Prospects in Mathematics, Oxford 2014

Research in Partial Differential Equations

John Ball

University of Oxford

Plan

1. Overview of research in PDE
2. Singularities and material defects (largely pictures)
3. The UK PDE scene

Overview of research in PDE

(according to MathSciNet the area in which the largest number of mathematicians work, and one which uses many different parts of mathematics, algebra, analysis, geometry, topology, numerical analysis ... as well as having connections with all branches of science)

Examples of PDE (deterministic)

1. *Laplace's* equation for $u = u(x)$, where $x \in \Omega \subset \mathbb{R}^n$,

$$\Delta u = 0,$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. If Ω is bounded we typically want to solve the equation subject to boundary conditions such as

$$u|_{\partial\Omega} = g,$$

where g is a given function. If $\Omega = \mathbb{R}^n$, say, we instead need conditions at infinity. This is the prototypical elliptic equation.

2. The *heat* or *diffusion* equation for $u = u(x,t)$

$$u_t = \Delta u.$$

Here as well as boundary conditions we need to prescribe an initial condition

$$u(x,0) = u_0(x),\ x \in \Omega.$$

This is the prototypical parabolic equation.

3. The *wave* equation for $u = u(x,t)$

$$u_{tt} = \Delta u,$$

for which we need two initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega.$$

This is the prototype hyperbolic equation.

These are *linear* PDE, which can be solved by writing down their solutions.

But most PDE arising in applications to science and other branches of mathematics are *nonlinear*. Linear PDE usually arise from more fundamental nonlinear PDE by linearization.

While there are a number of general methods for handling nonlinear PDE (e.g. the calculus of variations, weak convergence, integral estimates, degree theory ...) there is no general theory and each equation has to be treated individually.

4. *Navier-Stokes equations*

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \nu \Delta \mathbf{v} - \nabla p, \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned}$$

where the kinematic viscosity $\nu > 0$.

($1 million for proving – or disproving – that solutions are smooth for all time.)

5. *Elastodynamics*

$$\rho \mathbf{y}_{tt} = \mathrm{Div}\, D_{\mathbf{A}} W(D\mathbf{y}),$$

where $\mathbf{y} = \mathbf{y}(x,t)$ is the deformed position of a material point x, ρ is the density and $W = W(\mathbf{A})$ is the free-energy function of the material.

e.g. for rubber a simplified model is

$$W(\mathbf{A}) = \mu|\mathbf{A}|^2 + h(\det \mathbf{A}),$$

with h convex.

6. *Ricci flow*

$$\partial_t g_{ij} = R_{ij},$$

where g_{ij} is the metric tensor of a Riemannian manifold and R_{ij} is the Ricci tensor (the trace of the Riemann curvature tensor). The study of solutions of this equation was a key tool in the 2006 proof by Perelman of the Poincaré conjecture.

7. *Einstein equations*

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = \frac{8\pi G}{c^4} T_{ij},$$

where Λ is the cosmological constant and T the energy-momentum tensor.

Generic questions

1. How to define solutions (including choice of function space).
2. Do solutions exist?
3. Are they smooth, or do they have singularities?
4. How do they depend on parameters?
5. Asymptotic limits (as $t \to \infty, |x| \to \infty$, parameters tend to special values).

A cautionary tale is the zero dispersion limit $\varepsilon \to 0$ of KdV

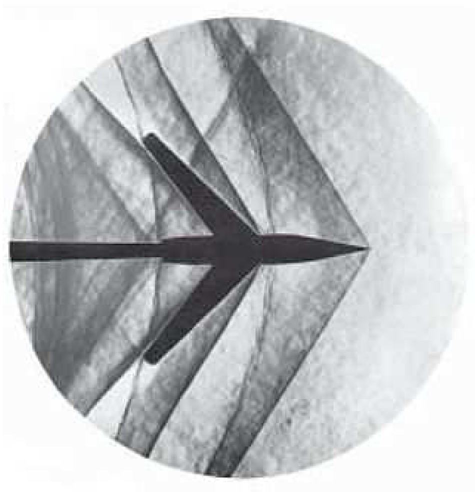
$$u_t + uu_x + \varepsilon u_{xxx} = 0$$

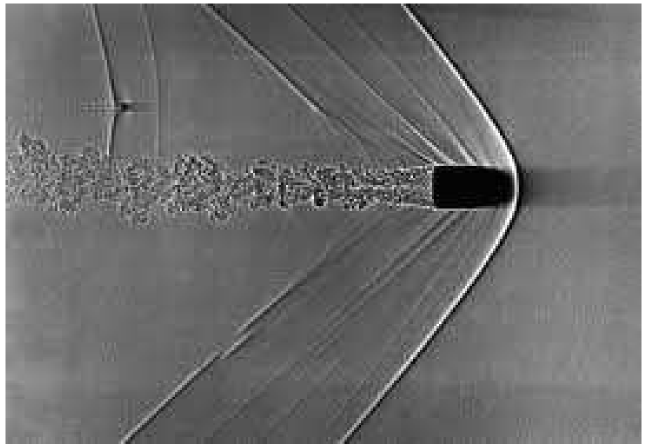
which was shown by Lax/Levermore 1979 NOT to be $u_t + uu_x = 0$.

Singularities and defects

Examples of defects in materials

Fluids

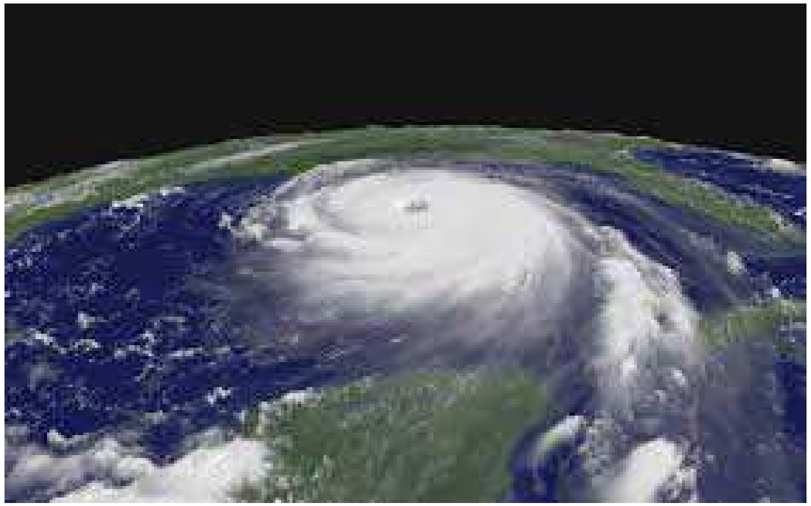




NASA

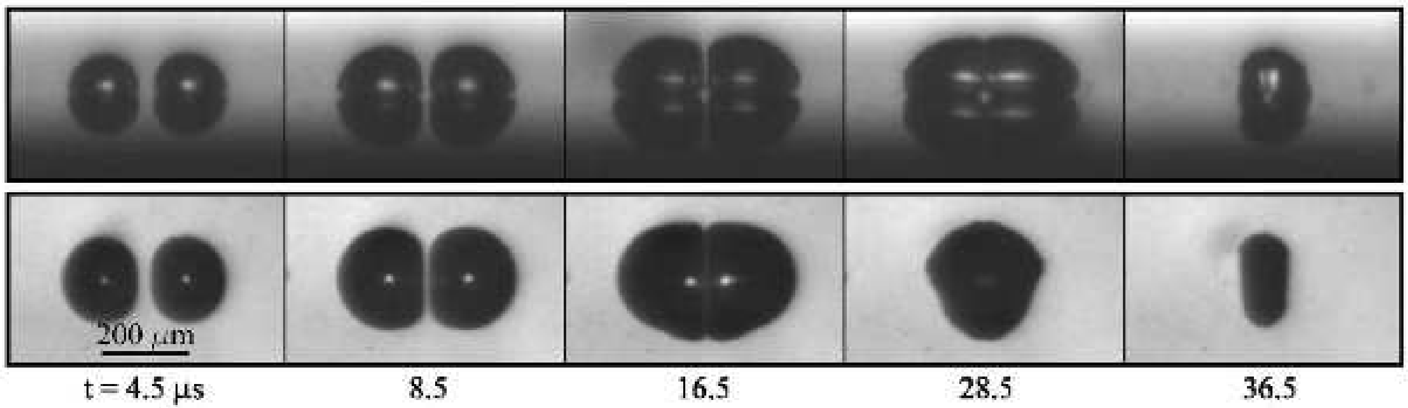
Shock waves - discontinuities in fluid velocity and other quantities

Vortices



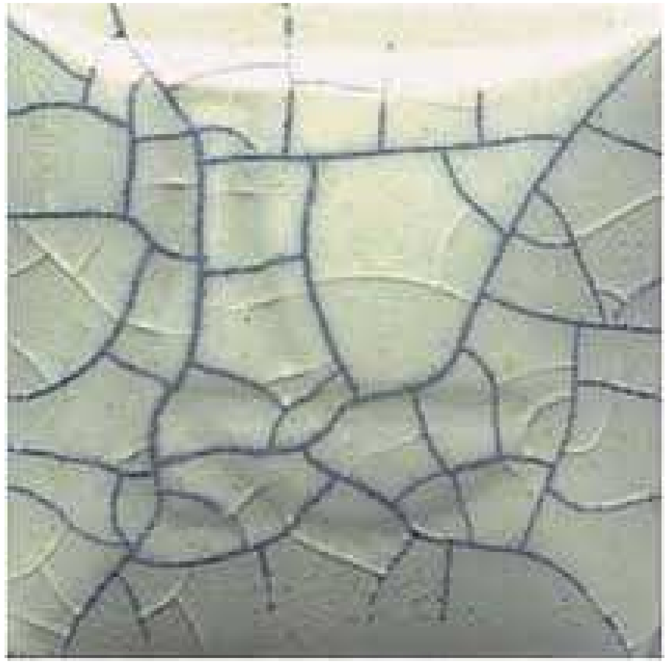
$\rightarrow$ turbulence

Cavitation



Nicolas Bremond, ESPCI, Paris

Solids



Cracks and fracture

Staffa,
Scotland

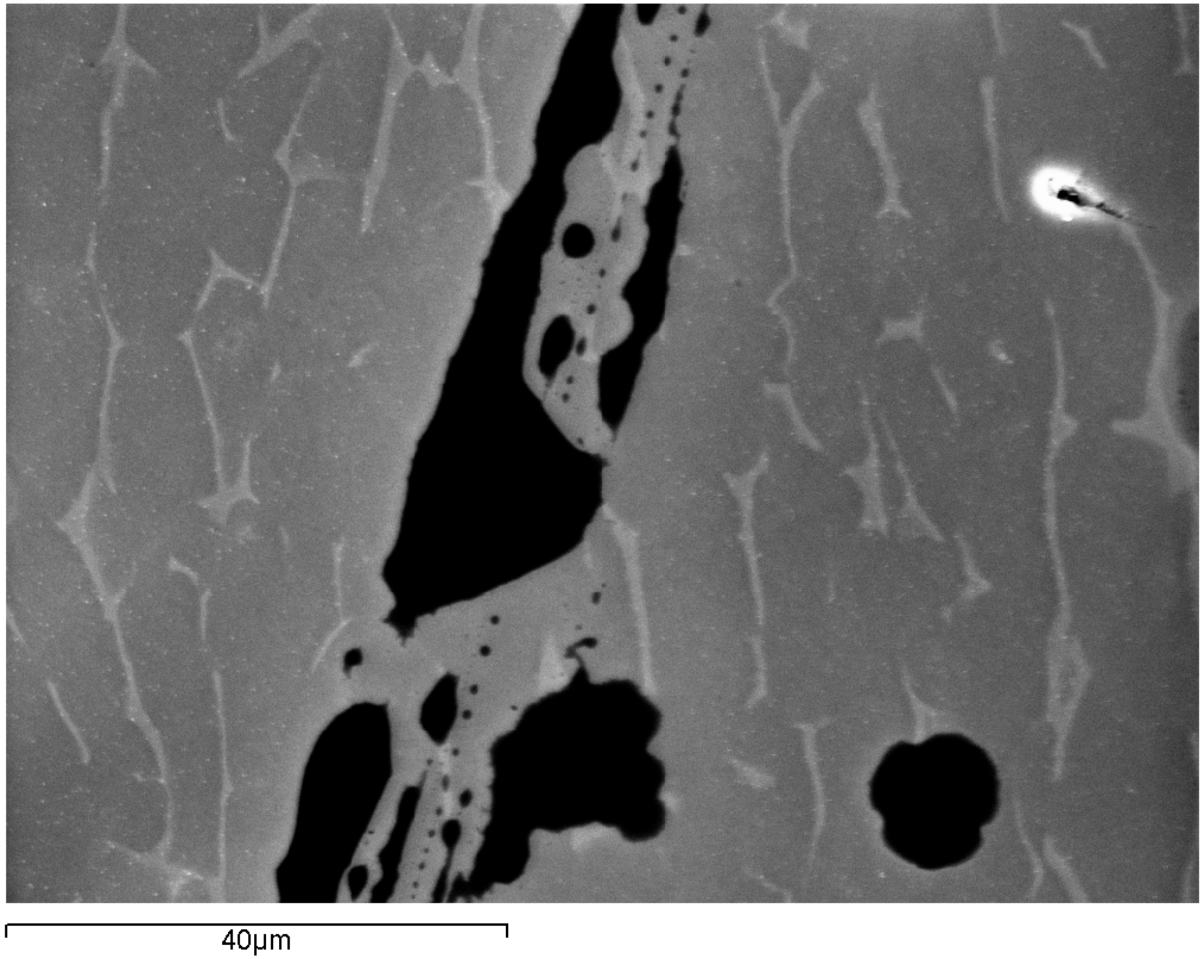
Giant's Causeway
Northern Ireland

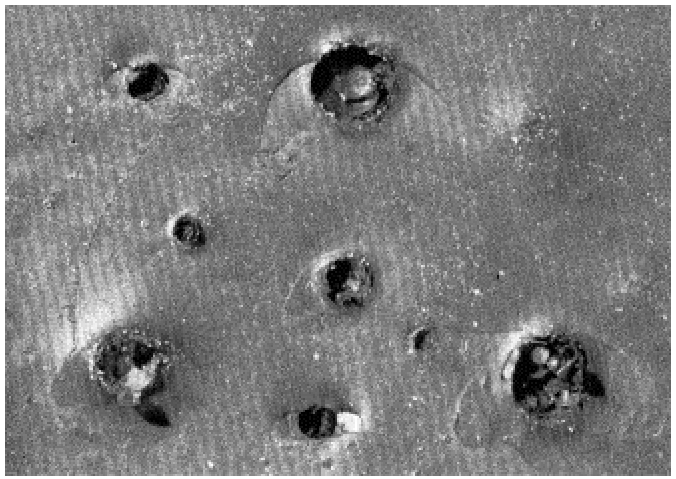
Columnar jointing in basalt

Cavitation

in TiAlV alloy

Petrinic,N., Curiel Soza,J.L.,
Siviour,C.R., Elliott,B.C.F.:
J. Phys. IV France 134 (2006)

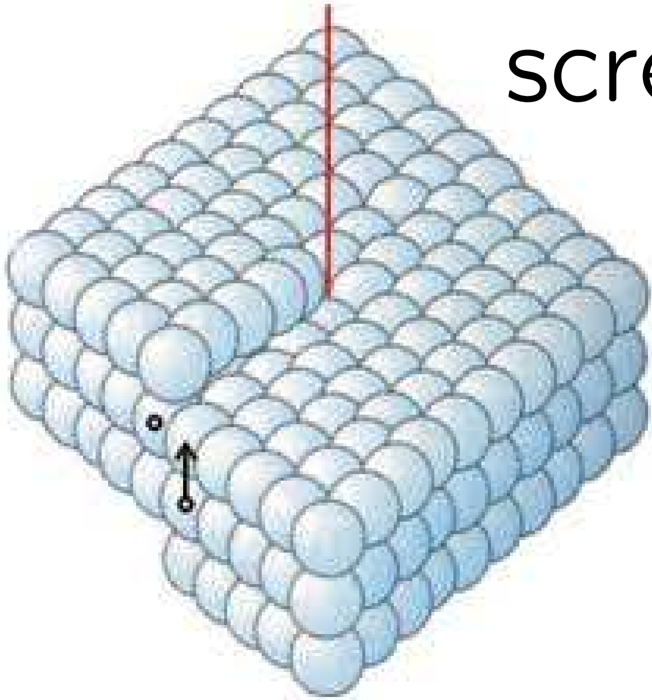




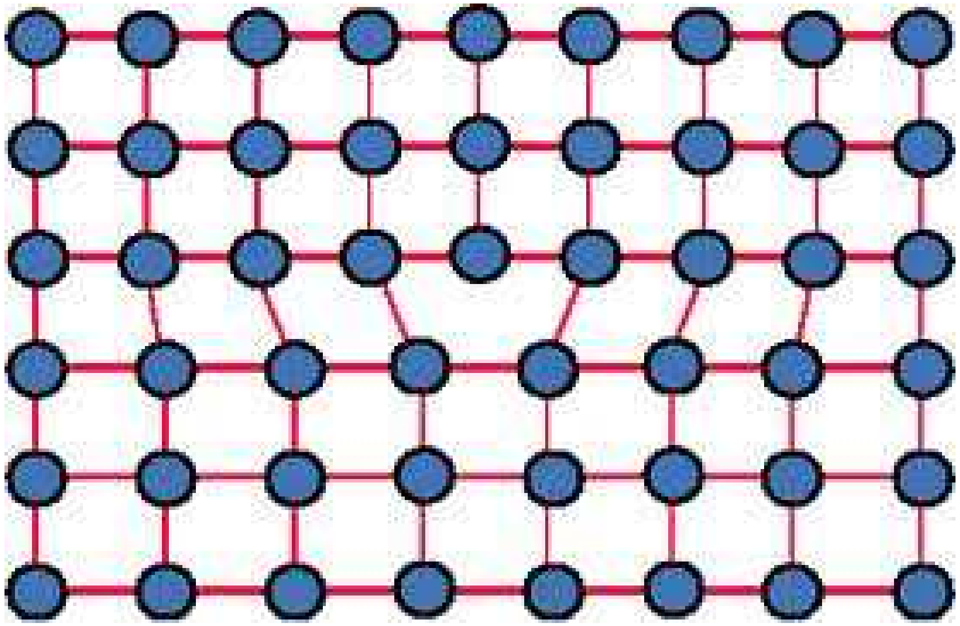
in rubber

South et al, Mechanics of Materials 2002

Dislocations in crystals

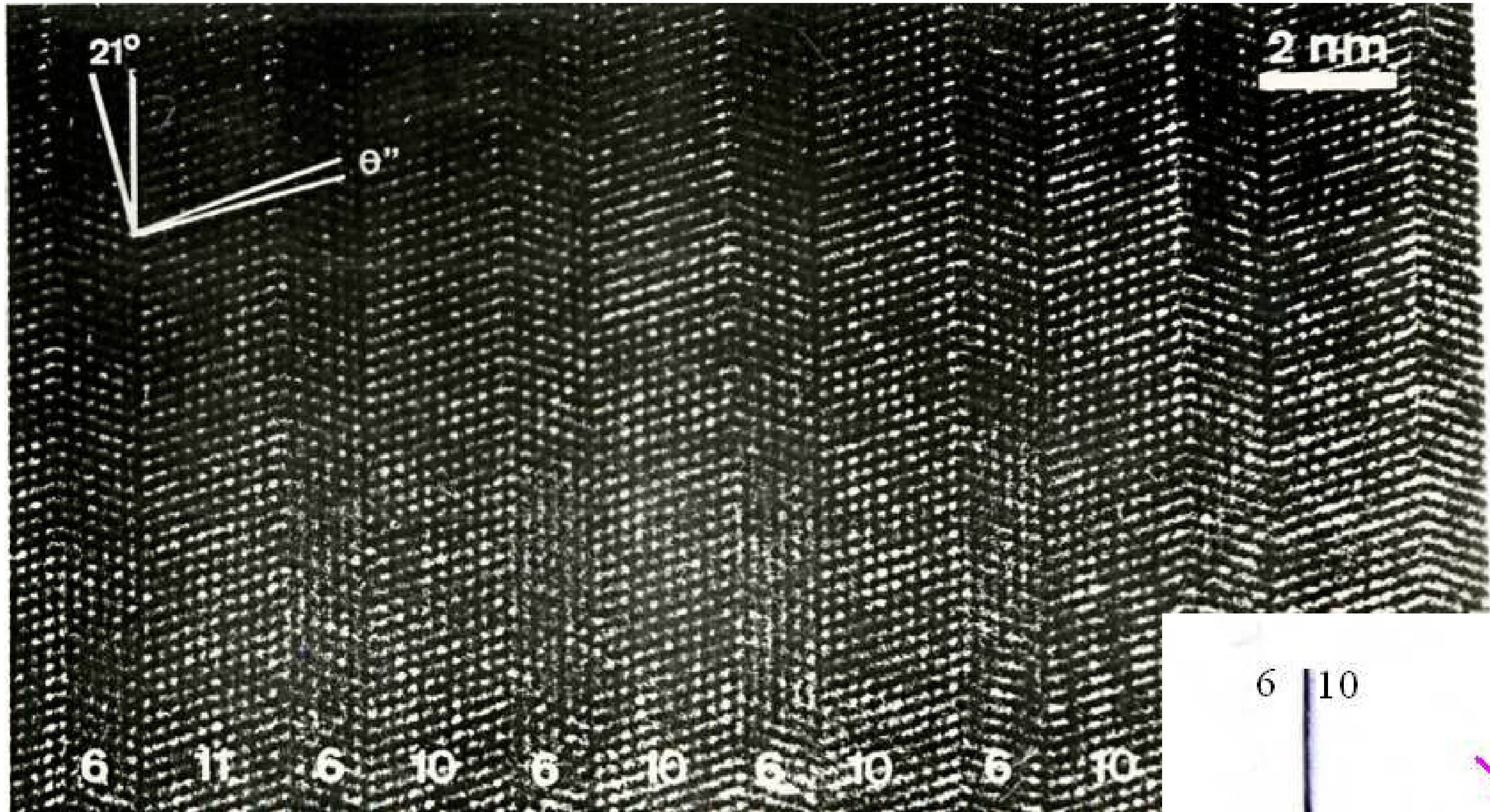


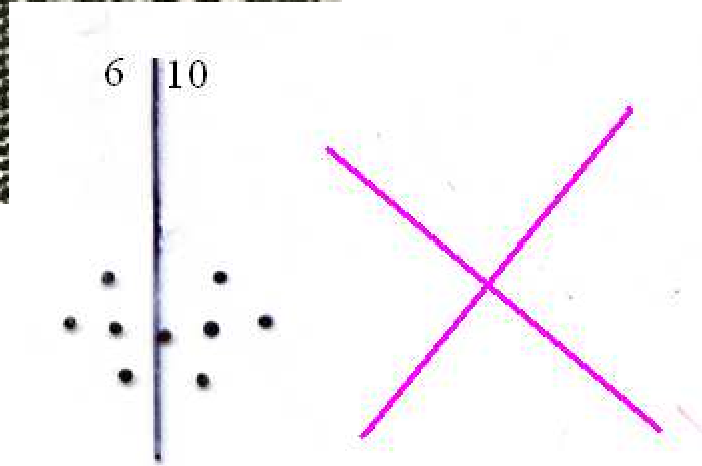
screw



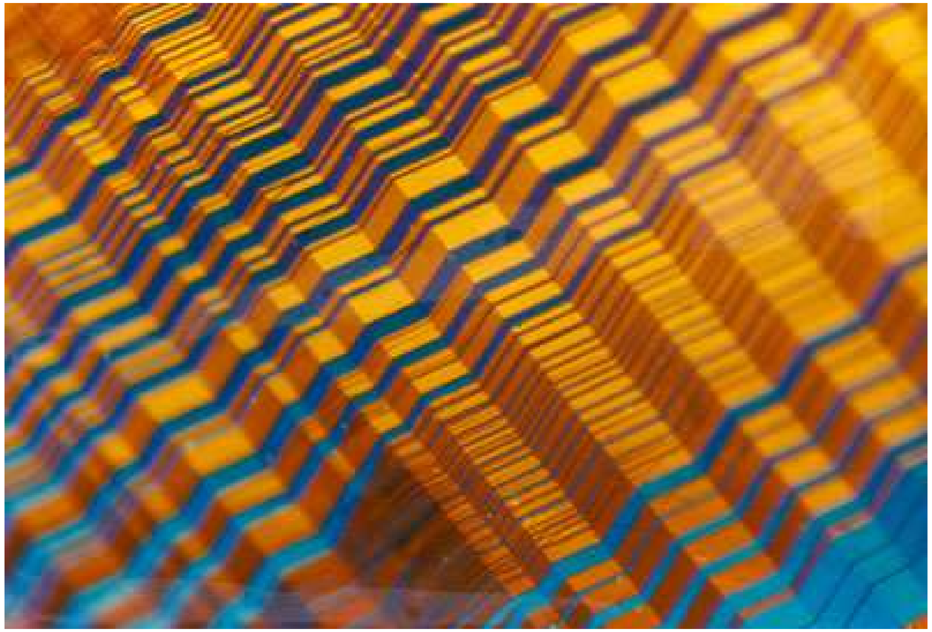
edge

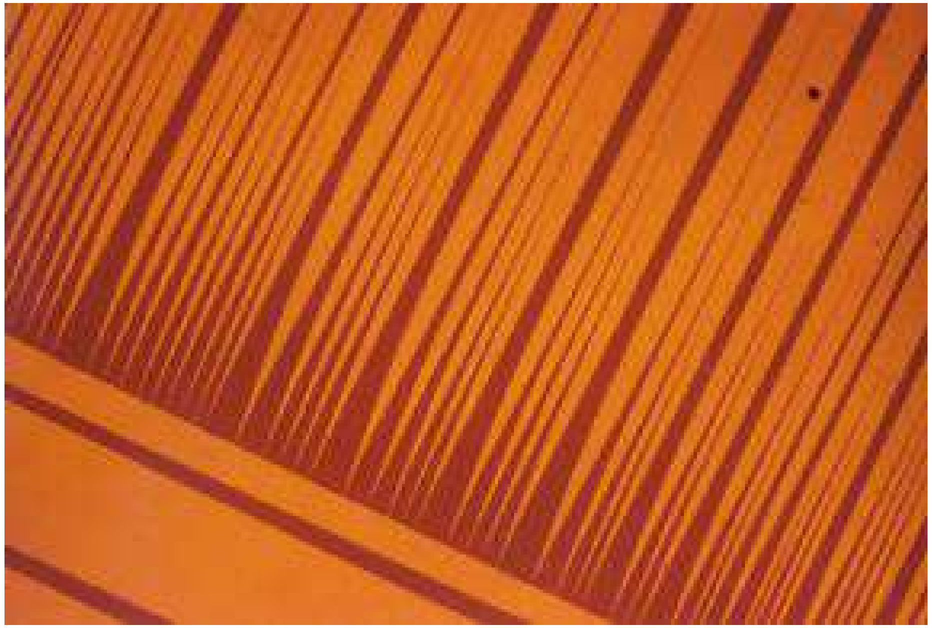
Phase boundaries in crystals and related microstructure





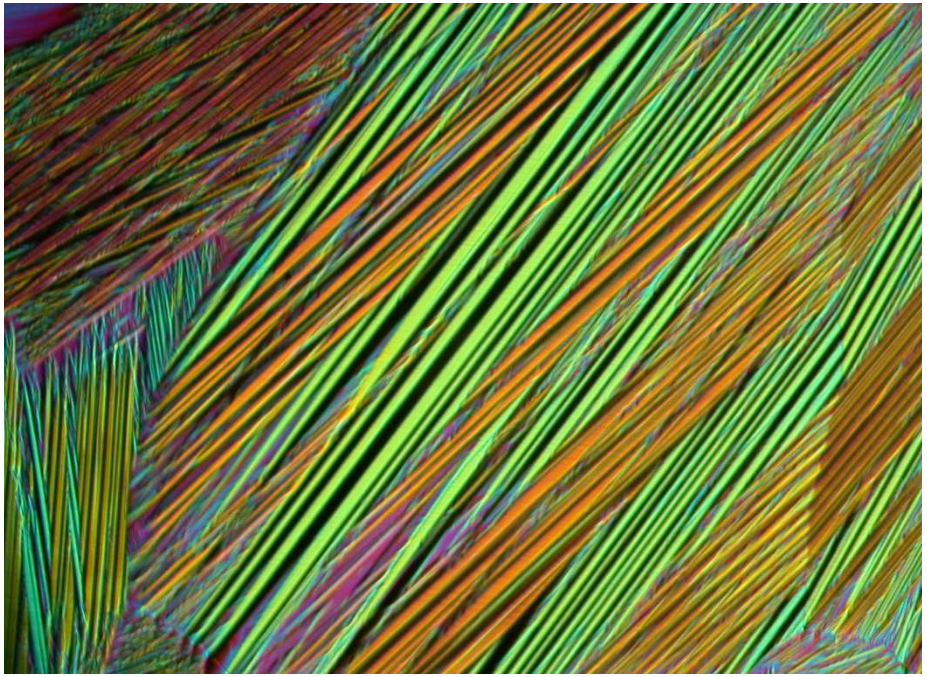
NiMn, Baele, van Tenderloo, Amelinckx





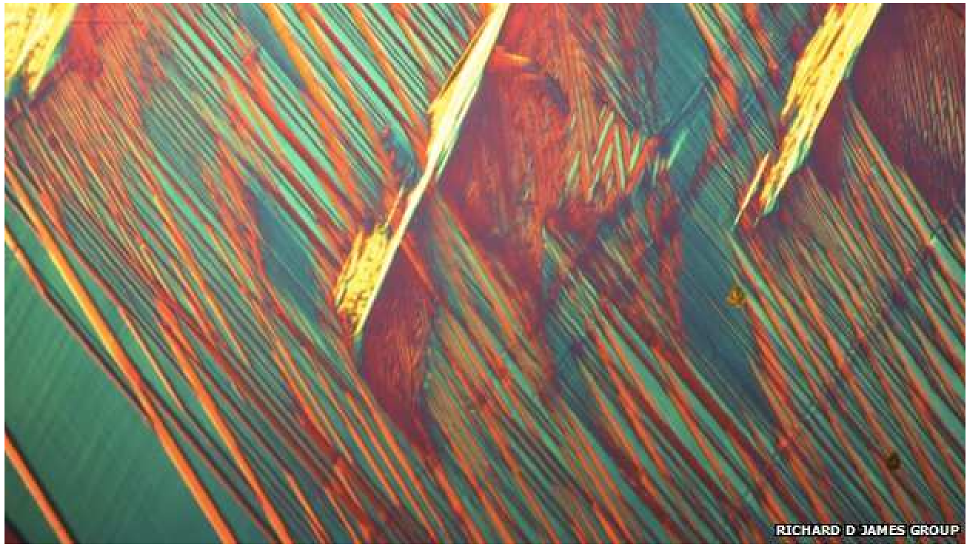
CuAlNi, Chu & James

CuZnAl Morin



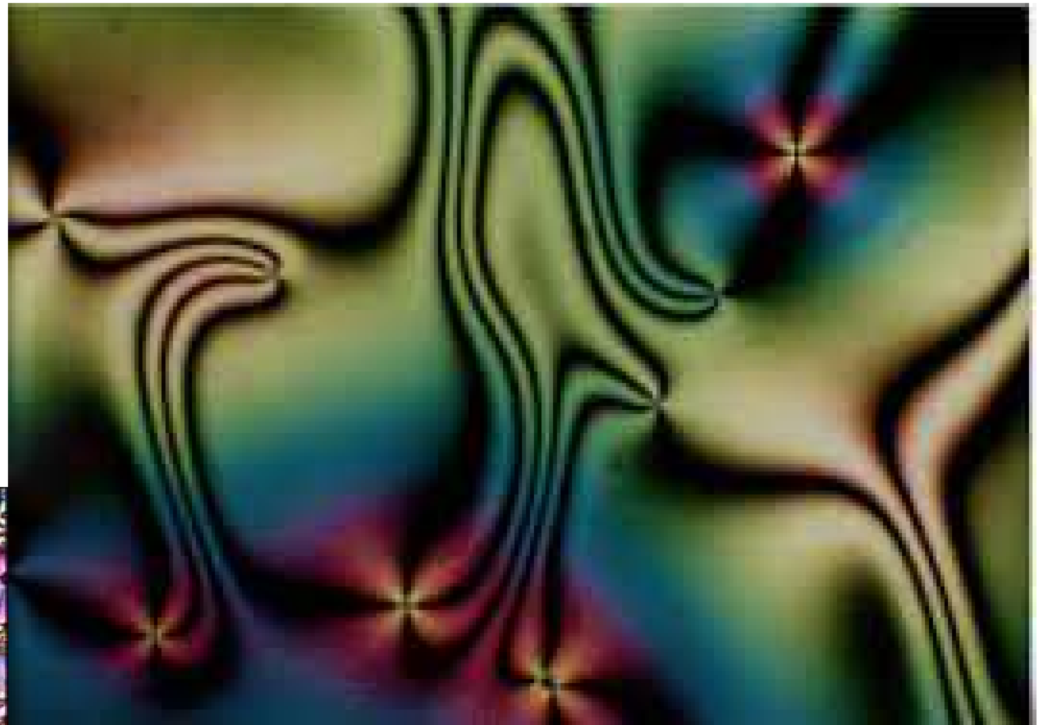
$Zn_{45}Au_{30}Cu_{25}$

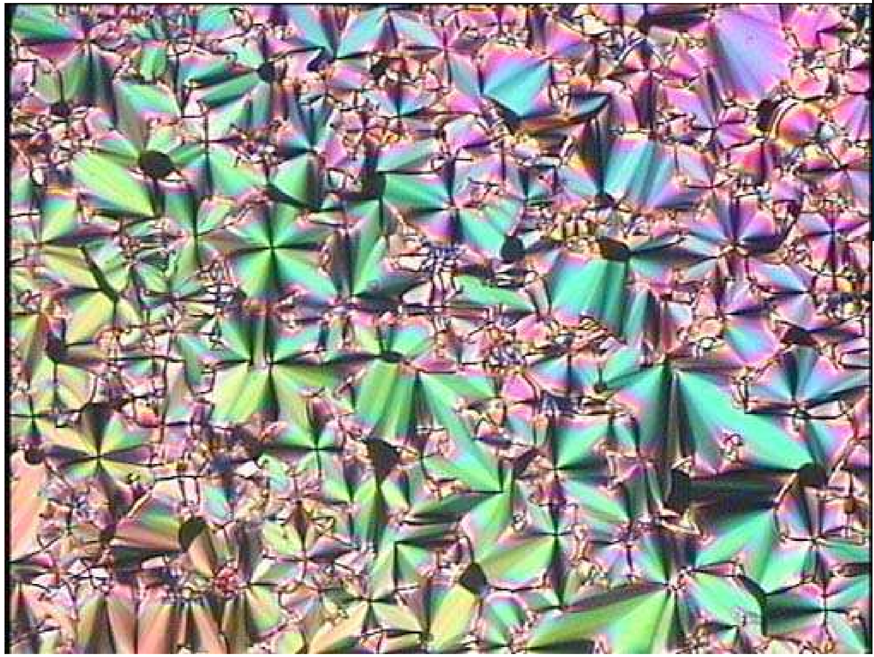
Song, Chen, Dabade, Shield & James, Nature 2013



Liquid crystals

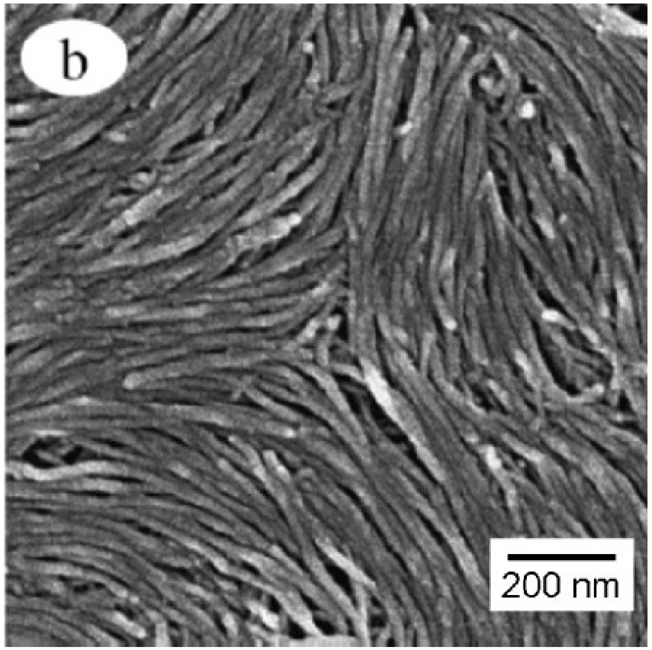
Boojums, O. Lavrentovich



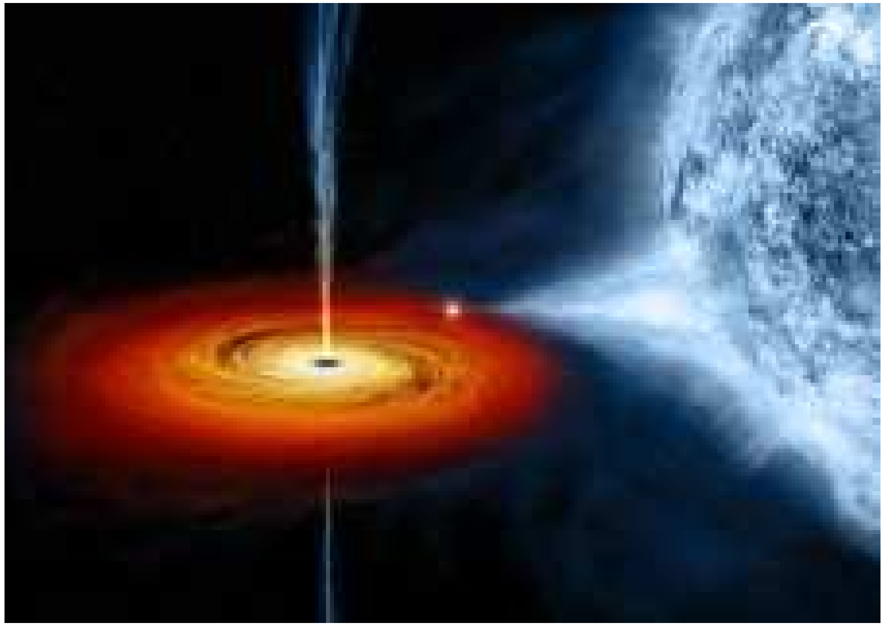


Smectic A, Lagerwall

Zhang/Kumar 2007
Carbon nano-tubes as liquid crystals



Black holes



All of these defects can be described by nonlinear PDE

For most of them there is a choice between ‘rougher’ models, in which the defects are described by singular solutions, in which state variables are discontinuous at points, lines, surfaces or patches, and ‘smoother’ models, in which the defect is endowed with more structure and ‘smoothed out’ (and thus the location of the defect is less easy to identify).

The UK PDE scene

Centres for Doctoral Training

Oxford: *Partial Differential Equations: Analysis and Applications*
Cambridge: *Cambridge Centre for Analysis*
Maxwell Institute (Edinburgh + Heriot-Watt): *Mathematical Analysis and its Applications*
Warwick: *Mathematics and Statistics (MASDOC)*

Other good groups and individuals at Bath, Bristol, Cardiff, Glasgow, Imperial, King's College London, Nottingham, Reading, Strathclyde, Surrey, Sussex, Swansea, UCL ...