On Support Sizes of Restricted Isometry Constants

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Joint with Jeff Blanchard (Grinnell)
Compressed Sensing

Let \( x \in \mathbb{R}^N \) be a given signal.

Suppose we obtain a vector \( b = Ax \in \mathbb{R}^n \) of linear measurements where \( A \in \mathbb{R}^{n \times N} \) is the measurement matrix.
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- $n < N \Rightarrow$ underdetermined system
- $x$ sparse with $k < n$ non-zeros
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Algorithm for the sparsest solution:

\[
\min \| x \|_0 \text{ subject to } Ax = b.
\]
$l_1$-Minimization and the RIP

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When does $l_1$ recover $l_0$?

\textbf{Restricted Isometry Constants:}

$$L_k := \min_{c \geq 0} c \text{ subject to } (1 - c)\|x\|_2^2 \leq \|Ax\|_2^2 \text{ for all } k\text{-sparse } x$$

$$U_k := \min_{c \geq 0} c \text{ subject to } (1 + c)\|x\|_2^2 \geq \|Ax\|_2^2 \text{ for all } k\text{-sparse } x$$

$$R_k := \max\{L_k, U_k\}$$
Conditions for $l_1$ to recover $l_0$

Chartrand (2007) \[ bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2 \]
\[ bL_{(b+1)k} + U_{bk} < b - 1; \quad b > 2 \]
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\[ bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2 \]
\[ \frac{b}{b-1} L_{(b+1)k} + \frac{1}{b-1} U_{bk} < 1; \quad b > 2 \]
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\[ R_{2k} < \sqrt{2} - 1 \approx 0.4142 \]
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\[ R_{2k} < \frac{2}{3 + \sqrt{2}} \approx 0.4531 \]
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What do these results mean quantitatively? Which is better?
Gaussian RIP Upper Bounds

(Blanchard, Cartis and Tanner, 2009)

**Theorem:** Let $A$ be a matrix of size $n \times N$ whose entries are drawn i.i.d. from $\mathcal{N}(0, \frac{1}{n})$.
Let $(k, n, N) \to \infty$ with $\frac{k}{n} \to \rho$ and $\frac{n}{N} \to \delta$.
Then there exist numerically computable functions $L(\delta, \rho)$ and $U(\delta, \rho)$ such that, for any $\epsilon > 0$,

$$\mathbb{P}\{L(k, n, N) < L(\delta, \rho) + \epsilon\} \to 1,$$
$$\mathbb{P}\{U(k, n, N) < U(\delta, \rho) + \epsilon\} \to 1.$$
Gaussian RIP Upper Bounds

$L(\delta, \rho)$

$U(\delta, \rho)$
Quantitative Sufficient Conditions

Chartrand (2007):

\[ bL([b + 1]k, n, N) + U(bk, n, N) < b - 1; \ b > 2 \]

Candès (2008):

\[(1 + \sqrt{2})L(2k, n, N) + U(2k, n, N) < \sqrt{2}\]

Foucart, Lai (2009):

\[
\frac{1 + U(2k, n, N)}{1 - L(2k, n, N)} < 4\sqrt{2} - 3
\]
Quantitative Sufficient Conditions

Chartrand (2007):

\[ bL(\delta, [b + 1]\rho) + U(\delta, b\rho) < b - 1; \quad b > 2 \]

Candès (2008):

\[ (1 + \sqrt{2})L(\delta, 2\rho) + U(\delta, 2\rho) < \sqrt{2} \]

Foucart, Lai (2009):

\[ \frac{1 + U(\delta, 2\rho)}{1 - L(\delta, 2\rho)} < 4\sqrt{2} - 3 \]
Comparison of $l_1$ Phase Transitions

The highest phase transitions are obtained by taking $b \approx 11$ in the result by Chartrand: $11L(12k, n, N) + U(11k, n, N) < 10$. 
Recent Results of Cai/Wang/Xu

\[ R_{k+a} + \sqrt{\frac{k}{b}} R_{k+a+b} < 1 \]

\[ L_{k+a} + \frac{1}{2} \sqrt{\frac{k}{b}} (L_{k+a+b} + U_{k+a+b}) < 1 \]

where \( 2a \leq b \leq 4a \).
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where \( 2a \leq b \leq 4a. \)

\[ R_a + tR_{a+b} < 1 \]
\[ L_a + \frac{1}{2} t(L_{a+b} + U_{a+b}) < 1 \]
where \( a \geq k, \ 8(a - k) \leq b, \) and
\[
t = \sqrt{\frac{a}{b}} + \frac{1}{4} \sqrt{\frac{b}{a}} - \frac{2(a-k)}{\sqrt{ab}}.\]
\[ \Rightarrow \ R_k < \frac{1}{1+\sqrt{5}} \approx 0.3090 \]
Comparison of $l_1$ Phase Transitions

Higher phase transitions can be obtained from the new results by again choosing large support sizes:

$$4L(2k, n, N) + L(6k, n, N) + U(6k, n, N) < 4.$$
Conclusions

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- We observe that results by Cai et al. with surprisingly high support sizes give the highest phase transitions for $l_1$ minimization.
Conclusions

• It is important to understand what RIP conditions mean quantitatively: the phase transition framework combined with RIP bounds for Gaussian matrices is a useful tool to investigate this.

• We observe that a results by Cai et al. with surprisingly high support sizes give the highest phase transitions for $l_1$ minimization.

• It is not always quantitatively beneficial to have RIP results with the smallest possible support sizes.
Thank you for your attention!