Reconstructing trees from small cards

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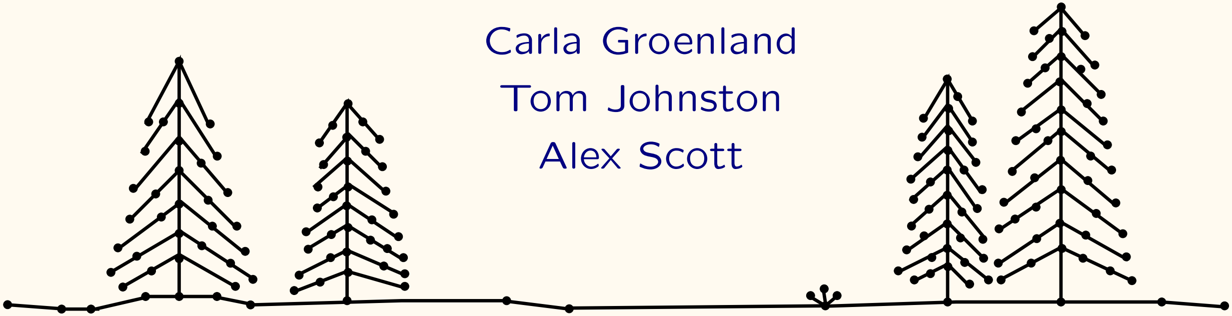
Joint with:

Carla Groenland

Tom Johnston

Alex Scott



BCC 8th July 2021

Classical reconstruction

Definition. *The deck* $\mathcal{D}(G)$ *of a graph* G *is the multiset of vertex-deleted subgraphs of* G*. i.e.* $\mathcal{D}(G) = \{G - v : v \in V(G)\}$ *(multiset!)*

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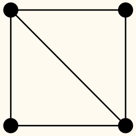

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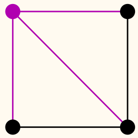

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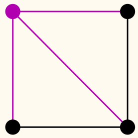

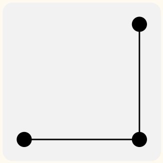

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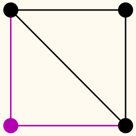

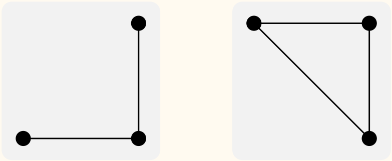

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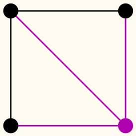

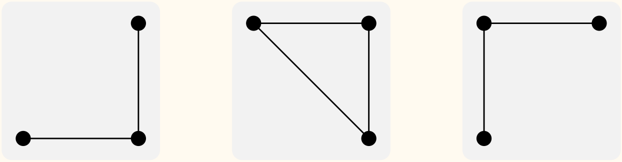

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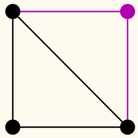

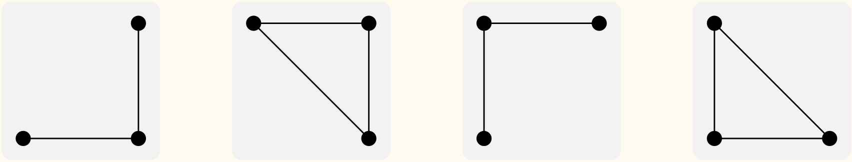

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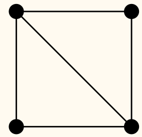

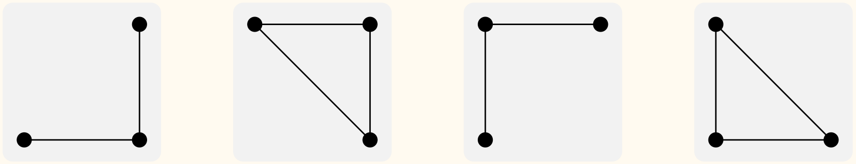

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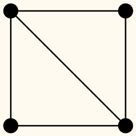

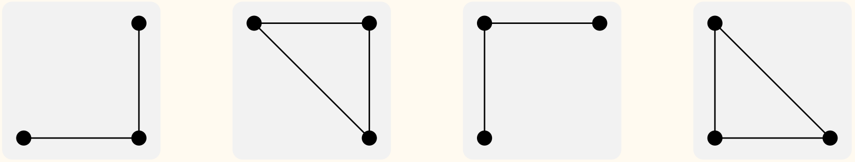

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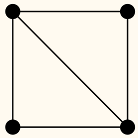

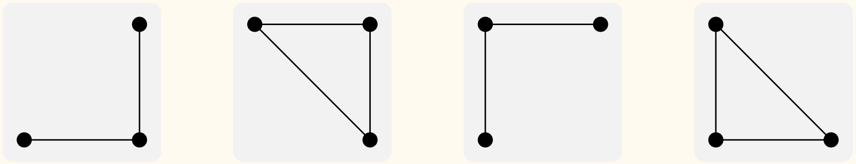

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There are many partial results reconstructing parameters and classes such as:

- # vertices
- # edges, degree sequence, regular graphs, subgraph counts, connectedness, disconnected graphs, trees (Kelly 1942, 1957)
- connectivity, unicyclic graphs (Manvel 1969, 1976)
- Tutte poly, chromatic poly, characteristic poly (Tutte 1967, 1979)
- outerplanar graphs (Giles 1974) maximal planar graphs (Fiorini, Lauri 1981)
- planarity (Bilinski, Kwon, Yu 2006)

Also verified for graphs on up to 13 vertices (McKay 2021).

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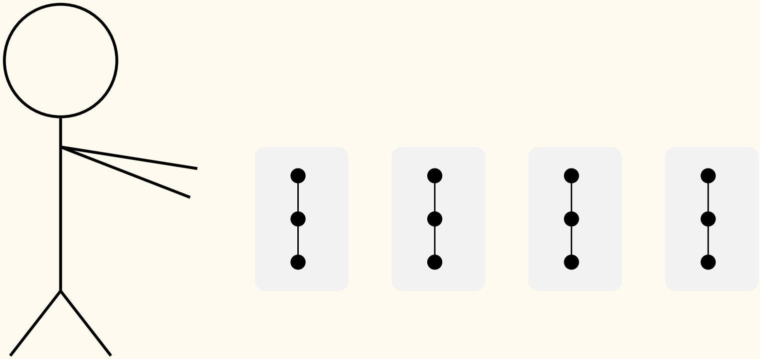

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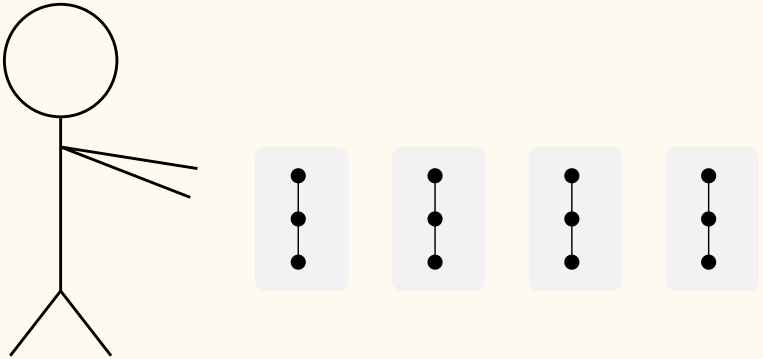

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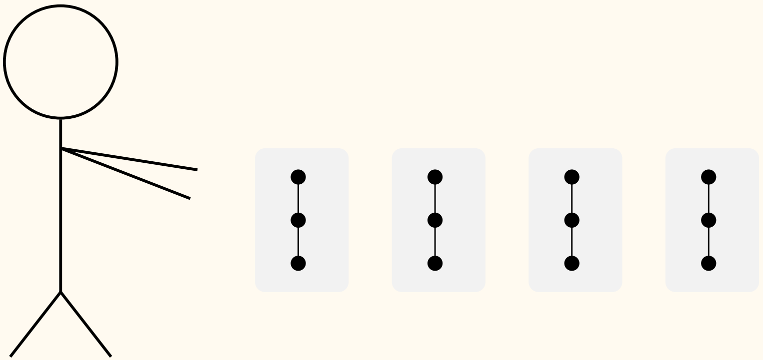

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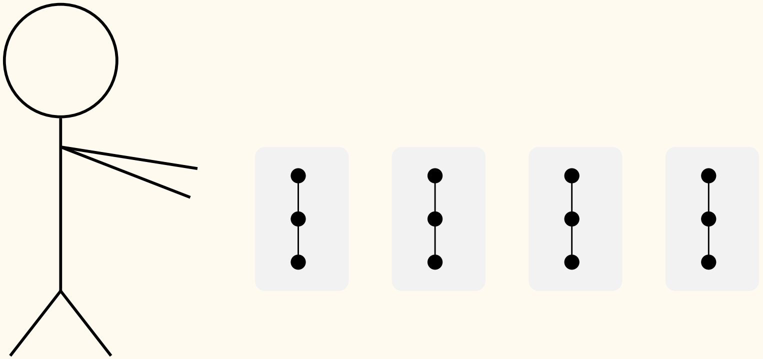

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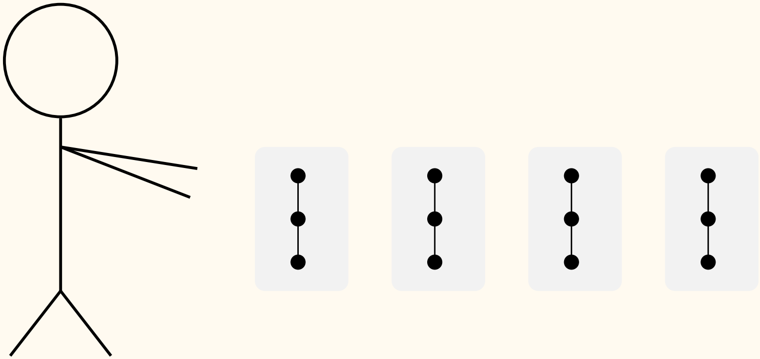

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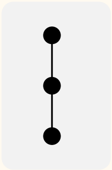

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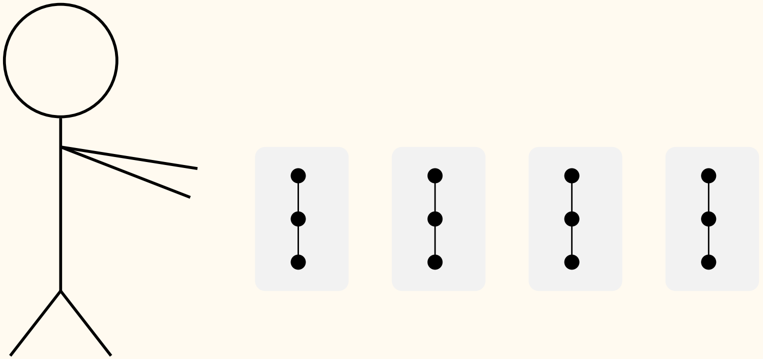

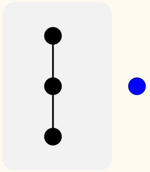

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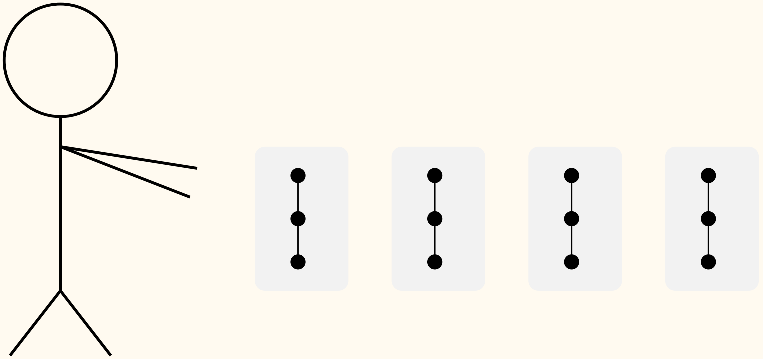

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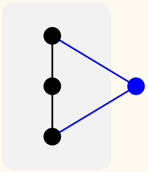

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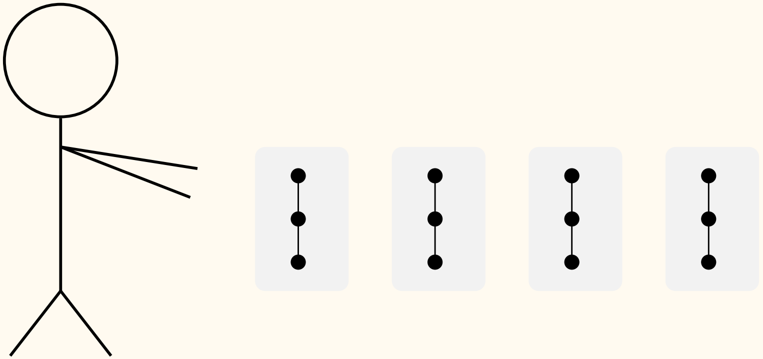

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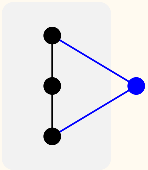

So $G \cong C_4$!

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(Kostochka, Nahvi, West, Zirlin) 3-regular graphs are 2-reconstructible.

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(Greenwell-Hemminger)
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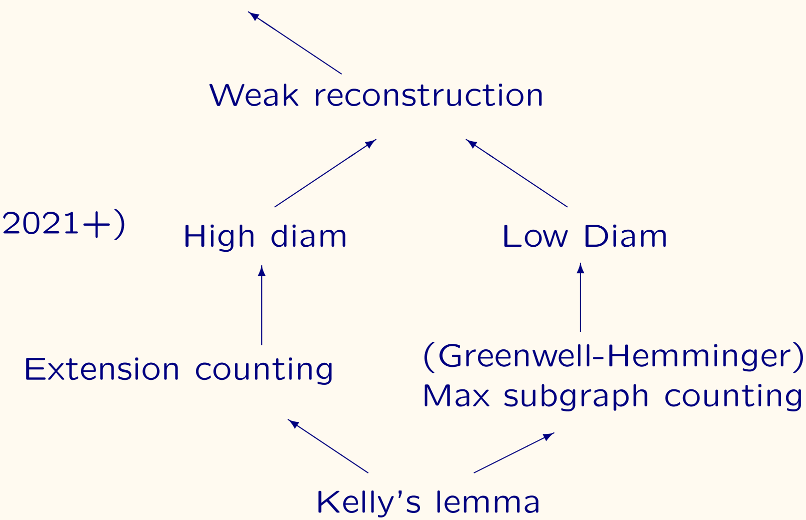

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Kelly's lemma

Proof outline - low diameter

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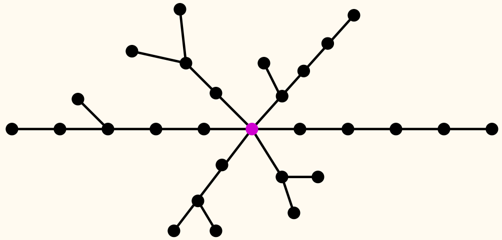

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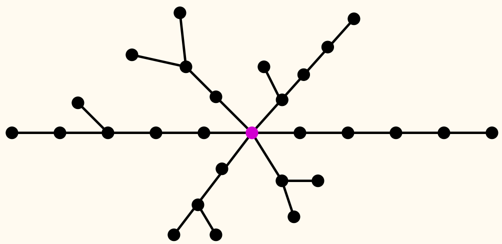

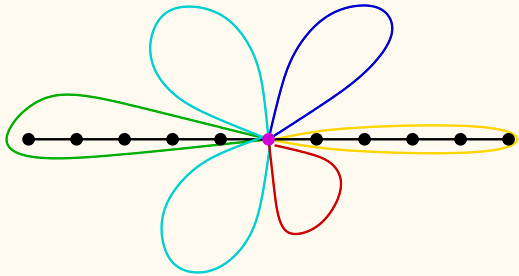

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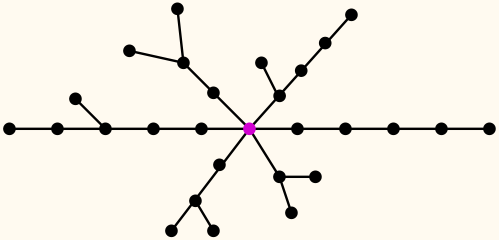
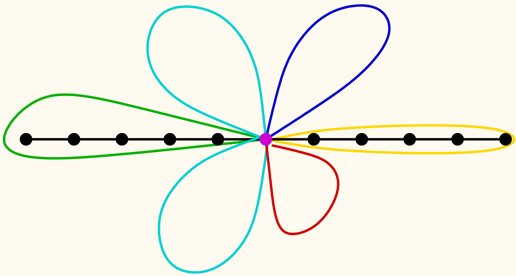
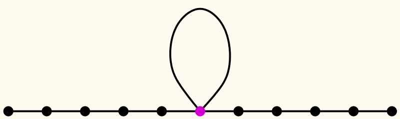
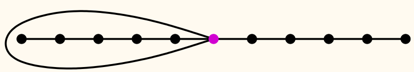




If there is only one P_{k+1}, we can reconstruct branches off the centre by counting maximal copies of

 and 

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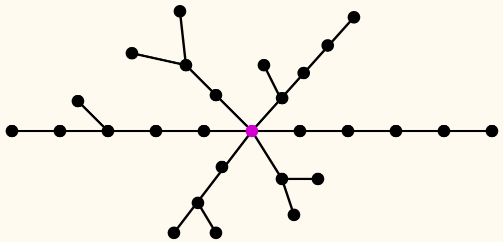

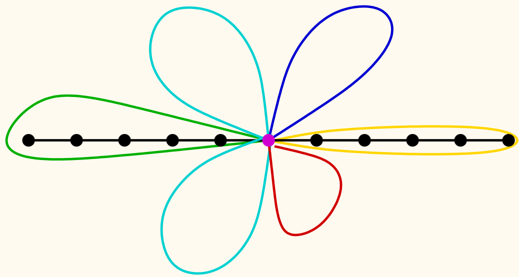

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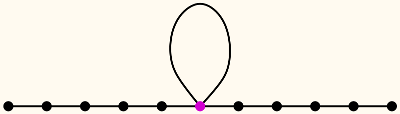

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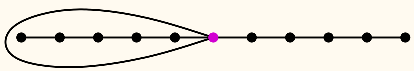

Essence of Lemma (Greenwell-Hemminger). *We can reconstruct the number of maximal copies of these two subgraph types from $D_\ell(G)$ provided the whole subgraph is small enough to be seen on a single card.*

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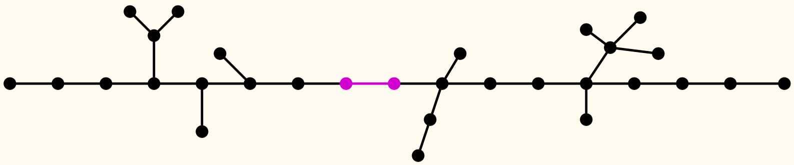

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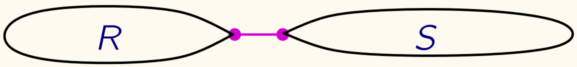

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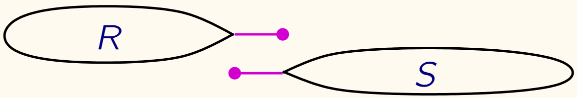

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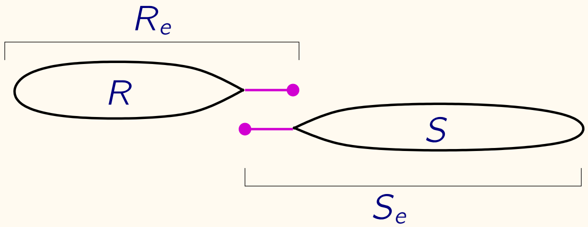
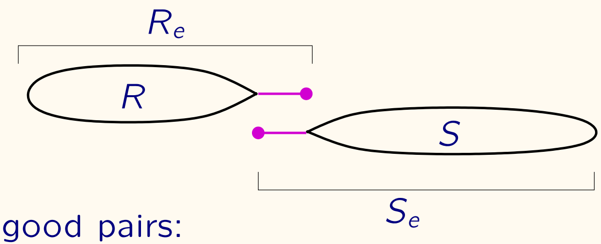


A good pair of leaf extensions

Proof outline - high diameter



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To find candidates for good pairs:

- List subtrees $R \subset T$ s.t. the neighbourhood around at least one copy of R in T contains only extra one vertex and one edge.
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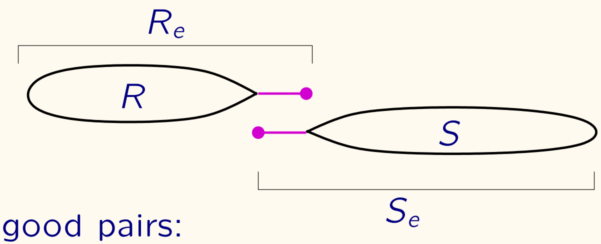
Proof outline - high diameter



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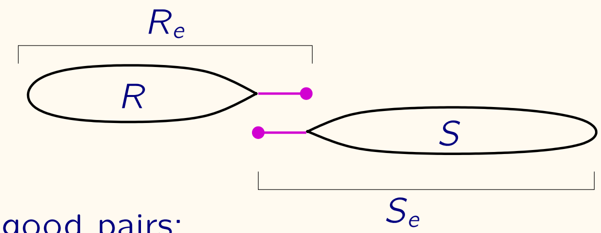

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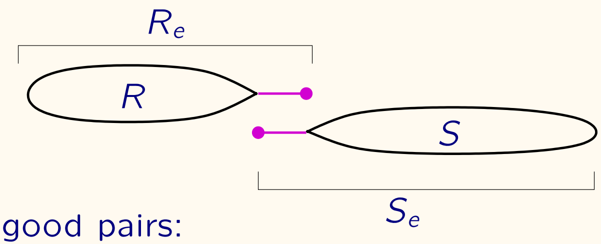

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Essence of Lemma (Extension-counting)**.** *We can count subtrees R whose 1-nbhd has exactly one extra edge and vertex (i.e. R_e) provided all nbhds of copies of R are small enough to fit on a single card.*

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Two non-isomorphic trees on 13 vertices with the same 7-deck:

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- Many other reconstruction variants...