Oxford 13 March 2017

Surgery on manifolds: the early days,

Or: What excited me in the 1960s.

C.T.C.Wall

In 1956 Milnor amazed the world by giving examples of smooth manifolds homeomorphic but not diffeomorphic to the 7-sphere S^7.

J. W. Milnor, On manifolds homeomorphic to the 7-sphere, *Ann. Math.* **64** (1956) 399–405.

Define $f_{h,j} : S^3 \to SO(4)$ by *$f_{h,j}(u)v = u^h v u^j$. Let $B_{h,j}$ be the associated D^4 bundle over S^4. The self-intersection of the central 4-sphere is $h + j$, so if $N_h := B_{h,1-h}$, the boundary $M_h := \partial N_h$ is homotopy equivalent and *in fact homeomorphic to S^7.

Define a closed manifold $\overline{N}_h$ by attaching an 8-disc along the boundary of N_h.

Then $\overline{N}_h$ has *signature $\tau = 1$, and *Pontrjagin number $p_1^2[N] = (2h-1)^2$. Now use the signature theorem obtained in

F. E. P. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, 165 pp, Springer-Verlag, 1956.

This tells us that for smooth 8-manifolds,

$$\tau = \frac{7p_2 - p_1^2}{45}.$$

Hence if $\overline{N}_h$ is smooth, $(2h-1)^2+45$ is divisible by 7: a contradiction if e.g. $h = 1$. Thus $\overline{N}_1$ is not smooth and M_1 is not diffeomorphic to S^7.

An elaboration of the argument shows that $(2h-1)^2 + 45$ (modulo 7) is a diffeomorphism invariant of M_h.

Milnor followed by papers using geometrical constructions to obtain significant results in homotopy theory. These were announced at a talk at the *ICM in 1958, which also marks the beginning of the collaboration with Kervaire. Bott's periodicity theorem was announced at the same meeting.

J. W. Milnor and M. Kervaire, Bernoulli numbers, homotopy groups and a theorem of Rohlin, *Proceedings of ICM (Edinburgh 1958)*, 1962.

Call M *almost parallelisable* (a.p.) if the tangent bundle of M with a point deleted is trivial. If M^{4k} is a.p., there is an obstruction in $\mathbb{Z}$ to the tangent bundle being stably trivial. This is measured by the signature τ, the Pontrjagin number $p_{4k}[M]$, or by $\pi_{4k-1}(SO)$. For explicit formulae, set $a_k = 2$ if k is odd, 1 if k is even, and write B_k for the k^{th} *Bernoulli number.

If x_0 generates $\pi_{4k-1}(SO)$, then
$p_{4k}(x_0)[M] = a_k(2k-1)!$,

$\tau[M] = 2^{2k}(2^{2k-1}-1)B_k p_{4k}[M]/(2k)!$.

By 1959, Smale had proved that any smooth manifold homotopy equivalent to S^n (with $n \geq 5$) is homeomorphic to it.

S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, *Ann. Math.* **74** (1961) 391–406.

Milnor now defined a group Θ_n of diffeomorphism classes of homotopy n-spheres, with sum given by connected sum. Also define $*P_n$ as the group of cobordism classes of framed n-manifolds with boundary a homotopy sphere.

There is then an exact sequence

$$P_{n+1} \xrightarrow{b} \Theta_n \to Coker J,$$

where $\pi_n^S := lim(\pi_{n+k}(S^k))$ is the stable homotopy group and $J : \pi_n(BSO) \to \pi_n^S$ the classical J-homomorphism.

To calculate Θ_n one next studies the image bP_{n+1}, and it is for the study of P_{n+1} that surgery was introduced. Several overlapping papers now appeared in quick succession.

J. W. Milnor, Differentiable structures on spheres, *Amer. J. Math.* **81** (1959) 962–972.

M. Kervaire, A manifold which does not admit any differentiable structure, *Comm. Math. Helv.* **34** (1960) 257–270.

J. W. Milnor, A procedure for killing the homotopy groups of differentiable manifolds, *Amer. Math Soc. Symp in Pure Math.* **3** (1961) 39–55.

C. T. C. Wall, Killing the middle homotopy groups of odd dimensional manifolds, *Trans. Amer. Math. Soc.* 103 (1962) 421–433.

M. Kervaire and J. W. Milnor, Groups of homotopy spheres I, Ann. Math. **77** (1963) 504–537.

The basic construction of surgery is to embed a product $S^r \times D^{n-r}$ in the manifold N^n, delete the interior of the image, and glue back in its place a copy of $D^{r+1} \times S^{n-r-1}$ (which has the same boundary), giving a new manifold N'. If all goes well, the homology or homotopy class of the sphere $S^r \times \{0\}$ in N is killed when we pass to N'. More precisely, suppose inductively that n is $(r-1)$-connected, choose an element $\xi \in \pi_r(N)$; then if $n \geq 2r+1$ there is no obstruction to representing ξ by an embedding $\overline{\phi} : S^r \to N$. Provided also that the tangent bundle of N pulls back to a (stably) trivial bundle over S^r, we can extend $\overline{\phi}$ to an embedding $\phi : S^r \times D^{n-r} \to N$ and perform the construction. Moreover, if $n > 2r+1$, $H_r(N')$ is the quotient of $H_r(N)$ by $[\xi]$. Iterating this, and taking a little care, we can start with a stably parallelisable manifold N^n and reduce it in a finite number of steps to an r-connected manifold.

The next step depends on the parity of n. Suppose that N^{2k} is a parallelisable manifold with boundary a homotopy sphere. Applying the above procedure (which does not change the boundary), we can reduce to the case when N is $(k-1)$-connected. If k is even, the quadratic form given by intersection numbers on $H_k(N;\mathbb{Z})$ is unimodular and even. Its signature (which, for any such form, is divisible by 8) is an obstruction to surgery to kill $H_k(N)$. If the signature vanishes, the homology group has a basis $\{e_i, f_i\}$ with all intersection numbers zero except $e_i \cdot f_i = 1$. If also $k \geq 3$, we can then perform surgery in turn on spheres representing the e_i (we will return to this point shortly), and this will make N contractible, so that ∂N is diffeomorphic to S^{2k-1}.

If $k \geq 3$ is odd, the intersection form is skew-symmetric; we can choose a basis $\{e_i, f_i\}$ as before, but now an embedded sphere representing e_1 has normal bundle given by an element

$$q(e_1) \in \pi_{k-1}(SO_k) \cong \mathbb{Z}_2.$$

These obstructions assemble to a quadratic map $q : H_k(N) \to \mathbb{Z}_2$, and such maps were studied by Arf who found that they had an invariant mod 2, which is given by

$$A(q) = \sum_i q(e_i)q(f_i).$$

The Arf invariant $A(q)$ is the Kervaire invariant of the surgery problem. If it vanishes, we can choose a new basis with each $q(e_i) = 0$, and then complete surgery as before.

When $n = 2k + 1$ is odd, we are again faced with a single remaining homology group $H_k(N; \mathbb{Z})$; we can choose any element and perform surgery, but the result of the surgery now depends on the choice of the trivialisation of the normal bundle of the embedded k-sphere. With some effort, we can first make $H_k(N; \mathbb{Z})$ finite, and then by induction on its order show that (provided $k \geq 3$) we can kill this group and conclude that $\partial N \cong S^{2k}$ and hence that $P_n = 0$.

Milnor also gave a ‘plumbing’ construction to construct an example of a parallelisable N^{4k} with boundary a sphere and signature 8 and of an N^{4k+2} with non-zero Kervaire invariant. This completes the calculation of P_n ($n \geq 6$):

n (mod 4)	0	1	2	3
P_n	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0

The size of the image bP_{4k} of $P_{4k} \to \Theta_{4k-1}$ is determined by calculations using the J homomorphism and the index formula. One finds

$$|\Theta_{4k-1}|/|\pi^S_{4k-1}| = a_k 2^{2k-2}(2^{2k-1} - 1)B_k/4k.$$

The calculation of bP_{4k+2} was a long standing problem and a major challenge to homotopy theorists. It was shown by Browder in 1969 that it is trivial if $k + 1$ is not a power of 2, and finally shown in 2009 by Hill, Hopkins and Ravenel that this image is trivial except for n equal to 2, 6 14, 30, 62 and possibly 126.

Next came two (independent) papers which generalised the method of surgery: instead of killing the homotopy groups of N, kill the relative homotopy groups of a map $N \to X$: thus constructing a manifold which approximates to X in a homotopy sense.

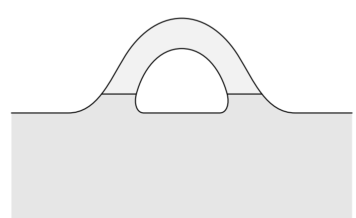
S. P. Novikov, Diffeomorphisms of simply-connected manifolds, *Dokl. Akad. Nauk SSSR* **143** (1962) 1046–1049.
W. Browder, Homotopy type of differentiable manifolds, in *Colloq. Alg. Top., Aarhus 1962*, 42–46.

We need a replacement for the hypothesis that N is parallelisable.

Define a normal map to consist of:
a map $f : N \to X$,
a vector bundle ν over X, and
a trivialisation F of $f^*\nu \oplus \tau(N)$.

Then any element $\xi \in \pi_{k+1}(f)$ yields a map $g : S^k \to N$, a nullhomotopy of $f \circ g$ and a stable trivialisation of $g^*\tau_N)$, and hence by immersion theory a regular homotopy class of immersions $S^k \times D^{n-k} \to N$.

For any embedding in this class, we can attach a handle $D^{k+1} \times D^{n-k}$ to $N \times I$ giving W, say,



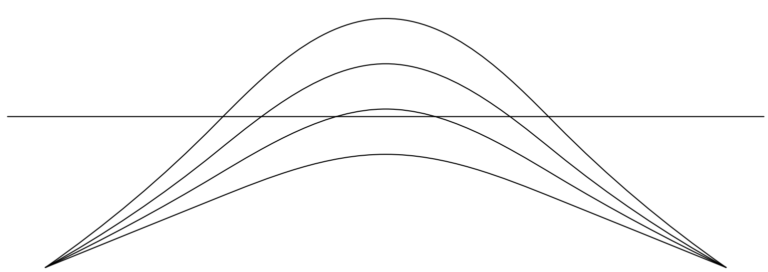
and extend f to a map $h : W \to X$ and F to a trivialisation of $h^*\nu \oplus \tau(W)$.

Since W is a cobordism of N to the manifold N' obtained by surgery, this is a complete analogue of what we had before.

If $n > 2k$, we can always find an embedding in the class of immersions and so do surgery, and as before there is no problem in doing surgery to make the map f k-connected. This is already a useful result, giving a method of construction of manifolds with certain properties and having applications to classification problems.

To go further requires more. If we are to obtain a homotopy equivalence from N to X, then X itself must satisfy Poincaré duality. If we assume this, and if also X is simply connected, then the obstruction to obtaining a homotopy equivalence is as before, so lies in $\mathbb{Z}, 0, \mathbb{Z}_2$ or 0 according as $n \equiv 0, 1, 2$ or 3 modulo 4; in the first case it is given by the signature, so may be calculated by standard methods. The Kervaire invariant case is much subtler, and to obtain a formula requires some delicate homotopy theory and the choice of a so-called 'Wu orientation'.

My own work goes in the direction of allowing a non-trivial fundamental group $\pi = \pi_1(X)$. First we need a precise formulation of Poincaré duality - I will pass over this. Next, when $n = 2k$, we have a regular homotopy class of immersions $S^k \times D^k \to N$ and seek an embedding in the class. Whitney's procedure for removing two intersections of opposite signs of k-manifolds M_1 and M_2 in N^{2k} is to choose one arc in each of M_1 and M_2 joining the two points: the two arcs form an embedded circle. If we can span this by an embedded 2-disc in N, we can then deform one arc across this disc to eliminate both intersections.



Provided $k \geq 3$, the only obstruction to doing this successfully is the class in $\pi_1(N)$ of the embedded circle.

Assembling over all (signed) intersections of M_1^k and M_2^k in N^{2k} gives a measure of the intersection, which lies in the group ring

$$R := \mathbb{Z}[\pi_1(N)].$$

There is an involution of R induced by $g \to g^{-1}$ in π (with a sign if g is orientation-reversing); if the intersection of M_1 and M_2 is given by z, the intersection of M_2 and M_1 is $(-1)^k\overline{z}$.

For self-intersections of M^k in N^{2k}, the same ideas lead to an invariant in the quotient R/R_0, where R_0 is the subgroup of elements $z-(-1)^k\overline{z}$.

Thus instead of studying forms over $\mathbb{Z}$, we require quadratic/hermitian forms over R. It is possible to make some direct calculations if π is finite, and also to say what happens if n is odd, but it is better to reformulate, and the pioneer here has been Ranicki.

A. A. Ranicki, Algebraic L-theory. I. Foundations, *Proc. London Math. Soc.* (3) 27 (1973), 101–125.

We start with an algebraic version of Poincaré duality. Instead of a manifold M we take a chain complex C, which consists of free finitely generated modules over a ring R. Duality must involve an isomorphism of C on the dual complex $C^* = \text{Hom}_R(C, R)$, regarded as an R-module using a preferred involution on R. Equivalently we have a pairing $C \times C \to R$, a map $C \otimes_R C \to R$, a map $R \to C^* \otimes_R C^*$ or just an element $A \in C^* \otimes_R C^*$.

This is also required to be symmetric, and this is where details require care. If C is merely a free module, write T for the automorphism of $C^* \otimes_R C^*$ which interchanges factors: then the symmetric pairings are given by the elements of Ker$(1-T)$, and we can define quadratic pairings to consist of elements of Coker$(1-T)$. When C is a chain complex we need also to introduce a free resolution F of $\mathbb{Z}$ as $\mathbb{Z}[\mathbb{Z}_2]$-module and work with $(C^* \otimes_R C^*) \otimes_{\mathbb{Z}[\mathbb{Z}_2]} F$, where the generator of $\mathbb{Z}_2$ acts by T. The details take several pages to write out.

We then obtain:
a bordism theory of 'symmetric bilinear forms' on chain complexes over R with class A of degree n, giving a bordism group $S_n(R)$, and another bordism theory of symmetric quadratic forms with a bordism group $Q_n(R)$.

Each of these is periodic in n with period 4. There is a natural map $s : Q_n(R) \to S_n(R)$ induced by symmetrisation.

The chain complex of a (closed) manifold M defines an element $c(M)$ of $S_n(\mathbb{Z}[\pi_1(M)])$, and a normal map $f : L \to M$ has surgery obstruction $\sigma(f)$ lying in $Q_n(\mathbb{Z}[\pi_1(M)])$.
If L is a manifold, $s(\sigma(f)) = c(L) - c(M)$.
In particular, if $\pi_1(M)$ is trivial and n is divisible by 4, $S_n(\mathbb{Z})$ and $Q_n(\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$, s is multiplication by 8, and $c(M)$ is the signature of M.