Loop Groups, Characters and Elliptic Curves

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ASPECTS of Topology in Geometry and Physics
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Overview

I will discuss ongoing joint work with David Nadler (in parts with Sam Gunningham, David Helm and Anatoly Preygel) applying gauge theory to representation theory.

Goal: introduce a 3d analog of the 2d topological field theory describing representations of loop groups.

Dedicated to Graeme Segal on the occasion of his 70th birthday, in gratitude for his crucial role in designing and constructing the fantastic playground in which we're playing.

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Motto

Theme: Representation theory is gauge theory in low dimensions.

Very low dimensions: focus on pt, S^1 and surfaces..

Traditionally: n-dimensional topological field theory[1] Z attaches numbers to n-manifolds, vector spaces to $n-1$-manifolds.
Extended TFT:[2] keep going – categories to $n-2$-manifolds,...

Bottom-up approach: the Cobordism Hypothesis:[3]

Input sufficiently finite object attached to a point,
"watch it grow": uniquely determines invariants of sufficiently low-dimensional manifolds and cobordisms.

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[2] Segal, Freed, Moore, Lawrence, Kazhdan, Reshetikhin, Turaev, Kontsevich, Soibelman, Costello

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2d Gauge Theory for Finite Groups

First case: 2d gauge theory $\mathcal{Z}$ with finite gauge group G.[4]

$\mathcal{Z}(X)$: linearization of space (orbifold/stack) of gauge fields on X

$$Loc_G(X) = \{\pi_1(X) \to G\}/G.$$

- $\mathcal{Z}(\Sigma) = \#Loc_G(\Sigma)$ (Σ closed surface): count of G-covers of Σ.
- $\mathcal{Z}(S^1) = \mathbb{C}[Loc_G(S^1)] = \mathbb{C}[G/G]$ class functions on G
- $\mathcal{Z}(pt) = Vect[Loc_G(pt)] = Vect[pt/G] = Rep\ G$
finite dimensional representations (modules for $\mathbb{C}[G]$)

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Characters of finite groups

The TFT formalism encodes the basic representation theory of G. In particular:

• For any TFT, $\mathcal{Z}(S^1)$ is cocenter or Hochschild homology of $\mathcal{Z}(pt)$: span of characters $char(M) \in \mathcal{Z}(S^1)$ of objects $M \in \mathcal{Z}(pt)$.

$\rightsquigarrow$ characters of G-representations are class functions $\mathcal{Z}(S^1) = \mathbb{C}[G/G]$.

• In any oriented 2d TFT $\mathcal{Z}(S^1)$ is also the center (Hochschild cohomology) of $\mathcal{Z}(pt)$

$\rightsquigarrow$ the center of $\mathbb{C}[G]$ is $\mathbb{C}[G/G]$.

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The Verlinde TFT

A 2d TFT $\mathcal{Z}_k$ encoding representations of loop group LG. [5]

Reduction of 3d Chern-Simons theory on a circle,
$\mathcal{Z}_k(X) = CS_k(X \times S^1)$.

G complex, simple, simply connected, G_c compact form

- $\mathcal{Z}_k(pt) = Rep_k(LG)$ category of level k positive energy representations of loop group:[6]

Smooth projective representations of LG of level k, extend to loop rotation $LG \rtimes \mathbb{C}^\times$ with weights bounded below and finite multiplicities.

Analogous to finite dimensional representations of G or G_c

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Highest weight theory

Both $Rep(G)$ and $Rep_k(LG)$ labelled by integral highest weights $\lambda \in \mathfrak{h}^*$ (in chamber or alcove)

Geometric construction (Borel-Weil):
Holomorphic sections of λ-line bundle on

flag manifold $G/B = G_c/T_c$ for G,

$B \subset G$ Borel subgroup (upper triangular matrices for $GL_n\mathbb{C}$)

affine flag manifold $LG/I = LG_c/T_c$ for LG

$I \subset LG_+ \subset LG$ Iwahori subgroup:
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Verlinde Algebra

- $\mathcal{Z}_k(S^1) = R_k(LG) = K(Rep_k(LG)) \otimes \mathbb{C}$, span of characters

Atiyah-Bott-Frobenius: Character values of induced representation $\leftrightarrow$ sum over fixed points.

Flag manifolds:
generic fixed points $\leftrightarrow$ Schubert cells $B \backslash G / B$
$\leftrightarrow$ Weyl group W $\rightsquigarrow$ Weyl character formula

Affine flag manifolds:
generic fixed points $\leftrightarrow$ affine Schubert cells $I \backslash LG / I$
$\leftrightarrow$ affine Weyl group W_{aff} $\rightsquigarrow$ Weyl-Kac character formula

$\rightsquigarrow$ Realize characters as modular forms[7]

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Modular Description

Quantum field theory explains modular behavior of affine characters. One formulation: realize $\mathcal{Z}_k(S^1) = CS_k(T^2)$.

Why torus? one loop for loop group, one loop for characters!

Direct route: describe characters as twisted class functions on LG:

Frightening space LG/LG has beautiful q-deformation:[8]

conjugacy classes of $LG \times \{q\} \subset LG \rtimes \mathbb{C}^\times$
$\leftrightarrow$ moduli space $Bun_G(E_q)$ of G-bundles on Tate elliptic curve

$$E_q = \mathbb{C}^\times / q^{\mathbb{Z}}.$$

$\mathcal{Z}_k(S^1)$ identified with holomorphic sections of natural level k line bundle on $Bun_G(E_q)$ – nonabelian theta functions in genus one.[9]

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Verlinde TFT on Surfaces

- $\mathcal{Z}_k(\Sigma)$ for closed (Riemann) surface: dimension of space of WZW conformal blocks on Σ – sections of level k line bundle on $Bun_G(\Sigma)$.[10]

- Calculated by Verlinde formula from pair of pants
$\leftrightarrow$ fusion ring structure on space of characters

- Refinement for punctured surface: $\mathcal{Z}_k(\Sigma \setminus x) \in \mathcal{Z}_k(S^1)$, character of representation of LG on theta functions on $Bun_G(\Sigma, \widehat{x})$ (bundles trivialized near x).

[10] Verlinde, Moore, Seiberg, Segal, Witten, Tsuchiya, Ueno, Yamada, Beauville, Laszlo, Faltings, Finkelberg, Teleman

Higher Algebra

Goal: 3d TFT encoding categorical representation theory of loop groups (reduction of 4d $\mathcal{N} = 4$ super-Yang-Mills on S^1)

We work in the context of Lurie's Higher Algebra but suppress all ∞-categorical technicalities.

For example: category will stand for an enhanced derived category (dg category).

One can do (homotopical) algebra with such categories – notions of tensor product, monoidal categories, module categories, centers, traces, characters . . .

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Categories in Algebra and Geometry

Algebraic source of categories: representations of a group or modules over an algebra.

Geometric representation theory seeks to describe such categories geometrically.

Geometric source of categories: sheaves or equivariant sheaves, i.e., sheaves on schemes or stacks X. Some variants:

- Coh : category of coherent sheaves on X (e.g., vector bundles)

- $\mathcal{D}(X)$: $\mathcal{D}$-modules on X – sheaves with flat connections (e.g., locally constant sheaves). Mainstay of geometric representation theory.

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Strong G-Categories

Fix G an algebraic group. What is a G-category?

To specify a class of representations, specify group algebra

Strong G-categories: modules for the monodial category $\mathcal{D}(G)$ of sheaves on G with convolution product – push forward along $m : G \times G \to G$

Key example: G algebraic group, X a G-space $\rightsquigarrow$ translation action of G on sheaves $\mathcal{D}(X)$

Elements of G act by functors, composing coherently and varying algebraically... and "locally constantly": derivative action of $\mathfrak{g}$ is trivialized (via flat connection)[11]

Idea: replace $Rep(G)$ or $Rep_k(LG)$ by highest weight $\mathcal{D}(G)-$mod or $\mathcal{D}(LG)-$mod!

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Hecke Algebras

Analog of highest weight theory for finite G:

look for representations in terms of functions $\mathbb{C}[G/K]$ on homogeneous space – or functions $\mathbb{C}_\lambda[G/K]$ twisted by a character λ of K

Representations of G appearing in decomposing $\mathbb{C}[G/K]$ identified with modules for Hecke algebra

$$H = End_G(\mathbb{C}[G/K]) = \mathbb{C}[K\backslash G/K] \subset \mathbb{C}[G]$$

(or λ-twisted version)

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categorification of Weyl group $W \leftrightarrow B\backslash G/B$.

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Likewise variants twisted by a weight $\lambda \in \mathfrak{h}^*$, and variants for partial flag manifolds ($B \rightsquigarrow P$ parabolic)

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The Automorphic Modules

Main source of LG-categories: sheaves on moduli of G-bundles

$Bun_G(\Sigma, \widehat{x})$ bundles with trivialization near x
$\rightsquigarrow LG$ symmetry

$Bun_G(\Sigma, x) = Bun_G(\Sigma, \widehat{x})/I$ parabolic bundles (B-reduction at x)
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Automorphic module:
$\mathcal{D}(Bun_G(\Sigma, \{x_i\}))$ (parabolic structures at $\{x_i\}$)
carries action of $\mathcal{H}_{aff}$ for each marked point,
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Geometric Langlands program: study of the automorphic modules.
Part of 4d $\mathcal{N} = 4$ super-Yang Mills.[12]

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The Automorphic Modules

Main source of LG-categories: sheaves on moduli of G-bundles

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The Character Theories

Theorem:[13] The assignment $\mathcal{Z}_{fin}(pt) = \mathcal{H}_{fin}-$ mod (2-category of highest weight G-categories) defines (via cobordism hypothesis) a TFT $\mathcal{Z}_{fin}$ through two dimensions, the *finite character theory*

i.e., $\mathcal{H}_{fin}$ satisfies strong finiteness properties like $\mathbb{C}[W]$. Deduce invariants: category $\mathcal{Z}_{fin}(S^1)$, vector space $\mathcal{Z}_{fin}(\Sigma)$ and operations for cobordisms.

Theorem:[14] The assignment $\mathcal{Z}_{aff}(pt) = \mathcal{H}_{aff}-$ mod (tamely ramified LG-categories) defines a TFT through two dimensions, the *affine character theory*

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Character Theory and Character Sheaves

$\mathcal{Z}_{fin}(S^1)$ and $\mathcal{Z}_{aff}(S^1)$: Hochschild homology categories of $\mathcal{H}$, $\mathcal{H}_{aff}$ $\leftrightarrow$ span of characters of highest weight G- and LG-categories.

Theorem:[15] $\mathcal{Z}_{fin}(S^1)$ is identified with the category of Lusztig's character sheaves.

Character sheaves: $\mathcal{D}_{nil}(G/G)$, "class sheaves" with only nilpotent characteristic directions (locally constant in semisimple directions).[16]

Thus character sheaves are categorical characters.

Definition: The category of affine character sheaves is the value $Ch_{aff} = \mathcal{Z}_{aff}(S^1)$ of the affine character theory on S^1.

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Character Sheaves as Representations

Lusztig: Character sheaves produce (as traces of Frobenius) all characters of representations of the finite groups $G(\mathbb{F}_{\mathbf{q}})$

Emerging affine version:[17]

$\boxed{\text{Affine character sheaves} \leftrightarrow \text{depth zero representations of } G(K)}$

K: nonarchimedean local field with residue field $\mathbb{F}_{\mathbf{q}}$
e.g., "loop group" $G(\mathbb{F}_{\mathbf{q}}((t)))$ or $G(\mathbb{Q}_p)$.
Depth zero representations: "come from" finite group $G(\mathbb{F}_{\mathbf{q}})$, e.g, highest weight.

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Springer Theory

Compute characters via categorified Atiyah-Bott formula: [18]

Character of standard module $\mathcal{D}(G/B)$ is the Springer sheaf $\mathcal{S} \in \mathcal{D}(G/G)$: stalks are cohomologies of fixed point loci on G/B (Springer fibers)

Character of $\mathcal{D}(LG/I) \rightsquigarrow$ the affine Springer sheaf.
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Automorphic description

Idea: Affine character sheaves $\overset{?}{\leftrightarrow}$ (nilpotent) "class sheaves on LG"

LG/LG frightening $\rightsquigarrow$ again q-deform to $Bun_G(E_q)$, G-bundles on Tate elliptic curve

Consider $\mathcal{D}_{nil}(Bun_G(E_q))$: category of $\mathcal{D}$-modules with nilpotent characteristic directions – elliptic character sheaves

"Categorified conformal blocks" – carries analog of Hitchin projective connection $\rightsquigarrow$ locally constant in q.

Calculate via nodal degeneration, i.e., in limit $q \to 0$:

Claim:[20] $\boxed{\mathcal{D}_{nil}(Bun_G(E_q)) \simeq \mathcal{Z}_{aff}(S^1)}$

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Langlands Duality

Langlands philosophy $\rightsquigarrow$ "Fourier transform":
representations of reductive group $\leftrightarrow$ geometry of dual group $G^\vee$

Physics: Montonen-Olive S-duality of $\mathcal{N} = 4$ super-Yang-Mills, equivalence of field theories with dual gauge groups.

Strongest known categorical duality:

$$\boxed{\mathcal{H}_{aff} = \mathcal{D}(I\backslash LG/I) \xrightarrow[\text{Bezrukavnikov}]{\simeq} Coh_{G^\vee}(X \times_Y X)}$$

convolution algebra of equivariant sheaves on Steinberg variety:
$Y \subset \mathfrak{g}^\vee$ nilpotent cone, $\quad X = T^*(G^\vee/B^\vee)$.

Duality for $\mathcal{H}_{aff}$, i.e., for $\mathcal{Z}_{aff}(pt) \rightsquigarrow$
Dual description of entire affine character theory

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Coherent Character Sheaves

Centers of "matrix algebras" $\mathbb{C}[X \times_Y X]$ easy to compute $\rightsquigarrow$

Theorem:[21] $\boxed{\mathcal{Z}_{aff}(S^1) \simeq Coh_{nil}(Loc_{G^\vee}(E))}$

coherent sheaves (with controlled singularities) on the commuting variety

$$Loc_{G^\vee}(E) = \{F, \sigma \in G^\vee :\ F\sigma = \sigma F\}/G^\vee,$$

(for fixed weight $\lambda \leftrightarrow$ generalized eigenvalue of σ)

Why torus? one loop (σ) for loop group,
one loop (F) for characters

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Geometric Langlands in Genus One

Combining the two dual descriptions of $\mathcal{Z}_{aff}(S^1)$ gives the geometric Langlands conjecture[22] in genus one[23] for arbitrary reductive G:

$$\boxed{\mathcal{D}_{nil}(Bun_G(E)) \xleftarrow[\text{automorphic}]{\simeq} \mathcal{Z}_{aff}(S^1) \xrightarrow[\text{spectral}]{\simeq} Coh_{nil}(Loc_{G^\vee}(E))}$$

[22] Arinkin, Gaitsgory

[23] for each fixed weight λ, i.e., generalized eigenvalues of monodromy σ

Character Theory on Surfaces

- Pair of pants $\rightsquigarrow$ braided multiplication on character sheaves
- $\mathcal{Z}_{fin}(\Sigma)$ closed surface: recover cohomology of character variety

$$Loc_G(\Sigma) = \{\pi_1(\Sigma) \to G\}/G$$

by "integrating" over highest weight $\lambda \in \mathfrak{h}^*$:

Theorem:[24]

$$\Gamma(\mathfrak{h}^*, \mathcal{Z}_{fin,\lambda}(\Sigma))^{W_{aff}} \xrightarrow{\simeq} \mathrm{H}^*(Loc_G(\Sigma))$$

"Fourier decomposition" into weights of cohomology of character variety

[24] Ben-Zvi, Gunningham, Nadler '13

Geometric Arthur-Selberg Trace Formula

Trace Formula: main tool in classical Langlands program. Describes character of automorphic module $L^2(\Gamma\backslash G/K)$ (traces of Hecke operators).

In Pursuit of Geometric Trace Formula:[25]

Describe character of automorphic module $\mathcal{D}(Bun_G(\Sigma))$

• geometrically (in terms of LG/LG) $\rightsquigarrow$ cohomology of Hitchin fibration on moduli of (grouplike) Higgs bundles.

• spectrally (in terms of decomposition as Hecke module) $\rightsquigarrow$ Langlands parameters $Loc_{G^\vee}(\Sigma \times S^1)$.

Natural context for Ngô's work on Fundamental Lemma (affine Springer theory, endoscopy etc.)

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Arthur-Selberg-Verlinde Formula

Conjecture:[26]

• Character of automorphic module is integral over weight λ of affine character theory on Σ:

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Analog of Verlinde formula for dimension of conformal blocks

• Langlands dual description of $\mathcal{Z}_{aff}(\Sigma)$ is spectral side of trace formula:

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Geometric Langlands on level of characters is captured by $\mathcal{Z}_{aff}$

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Thank You for Listening

Happy Birthday, Graeme!