# Cobordisms: old and new

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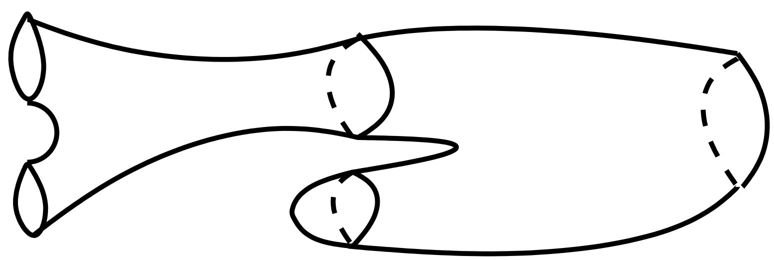
Second Abel Conference: A Celebration of John Milnor, 2012

## Classical Cobordism Theory

**Motivation:** Classification of compact smooth (oriented, spin, framed, almost complex, ...) manifolds.

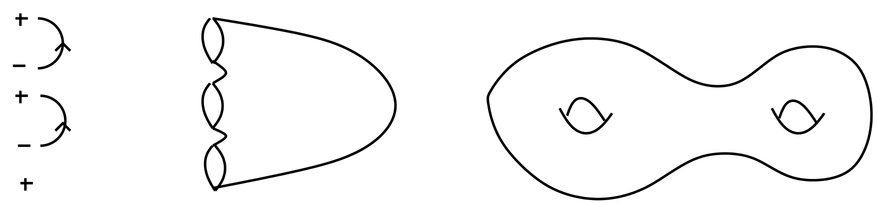
**Definition:** Two closed oriented $(d-1)$-dimensional manifolds $M_0$ and $M_1$ are *cobordant* if there exists a compact oriented $d$-dimensional manifold $W$ with boundary $\partial W = \bar{M}_0 \sqcup M_1$.



$$M_0 \longrightarrow W \longleftarrow M_1 \longrightarrow W' \longleftarrow M_2$$

- *equivalence relation*; equivalence classes $=: \mathfrak{N}^+_{d-1}$
- group with *product* $\coprod$ and *inverse* $M^{-1} = \bar{M}$;
- graded ring $\bigoplus_{d>0} \mathfrak{N}^+_{d-1}$ with *multiplication* $\times$.

**Examples:** $\mathfrak{N}_0^+ = \mathbb{Z} \qquad \mathfrak{N}_1^+ = \{0\} \qquad \mathfrak{N}_2^+ = \{0\}$




**Theorem (Thom)** $\mathfrak{N}_{d-1}^+ = \pi_{d-1}\Omega^\infty\mathbf{MSO}$

where $\Omega^\infty\mathbf{MSO} := \lim_{n\to\infty} \lim_{k\to\infty} \text{maps}_*(S^n, (U_{n,k})^c)$

and $U_{n,k} \to Gr^+(n,k)$ is the universal $n$-dimensional bundle over the Grassmannian manifold of oriented $n$-planes in $\mathbb{R}^{n+k}$.

$M \subset$ tubular neighbourhood $N(M) \subset \mathbb{R}^{d-1+n}$

$\rightsquigarrow \; f_M : S^{d-1+n} = (R^{d-1+n})^c \xrightarrow{collapse} (N(M))^c \xrightarrow{\phi_{N(M)}} (U_{n,k})^c$

**Theorem (Thom)** $\mathfrak{N}_*^+ \otimes \mathbb{Q} \simeq \mathbb{Q}\,[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$.

**Proof:** For fixed $*$ and large $n$ and $k$,

$$\begin{aligned} \pi_*(\Omega^\infty \mathrm{MSO}) \otimes \mathbb{Q} &= \pi_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} && \text{by definition} \\ &= H_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} && \text{by Serre} \\ &= H_*(Gr^+(n,k)) \otimes \mathbb{Q} && \text{by Thom.} \end{aligned}$$

Wall computed the 2-torsion.
Milnor showed there is no odd torsion in $\mathfrak{N}^+$, and no torsion at all in the complex analogue $\mathfrak{N}^{\mathbb{C}}$.

**Theorem (Milnor)** $\mathfrak{N}_*^{\mathbb{C}} \simeq \mathbb{Q}\,[\mathbb{C}P^1, \mathbb{C}P^2, \ldots]$.

# ON THE COBORDISM RING $\Omega^*$ AND A COMPLEX ANALOGUE, PART I.*

By J. Milnor.

This paper will prove that the cobordism groups $\Omega^i$, defined by Thom [15], have no odd torsion.[1] Furthermore, it is shown that certain related groups $\pi_{i+2n}M(U_n)$ have no torsion at all; providing that $n$ is large. The proofs are based on a spectral sequence due to J. F. Adams [1, 2].

The following is a brief summary of Thom's constructions. Let $G$ be a subgroup of the orthogonal group $O_n$. (More generally one could start with any Lie group $G$, together with a specified representation into $O_n$.) Beginning with a universal bundle for $G$ we can form:

1) The weakly associated bundle having the disk $D^n$ as fibre. Let $\pi\colon E \to B(G)$ denote the projection map of this bundle.

2) The weakly associated bundle having the sphere $S^{n-1}$ as fibre. Let $\partial E \subset E$ denote the total space.

The *Thom space* $M(G)$ is now defined as the identification space obtained from $E$ by collapsing $\partial E$ to a point.

Taking $G$ to be the rotation group $SO_n \subset O_n$, Thom showed that the homotopy group $\pi_{i+n}M(SO_n)$ is independent of $n$, providing that $n$ is large. He showed that this group is isomorphic to the "cobordism group" $\Omega^i$; and determined its structure up to torsion. The 2-torsion subgroup of $\Omega^i$ has recently been determined by C. T. C. Wall. Hence the assertion that $\Omega^i$ has no odd torsion completes the description of this group.

Let $M(U_n)$ denote the Thom space for the unitary group $U_n \subset O_{2n}$. In Part II of this paper it will be shown that the stable homotopy group $\pi_{i+2n}M(U_n)$ can be interpreted as a "complex cobordism group." Part I will determine the structure of this group without attempting to interpret it.

* Received July 27, 1959.

[1] Added in proof. This result has been obtained independently by B. G. Averbuch, *Doklady Akademii Nauk SSSR*, vol. 125 (1959), pp. 11-14. The results on complex cobordism have been obtained independently by Novikov.

## Topological Field Theory

Let $\mathcal{C}ob_d^\delta$ be the discrete cobordism category with objects compact, closed, oriented $d-1$ dimensional manifolds. A $d$-dimensional cobordism $W$ with $\partial W = \bar{M}_0 \sqcup M_1$ defines a morphisms from $M_0$ to $M_1$. Another cobordism $W'$ with $\partial W = \partial W'$ defines the same morphisms if there is a diffeomorphisms relative to the boundary taking $W'$ to $W$.

**Definition:** A $d$-dimensional TQFT is a symmetric monoidal functor

$$\mathcal{F} : \mathcal{C}ob_d^\delta \longrightarrow \mathcal{V}$$

to the category of vector spaces that takes disjoint union of manifolds to tensor products of vector spaces.

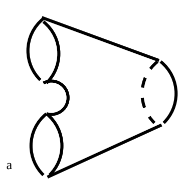
$h$-cobordisms as morphisms in a category.

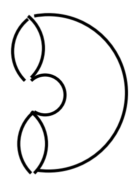
**Motivation:** $d$-dimensional TQFTs define topological invariants for $d$-dimensional closed manifolds: If $\partial W = \varnothing$ then

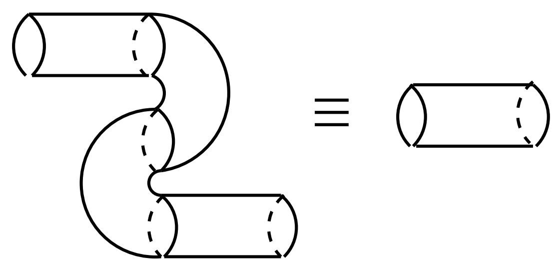
$$\mathcal{F}(W) : \mathcal{F}(\varnothing) = \mathbb{C} \longrightarrow \mathcal{F}(\varnothing) = \mathbb{C}$$

assigns a number to $W$ depending only on its topology.

**Folk Theorem:** 2-dimensional TQFTs are in one-to-one correspondence with finite dimensional, commutative Frobenius algebras:
Let $\mathcal{F}(S^1) = A$ and $\mathcal{F}(\coprod_n S^1) = A^{\otimes n}$.





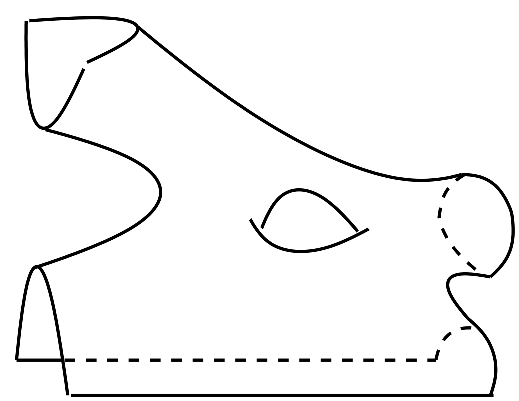


product bilinear form non-degeneracy

**Physical inspiration:**
Quantum field theories are local $\Longrightarrow$

*Categorification:* starting with points, we take cobordisms, cobordisms of cobordisms, ... . $\Longrightarrow$

Replace $\mathcal{C}ob_d^\delta$ by the $d$-fold category $ex\mathcal{C}ob_d^\delta$, and vector spaces by some $d$-fold symmetric monoidal category $\mathcal{V}_d$, and study functors between them, the so called *extended* TQFTs.

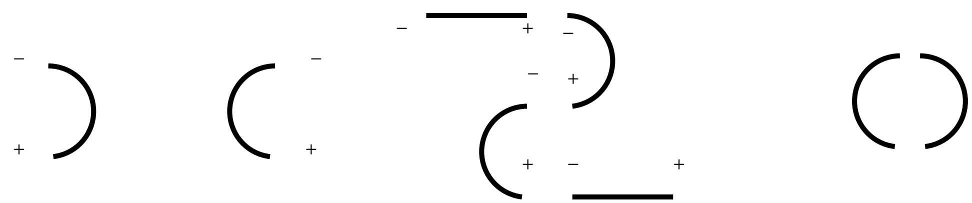


**Cobordism hypothesis (Baez-Dolan)**

Extended TQFTs are determined by their value on a point.

This is certainly so for 1-dimensional theories:

Let $\mathcal{F}(*_+) = V$ and $\mathcal{F}(*_-) = V'$.



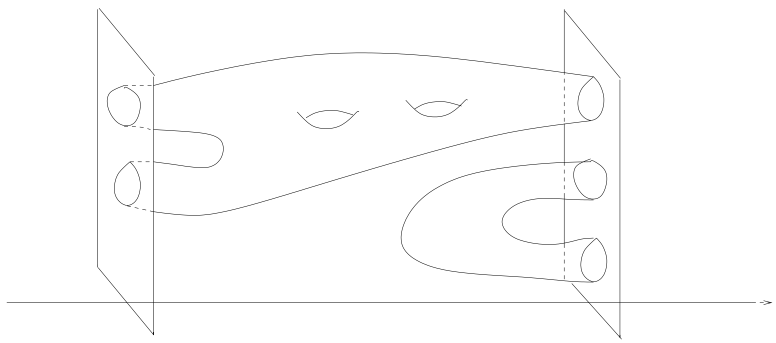
- evaluation $e : V \otimes V' \to \mathbb{C}$
- co-evaluation $e^* : \mathbb{C} \to V' \otimes V$
- $V$ is finite dimensional as

$$id : V \overset{id \otimes e^*}{\longrightarrow} V \otimes V' \otimes V \overset{e \otimes id}{\longrightarrow} V$$

- $e \circ e^* = \dim(V) : \mathbb{C} \to \mathbb{C}$

## Enriched TQFTs

Consider moduli spaces of all compact $(d-1)$- and $d$-manifolds embedded in $\mathbb{R}^{d+n}, n \to \infty$, to form the topological category $\mathcal{C}ob_d$.



$$M_0 \longrightarrow W \longleftarrow M_1$$

The homotopy type of the space of morphisms is

$$\mathit{morph}_{\mathcal{C}ob_d}(M_0, M_1) \simeq \coprod_W B\mathrm{Diff}(W; \partial)$$

where the disjoint union is taken over all diffeomorphism classes of cobordisms $W$.

**Theorem (Hopkins-Lurie ($n = 2$), Lurie (general)):** The cobordism hypothesis holds for extended and enriched TQFTs: symmetric monoidal functors

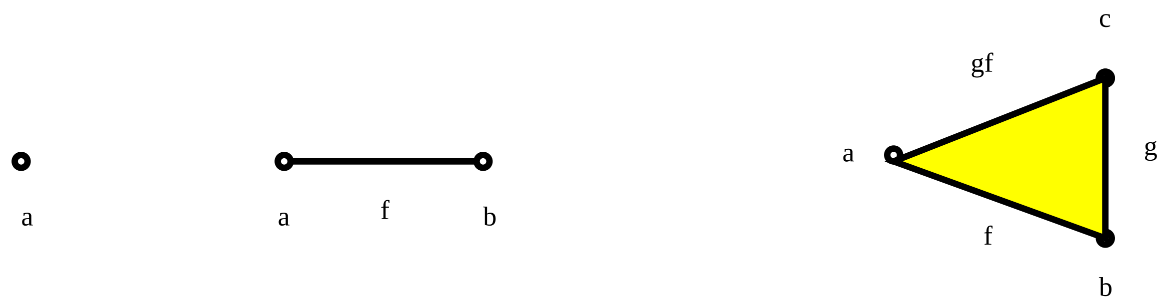
$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \to d\mathcal{V}$$

are determined by $\mathcal{F}(*)$, the value on a point, and any object in $\mathcal{V}_d$ satisfying certain duality and non-degeneracy properties gives rise to a TQFT.

More general: for non-orientable, oriented, spin, ... $\mathcal{F}$ is still determined by $\mathcal{F}(*)$ but there are group actions that have to be considered.

## Classifying space

$B$ : Topological Categories $\longrightarrow$ Spaces, $\mathcal{C} \mapsto B\mathcal{C}$




- morphisms $\mapsto$ paths *which are homotopy invertible!*

. for every $* \in ob_{\mathcal{C}}$, there is a characteristic map

$$\alpha : morph_{\mathcal{C}}(*, *) \longrightarrow \text{maps}([0,1], \partial; B\mathcal{C}, *) = \Omega B\mathcal{C}$$

- monoidal cats $\mapsto$ $E_1$-spaces ($\Omega$-spaces)
- symmetric monoidal cats $\mapsto$ $E_\infty$-spaces ($\Omega^\infty$-spaces)

**Theorem (Galatius, Madsen, T., Weiss)**

$$\Omega B(\mathcal{C}ob_d) \simeq \Omega^\infty \mathbf{MTSO}(d) = \lim_{n\to\infty} \Omega^{d+n}((U_{d,n}^\perp)^c)$$

where $U_{d,n}^\perp$ is the orthogonal complement of the universal bundle $U_{d,n} \to Gr^+(d,n)$.

*Note: the Thom class is in dimension $-d$!*

The characteristic map:
*morph*$_{\mathcal{C}ob_d}(\varnothing,\varnothing) \ni W \subset N(W) \subset \mathbb{R}^{d+n}$,

$$\alpha(W) : S^{d+n} = (R^{d+n})^c \xrightarrow{collapse} N(W)^c \xrightarrow{\phi_{T(W)}} (U_{d,n}^\perp)^c$$
$$(x,v) \mapsto (T_xW, v).$$

*In Thom's theory:* $(x,v) \mapsto (N_xW, v) \in (U_{n,d})^c$.

$$H^*(\Omega_0^\infty \mathbf{MTSO}(d), \mathbb{Q}) \simeq \mathbf{\Lambda}^*(H^{*>0}(BSO(d); \mathbb{Q})[-d])$$

**Theorem (Barrett-Priddy, Quillen, Segal)**
For $d = 0$: $B\Sigma_n \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(0) \simeq \Omega^\infty S^\infty$ is a homology isomorphism in degrees $* \leq n/2$.

**Theorem (Madsen-Weiss)**
For $d = 2$: $B\mathrm{Diff}(F_g) \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(2)$ is a homology isomorphism in degrees $* \leq (2g-2)/3$.
$\implies$ Mumford's conjecture

**Note**
For $d = 1$: $B\mathrm{Diff}(S^1) \simeq \mathbb{C}P^\infty \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(1) \simeq \Omega^{\infty+1} S^\infty$ is trivial in rational homology.

**Theorem (Ebert)**
For $d = 3$: $B\mathrm{Diff}(W^3) \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(3)$ is trivial in rational homology.

## Filtration of classical cobordism theory

The inclusion of multi-categories

$$ex\mathcal{C}ob_1 \subset \cdots \subset ex\mathcal{C}ob_{d-1} \subset ex\mathcal{C}ob_d \subset \ldots$$

induces on taking multi-classifying spaces a filtration

$$\Omega^\infty S^\infty \to \cdots \to \Omega^{\infty-(d-1)}\mathbf{MTSO}(d-1) \to \Omega^{\infty-d}\mathbf{MTSO}(d) \ldots$$

of Thom's space $\Omega^\infty\mathbf{MSO}$ which respects the additive and multiplicative structure.
*All Thom classes are in degree zero!*

For framed manifolds, this is the constant filtration

$$B(ex\mathcal{C}ob_d^{fr}) = \lim_{n\to\infty} \Omega^n(\tilde{U}_{d,n}^\perp)^c \simeq \Omega^\infty S^\infty$$

where $\tilde{U}_{d,n}$ is the universal bundle over the Stiefel manifold of framed $d$-planes in $\mathbb{R}^{d+n}$.

## Fibration sequence

$$\Omega^\infty \mathbf{MTSO}(d) \longrightarrow \Omega^\infty \Sigma^\infty (BSO(d)_+) \longrightarrow \Omega^\infty \mathbf{MTSO}(d-1).$$

Genauer proves that this corresponds to natural maps of cobordism categories:

$\mathcal{C}ob_d$: $d$-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times (0, \infty)$

$\bigcap$

$\mathcal{C}ob_d^\partial$: $d$-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times [0, \infty)$

$\downarrow$

$\mathcal{C}ob_{d-1}$: $d-1$-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d-1+n} \times \{0\}$

## Cobordism Theorem for invertible theories

An extended framed TQFT

$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \longrightarrow \mathcal{V}_d$$

induces a map of infinite loop spaces of classifying spaces

$$B\mathcal{F} : B(ex\mathcal{C}ob_d^{fr}) \simeq \Omega^\infty S^\infty \longrightarrow B(\mathcal{V}_d).$$

$\Omega^\infty S^\infty$ is the free infinite loop space on one point.
$\Longrightarrow$ $B\mathcal{F}$ is determined by its value on that point, $B\mathcal{F}(*)$.

If $\mathcal{F}$ is *invertible* (in the sense that the images of all morphisms are invertible) it 'factors' through $B\mathcal{F}$.

## Sketch of proof

Starting with Madsen-Weiss, the proof has been continuously simplified and the theorem generalised (Galatius-Madsen-T.-Weiss, Galatius, Bökstedt-Madsen).

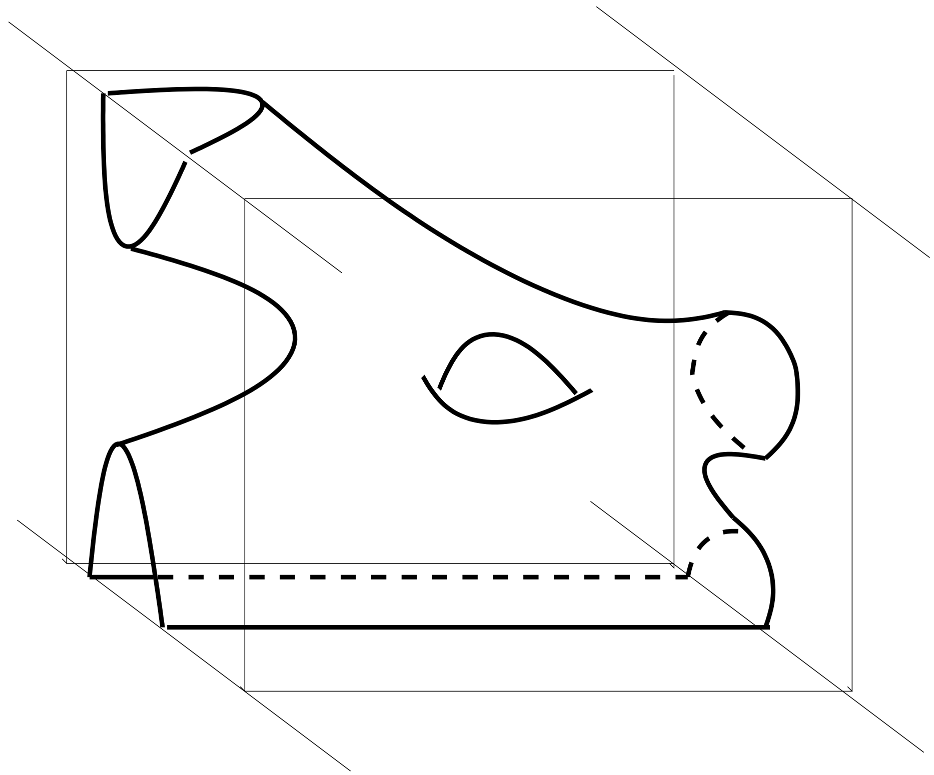
$\Phi_{d,n}$: space of all embedded $d$-manifolds without boundary which are closed as a subset in $\mathbb{R}^{d+n}$; base point $\varnothing$; topologized so that manifolds can disappear at infinity.

$\Phi^k_{d,n}$: subspace of manifolds embedded in $\mathbb{R}^k \times (0,1)^{n+d-k}$.

$$\Phi_{d,n} = \Phi^{d+n}_{d,n} \supset \cdots \supset \Phi^k_{d,n} \supset \cdots \supset \Phi^0_{d,n} \simeq \coprod_{W,\,\partial=\varnothing} B\mathrm{Diff}(W)$$

$\mathcal{C}ob^k_{d,n}$: $k$-fold cobordisms category of $d$-manifolds embedded in $\mathbb{R}^{d+n}$

$\lim_{n\to\infty} \mathcal{C}ob^1_{d,n} = \mathcal{C}ob_d$ and $\lim_{n\to\infty} \mathcal{C}ob^d_{d,n} = ex\mathcal{C}ob_d$.



A 2-morphism in $\mathcal{C}ob^2_{2,1}$.

*Step 1:* $\Phi^k_{d,n} \simeq \Omega^{d+n-k}\Phi_{d,n}$ for $k > 0$
Scanning! (Uses Gromov's theory of microflexible sheaves.)

*Step 2:* $(U^{\perp}_{d,n})^c \simeq \Phi_{d,n}$
Tangential information: $(P, v) \mapsto v - P$.

*Step 3:* $B(\mathcal{C}ob^k_{d,n}) \simeq \Phi^k_{d,n}$ for $k > 0$
Nature of classifying spaces.

$\Longrightarrow$
$$B(\mathcal{C}ob^k_{d,n}) \simeq \Omega^{d+n-k}(U^{\perp}_{d,n})^c$$

for $k = 1$ and $n \to \infty$, $B(\mathcal{C}ob_d) \simeq \Omega^{\infty-1}\mathbf{MTSO}(d)$
for $k = d + n$ and $n \to \infty$, $B(ex\mathcal{C}ob_d) \simeq \Omega^{\infty-d}\mathbf{MTSO}(d)$

This gives an even finer filtration:

$$\begin{aligned}\Omega^\infty \mathbf{MSO} &\simeq \lim_{n\to\infty} \lim_{d\to\infty} \Omega^n(U_{n,d})^c \\ &\simeq \lim_{d\to\infty} \lim_{n\to\infty} \Omega^n(U_{d,n}^\perp)^c \\ &\simeq \lim_{d\to\infty} \lim_{n\to\infty} B(\mathcal{C}ob_{d,n}^d)\end{aligned}$$

Thank you!