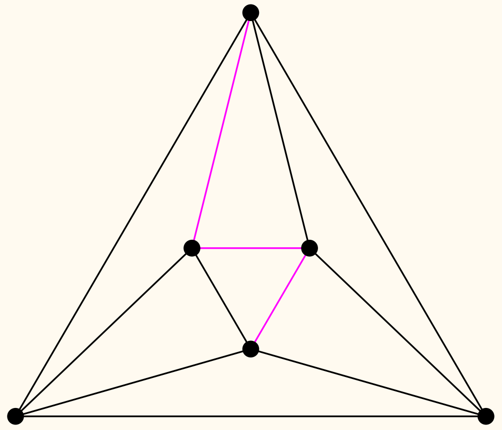


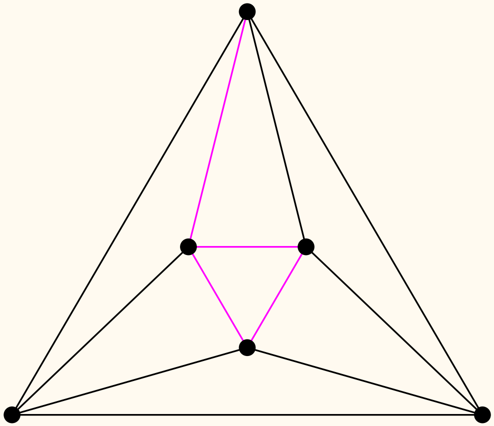
42ACCMCC, December 2019

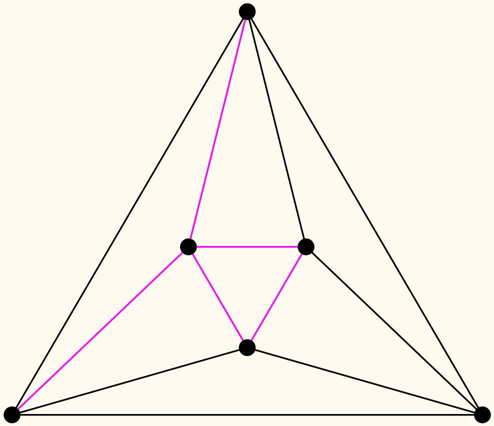


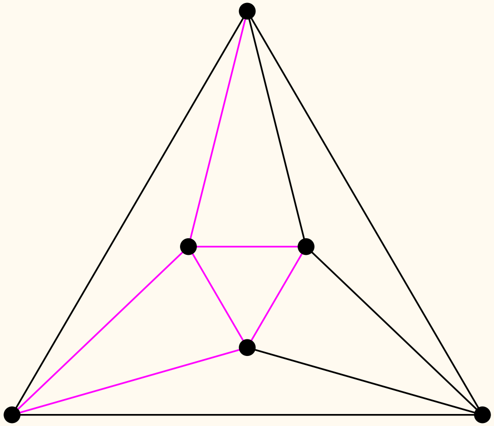


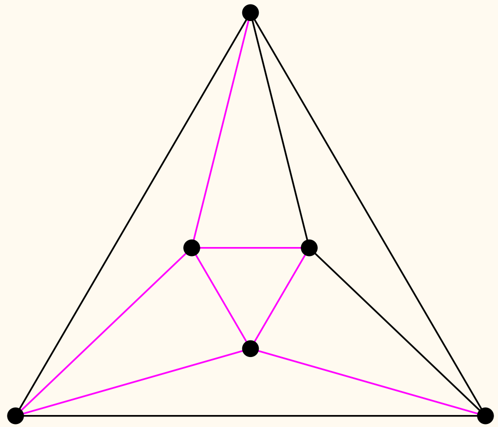


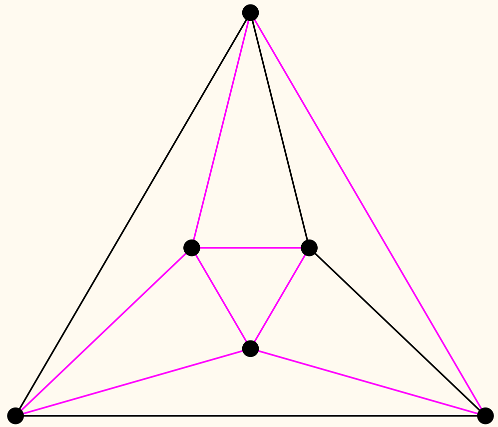


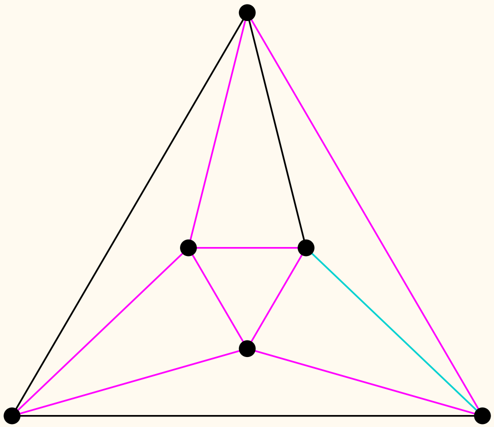


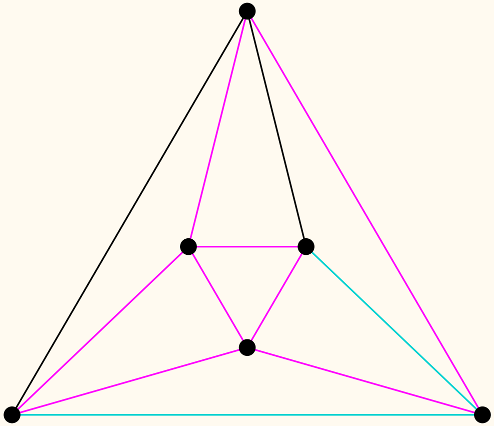


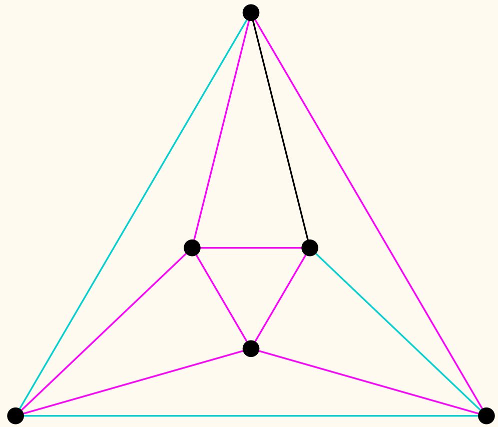


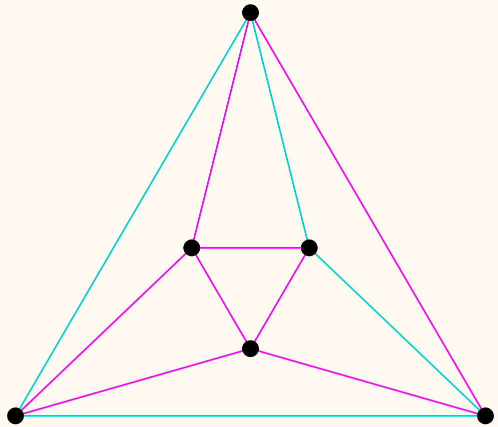












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- (Le, 2019) High minimum degree is sufficient to guarantee a k-lsa EC

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Let $P_{\ell+1}$ be the path of length ℓ (on $\ell + 1$ vertices)

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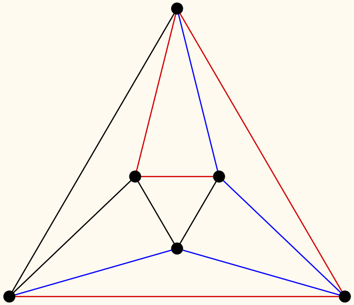

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- ℓ-Isa Eulerian circuit + divisibility guarantees a $P_{\ell+1}$-decomposition

Main results

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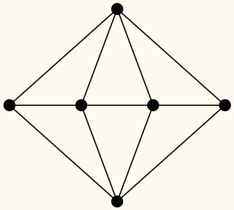

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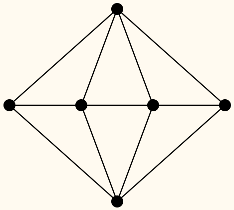

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Corollary. *Every connected quartic planar of even order that does not contain* F_6 *admits a* P_5*-decomposition.*

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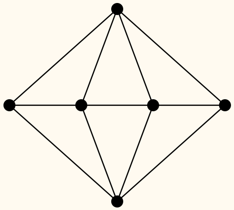

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Corollary. *A connected quartic planar graph admits a* P_5*-decomposition iff it has even order.*

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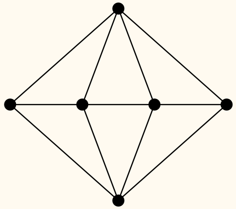

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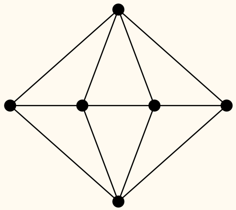

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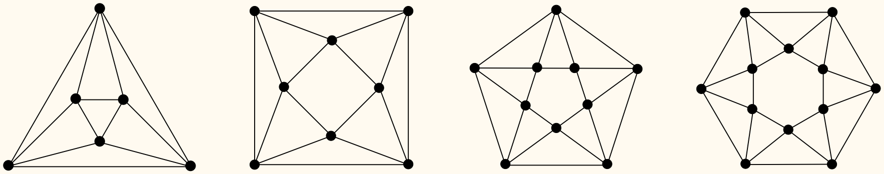

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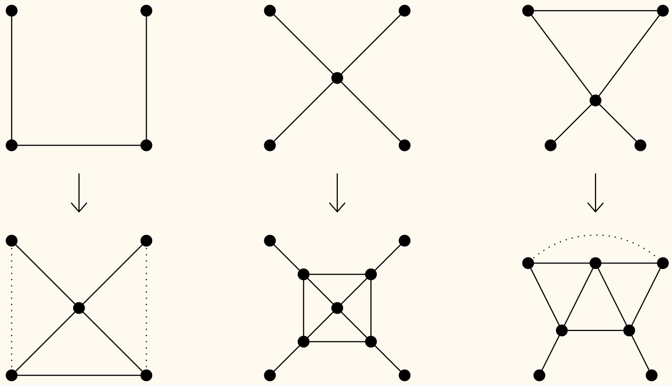

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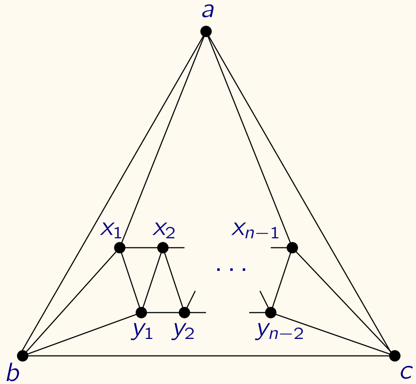

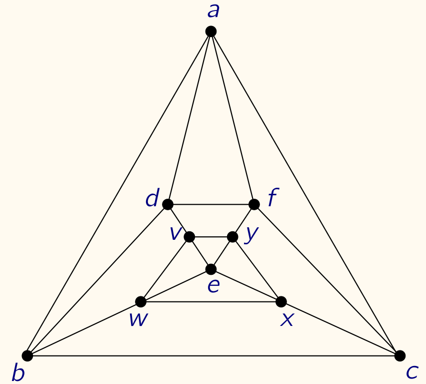

$abwevyxcbdafyexwvdfca$

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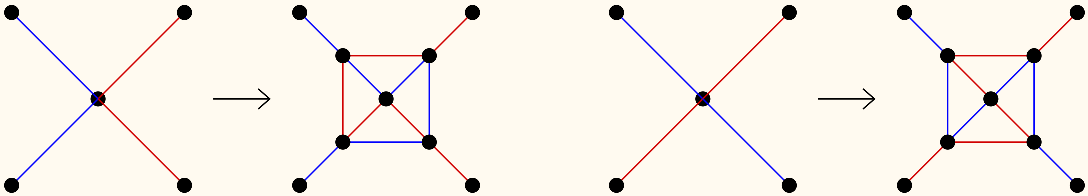

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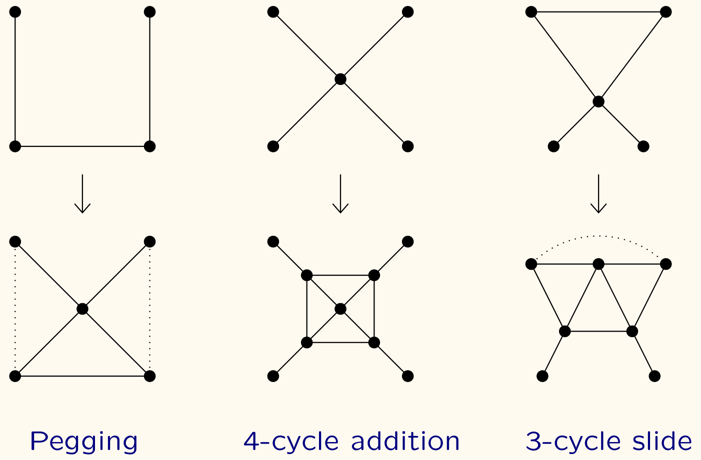

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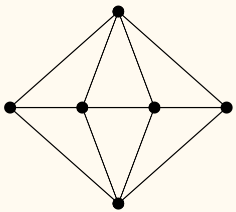

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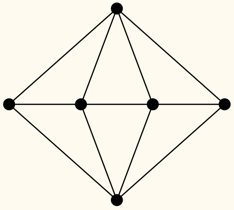

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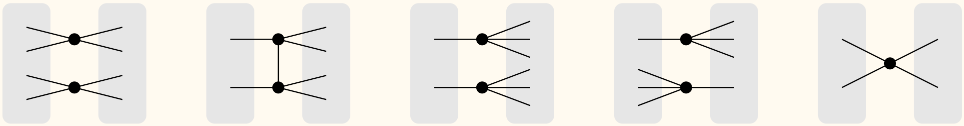

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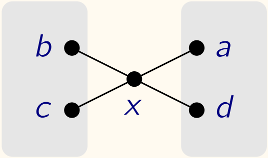

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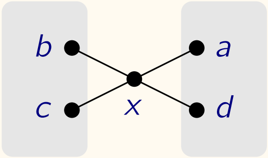

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Theorem. *A connected quartic planar graph has a 4-locally self-avoiding Eulerian circuit iff it does not contain* $F_6 := \overline{P_2 \cup P_4}$ *as a subgraph.*

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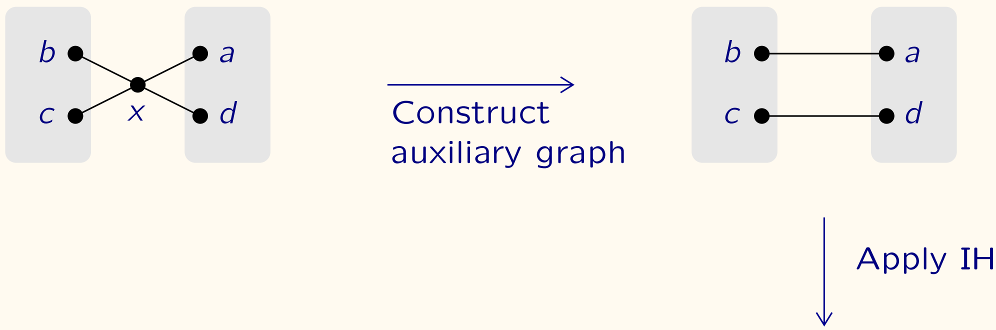

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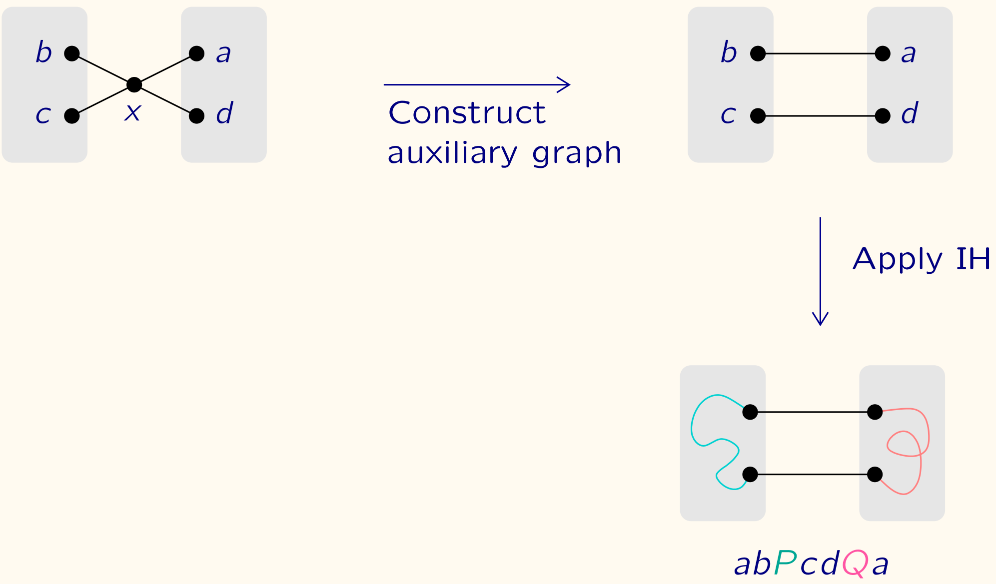

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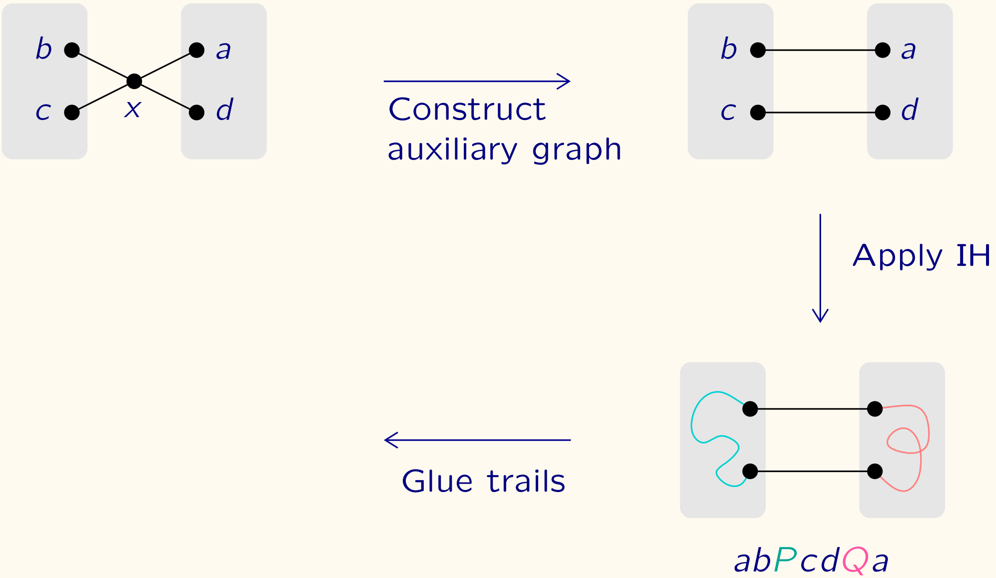

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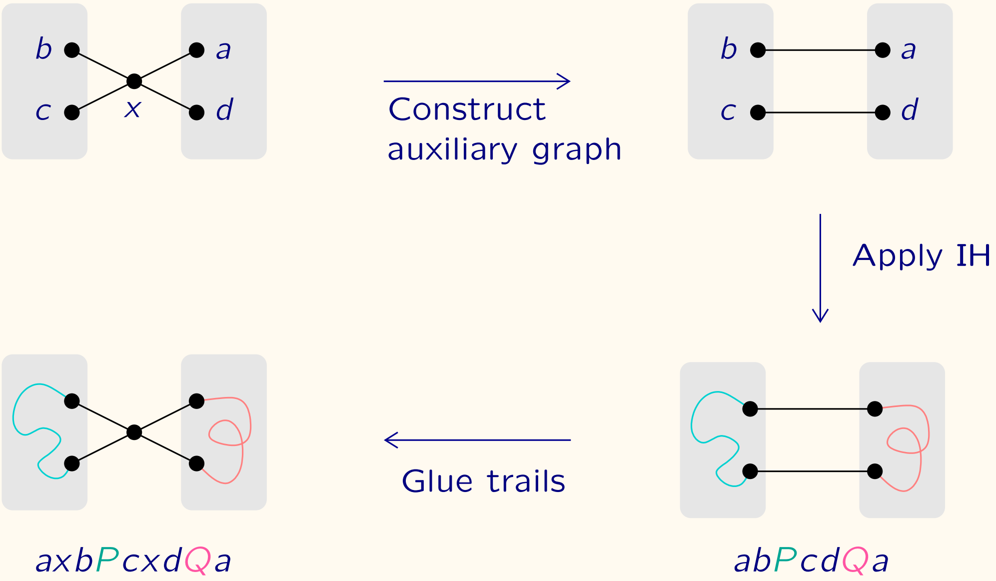

Outline

Theorem. *A connected quartic planar graph has a 4-locally self-avoiding Eulerian circuit iff it does not contain $F_6 := \overline{P_2 \cup P_4}$ as a subgraph.*

$\uparrow$

2-connected . . . does not contain F_6

$\uparrow$

3-edge-connected . . . is not the octahedron

$\uparrow$

3-connected . . . is not the octahedron