Homology Stability of Groups in Algebra and Geometry

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The Legacy of Daniel Quillen, Cambridge 2012

Motivation

- to study of smooth manifolds W^d of dimension d;
- to study their automorphism groups $\mathrm{Diff}(W)$;
- to study them in families and their *moduli spaces*:

$$\mathcal{M}(W) := \mathrm{Emb}(W;\mathbb{R}^\infty)/\mathrm{Diff}(W) \;\; (= B\mathrm{Diff}(W))$$

Question:
What are the characteristic classes? what is $H^*(\mathcal{M}(W))$?

Difficult!

Stabilization

$W \rightsquigarrow W \times I \rightsquigarrow \cdots \rightsquigarrow W \times I^k$

pseudo-isotopies:

$$P(W) := \mathrm{Diff}(W \times I \text{ rel } \partial W \times I \cup W \times \{0\})$$

stable pseudo-isotopies: $\mathcal{P}(W) := \lim_{k\to\infty} P(W \times I^k)$

studied via Waldhausen K-theory:

$$\begin{aligned} A(W) &\simeq B^2\mathcal{P}(W) \times \Omega^\infty\Sigma^\infty(W_+) \\ &\simeq \mathbb{Z} \times B\,\mathrm{GL}(\Omega^\infty\Sigma^\infty(\Omega W_+))^+ \end{aligned}$$

$W \rightsquigarrow W\#Q \rightsquigarrow \cdots \rightsquigarrow W\#_k Q$

where Q is another d dimensional manifold

Example: $d = 0$

$\Sigma_n = \text{Diff}(n \text{ points})$

$\Sigma_n \hookrightarrow \Sigma_{n+1}$

Barratt-Priddy-Quillen:

$$\lim_{n\to\infty} H_*(B\Sigma_n) = H_*(\Omega_0^\infty S^\infty).$$

Homology stability:

$$H_*(B\Sigma_n) \to H_*(B\Sigma_{n+1})$$

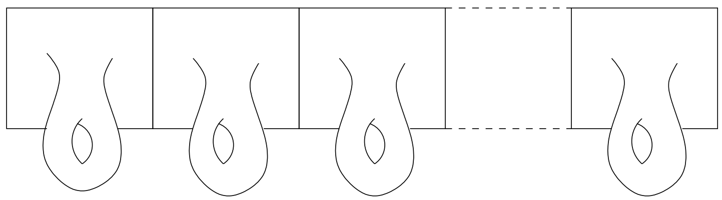
is an isomorphism for $* \leq n/2$.

$\Rightarrow$ $\tilde{H}_*(\Omega^\infty S^\infty)$ is all torsion

$\Rightarrow$ $\tilde{\pi}_*^s S^0$ is all torsion (Serre)

Example: $d = 2$

$\Gamma_{g,1} = \pi_0 \operatorname{Diff}(F_{g,1} \operatorname{rel} \partial) \simeq \operatorname{Diff}(F_{g,1} \operatorname{rel} \partial)$ for $g > 0$



$\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$

Homology stability:

$H_*(B\Gamma_{g,1}) \to H_*(B\Gamma_{g+1,1})$ is an isom. for $* \leq (2g-2)/3$

Madsen-Weiss:

$\lim_{g\to\infty} H_*(B\Gamma_{g,1}) \simeq H_*(\Omega_0^\infty MTSO(2))$

$\Rightarrow \lim_{g\to\infty} H_*(B\Gamma_{g,1}; \mathbb{Q}) \simeq \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $|\kappa_i| = 2i$

(Mumford conjecture)

Scanning map:

Let W be oriented and $W' \in \mathcal{M}(W)$.
Choose a tubular neighborhood $W' \subset N(W') \subset \mathbb{R}^{d+n}$.
Define

$$\begin{aligned}\alpha(W') : S^{d+n} = (R^{d+n})^c \stackrel{collapse}{\longrightarrow} N(W')^c &\stackrel{\phi_{T(W')}}{\longrightarrow} (U_{d,n}^{\perp})^c \\ (x,v) &\mapsto (T_x W', v).\end{aligned}$$

Here $U_{d,n}$ is the universal d bundle over the Grassmannian manifold $Gr(d,n)$ of oriented d planes in $d+n$.

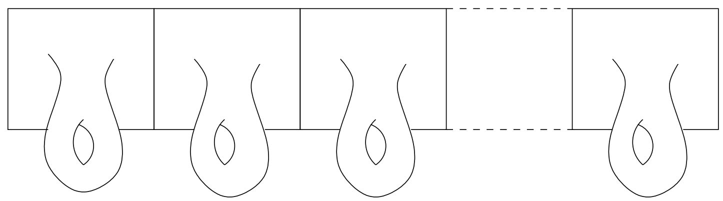
$\Rightarrow$ This gives a map

$$\alpha : \mathcal{M}(W) \longrightarrow \lim_{n\to\infty} \Omega^{d+n}(U_{d,n}^{\perp})^c =: \Omega^\infty \mathbf{MTSO}(d).$$

$$H^*(\Omega^\infty \mathbf{MTSO}(d); \mathbb{Q}) = \mathbb{Q}[H^{*>d}(BSO(d); \mathbb{Q})[-d]]$$

Higher dimensional examples

$F_{g,1} = \sharp_g \, (S^1 \times S^1) \setminus \operatorname{int} D^2$



$W_{g,1} := \sharp_g \, (S^k \times S^k) \setminus \operatorname{int} D^{2k}$, where $k > 2$.

Galatius & Randal-Williams:

(L) $\lim_{g \to \infty} H_*(B\mathrm{Diff}(W_{g,1} \operatorname{rel} \partial)) = H_*(\Omega^\infty \mathbf{MTSO}(2k)^{<k>})$

(S) $H_*(B\mathrm{Diff}(W_{g,1} \operatorname{rel} \partial)) \to H_*(B\mathrm{Diff}(W_{g+1,1} \operatorname{rel} \partial))$
. is a homology isomorphism for $* \leq (g-4)/2$.

Hatcher:

$H_{g,1} := \sharp_g(S^1 \times D^2)$ handlebody of genus g with a disk marked on the boundary
(L) $\lim_{g\to\infty} H_*(B\mathrm{Diff}(H_{g,1}\,\mathrm{rel}\,D^2)) = H_*(\Omega^\infty\Sigma^\infty(BSO(3)_+))$

$M_{g,1} := \sharp_g(S^1 \times S^2)$ with a marked disk D^3
(L) $\lim_{g\to\infty} H_*(B\mathrm{Diff}(M_{g,1}\,\mathrm{rel}\,D^3))) = H_*(\Omega^\infty\Sigma^\infty(BSO(4)_+))$

Hatcher-Wahl:

$\Gamma_{g,1} \supset \mathcal{H}_{g,1} := \pi_0\,(\mathrm{Diff}(H_{g,1}\,\mathrm{rel}\,D^2)) \simeq \mathrm{Diff}(H_{g,1}\,\mathrm{rel}\,D^2)$
(S) $H_*(\mathcal{H}_{g,1}) \to H_*(\mathcal{H}_{g+1,1})$ is an isom. for $* \le (g-2)/2$.

(S) Homology stability for $\mathrm{Diff}(M_{g,1}\,\mathrm{rel}\,D^3)$??

Homology stability for discrete groups

Nakaoka: Σ_n

Borel, Quillen: matrix groups

Charney, Dwyer, van der Kallen, Vogtmann:
more matrix groups

Harer, Ivanov, Wahl, Boldsen, Randal-Williams:
mapping class group of surfaces: $\Gamma_{g,n}^k$, $\mathcal{N}_{g,n}^k$, ...

Hatcher, Vogtmann, Wahl: $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$

Hatcher-Wahl: mapping class groups of 3-manifolds,
including $\mathcal{H}_{g,1}$

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Inventiones mathematicae

Homology Stability for Linear Groups

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Summary. Let R be a commutative finite dimensional noetherian ring or, more generally, an associative ring which satisfies one of Bass' stable range conditions. We describe a modified version of H. Maazen's work [18], yielding stability for the homology of linear groups over R. Applying W.G. Dwyer's arguments (cf. [9]) we also get stability for homology with twisted coefficients. For example, $H_2(GL_n(R), R^n)$ takes on a stable value when n becomes large.

§ 1. Introduction

1.1. Our motivation for this work has been to prove stability for algebraic K-theory in BGL^+ context. Thanks to the recent work of Dwyer we actually get much more general statements. These imply a result which seems to be of interest to geometric topologists. Namely, we find that the twisted homology groups $H_k(GL_n(R), \rho_n)$, considered by Dwyer in [9], stabilize with respect to n not only when R is a PID, but also when R is the group ring $\mathbb{Z}[\pi]$ of a finite group π. This fits in with work of W.G. Dwyer, Wu-Chung Hsiang and R.E. Staffeldt on Waldhausen's rational algebraic K-groups of a space.

1.2. Let us remind the reader what sort of families $\{\rho_n\}$ are considered by Dwyer, leaving out all technicalities and using some suggestive but unexplained terminology. A basic example is the family $\lambda = \{\lambda_n\}$, where λ_n denotes (the standard representation of $GL_n(R)$ in) the right R-module R^n of column vectors of length n over R. This system λ grows linearly with n. Note that the difference between λ_{n+1} and λ_n is equal to R for all n, so that the system of differences is constant in this case. More generally Dwyer considers systems that grow polynomially with n, such as the system $\mu = \{\mu_n\}$, where μ_n denotes (the representation by conjugation of $GL_n(R)$ in) the space of n by n matrices over R. The system μ grows quadratically with n, which can be rephrased by saying that its system of third iterated differences is zero, while its system of second iterated differences is not zero. (To make sense of all this, one has to add more structure

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1.5. By way of a Hurewicz argument stability for the $\pi_m(BGL_n^+(R))$ follows from stability for the $H_m(E_n(R))$ ($m \geqq 2$). (See 4.12 below.) As a first approximation to stability for the $H_m(E_n(R))$ one may study the simpler problem of stability for the $H_m(GL_n(R))$. Quillen (unpublished) has shown that, when R is a field different from $\mathbb{F}_2$, the map $H_m(GL_n(R)) \to H_m(GL_{n+1}(R))$ is an isomorphism for $n \geqq m+1$. As the present work follows the same general principles, let us sketch Quillen's approach, stressing features that are relevant to us. Suppose G is a group, H a subgroup, and suppose there is a nice sort of geometry associated with the set of right cosets G/H. (For example, when $G = GL_n(k)$ where k is a field, choose a non-zero vector v in k^n and let H be the stabilizer of v in G. The set G/H may be identified with the orbit of v, which is almost the same as k^n, and in this case we may associate with G/H the geometry of linear n-space k^n.) Now construct a simplicial complex T, based on combinatorial properties of the geometry, such that G acts naturally on T and H is the stabilizer of some 0-simplex. When G acts transitively on simplices of fixed dimension, for each dimension, and moreover T is highly connected, one gets a spectral sequence relating the homology of G with the homology of the stabilizers in G of simplices of T. This spectral sequence may be useful in an inductive argument, e.g. when one wants to show that in a certain range the homology of G is the same as the homology of H. (Compare with the following situation which one meets when studying homotopy groups of the Lie groups $SO_n(\mathbb{R})$: There is a fibration $SO_n(\mathbb{R}) \to SO_{n+1}(\mathbb{R}) \to S^n$ and the fact that S^n is $(n-1)$-connected makes that $\pi_i(SO_n(\mathbb{R})) \to \pi_i(SO_{n+1}(\mathbb{R}))$ is an isomorphism for $i \leqq n-2$.)

Quillen tried several simplicial complexes. One is the Tits building, which is known to be highly connected by the Solomon-Tits theorem. Another one was based on unimodular sequences of vectors. Quillen showed it to be highly connected in the case of local rings and he conjectured a similar result for finite dimensional noetherian rings. (See [33], Sect. 1.) The proof of this conjecture is one of the goals of Sect. 2 below.

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W_a/W_{a-1} it is a sum $\oplus_\alpha V_\alpha$ of the eigenspaces V_α of distinct, non-trivial characters $\alpha : D^* \to k^*$.

Let $\pi : W_a \to W_a/W_{a-1}$ be the projection and for $b \geq 0$ let $Y_b \subset W_a$ be the D^*-invariant subset of those z of the form (6.1) such that $\pi(x_\sigma)$ has at most b non-zero components in $\oplus_\alpha V_\alpha$. We will show that any z of the form (6.1) with $x_\sigma \in W_a$ goes to zero by showing inductively over b that z goes to zero when $x_\sigma \in Y_b$. Clearly if $b = 0$, then $x_\sigma \in W_{a-1}$ and hence z goes to zero by the second induction hypothesis. So suppose $x_\sigma \in Y_b$.

Third Induction Hypothesis. For $0 \leq b' < b$ any chain z of the form (6.1) goes to zero provided $x_\sigma \in Y_{b'}$.

Let $\alpha : D^* \to k^*$ be one of the non-trivial characters such that $\pi(x_\sigma)$ has a non-zero component in V_α. Choose an $\eta \in D^*$ such that $\alpha(\eta) \neq 1$. Then $\eta \cdot x_\sigma = \alpha(\eta)x_\sigma + v$ where $\eta \cdot x_\sigma$ is the diagonal representation action of (2.1); $\alpha(\eta)x_\sigma$ is scalar multiplication in the k-vector space $H_q(BU_F; k)$; and, finally, $v = \eta \cdot x_\sigma - \alpha(\eta)x_\sigma$ lies in Y_{b-1}. The chain $\eta \cdot z - \alpha(\eta)z \in C_p(\mathcal{P}^m; \mathcal{H}_q)$ is a cycle and furthermore

$$\begin{aligned}\eta \cdot z - \alpha(\eta)z &= \eta \cdot y + (\eta \cdot x_\sigma)\cdot\sigma - \alpha(\eta)y - (\alpha(\eta)x_\sigma)\cdot\sigma \\ &= \{\eta \cdot y - \alpha(\eta)y\} + (\alpha(\eta)x_\sigma + v)\cdot\sigma - (\alpha(\eta)x_\sigma)\cdot\sigma \\ &= \{\eta \cdot y - \alpha(\eta)y\} + v \cdot \sigma.\end{aligned}$$

Since $v \in Y_{b-1}$, the third induction hypothesis implies that $\eta \cdot z - \alpha(\eta)z$ goes to zero in $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$. Hence

$$\begin{aligned}0 &= i_*(\eta \cdot z - \alpha(\eta)z) \\ &= i_*(\eta \cdot z) - i_*(\alpha(\eta)z) \\ &= \eta \cdot i_*(z) - \alpha(\eta)i_*(z) \\ &= i_*(z) - \alpha(\eta)i_*(z) && \text{(by (5.1))} \\ &= (1-\alpha(\eta))i_*(z).\end{aligned}$$

Since $1 - \alpha(\eta) \neq 0$, $i_*(z) = 0$ in the k-vector space $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$.

Appendix*

Let A be an associative ring with identity. Let GL_n be the group of invertible $n \times n$ matrices over A and let $GL_\infty = \lim_n GL_n$. For $1 \leq m, n \leq \infty$ let

* By D. Quillen.

$$GL_{m,n} = \left(\begin{array}{c|c} GL_m & N \\ \hline 0 & GL_n \end{array}\right)$$

where N is the set of $m \times n$ matrices over A. In case m or n is infinite, take N to be those matrices with only finitely many non-zero entries. There are homomorphisms

$$GL_m \times GL_n \xrightarrow[i_{m,n}]{} GL_{m,n} \xrightarrow[\pi_{m,n}]{} GL_m \times GL_n$$

with $\pi_{m,n} \circ i_{m,n} =$ identity.

Theorem A.1. *Let k be a field and assume there is a prime number l invertible in A which divides* char(k), *hence either* char$(k) = l$ *or* char$(k) = 0$. *Then*

$$(\pi_{m,n})_* iH_*(GL_{m,n}; k) \to H_*(GL_m \times GL_n; k)$$

is an isomorphism.

When char$(k) = 0$ this theorem applies to any ring in which some prime number is invertible. Let L be the set of prime numbers invertible in A. Assuming this is non-empty, the theorem says $\pi_{m,n}$ induces isomorphisms on homology with coefficients in Q and Z/l for all l in L. By standard universal coefficient arguments, it follows that $\pi_{m,n}$ induces isomorphisms on homology and cohomology with coefficients in any abelian group which is uniquely p-divisible for all primes p not in L. For example, $\pi_{m,n}$ induces isomorphisms on integral homology when A is an algebra over Q.

Let $i = i_{\infty,\infty}$ and $\pi = \pi_{\infty,\infty}$ when $m = n = \infty$.

Theorem A.2. *The homomorphism π_* is an isomorphism for homology with integral coefficients.*

Thus the subgroup $N = \ker \pi$ disappears for homology in the stable case. This is the algebraic analogue of the fact that N is contractible in the situation where, say, A is the real numbers and $GL_{m,n}$ and $GL_m \times GL_n$ have the usual topology yielding a homotopy equivalence $BGL_{m,n} \to BGL_m \times BGL_n$.

Proof of Theorem A.1. Consider the spectral sequence

$$E^2_{st} = H_s(GL_m \times GL_n; H_t(N; k)) \Longrightarrow H_{s+t}(GL_{m,n}; k)$$

corresponding to the extension

$$0 \to N \to GL_{m,n} \xrightarrow[\pi_{m,n}]{} GL_m \times GL_n \to 1.$$

The abelian group N is an A-module via scalar multiplication on the rows. Since l is invertible in A, N is also a module over $D = Z[l^{-1}]$. If char$(k) = l$, one has

Homology stability for diffeomorphisms groups

Galatius & Randal-Williams:
Homology stability for $W_{g,1} = \sharp_g(S^n \times S^n) \setminus \mathrm{int} D^{2n}$ for degrees $\leq (g-4)/2$.

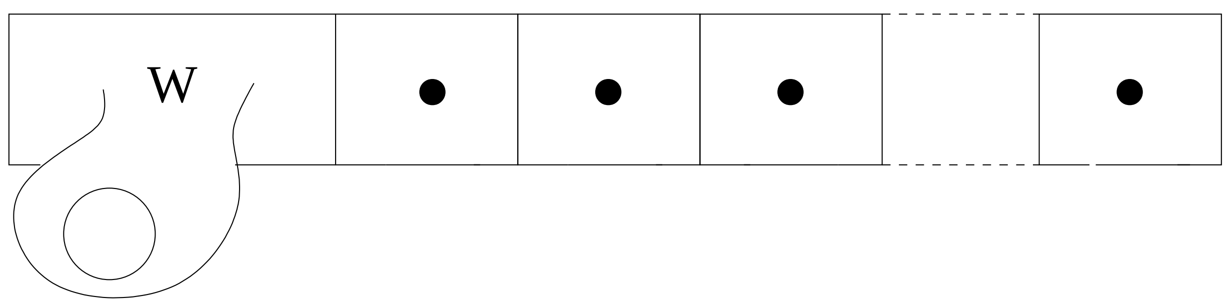
Sketch of proof:

- The simplicial sets are now replaced by simplicial spaces.
- The arc complex in the case of surfaces is now replaced by spaces of maps of $W_{1,1}$ to $W_{g,1}$.
- High connectivity of the simplicial space is deduced from a theorem of Charney: a certain simplicial complex built out of hyperbolic submodules $(H_n(S^n \times S^n))$ in quadratic modules $(H_n(W_{g,1}))$ is highly connected.
- (Need $n > 2$ so that elements in $\pi_n(W_{g,1}) = H_n(W_{g,1})$ can be represented by embedded spheres.)

Symmetric diffeomorphism groups

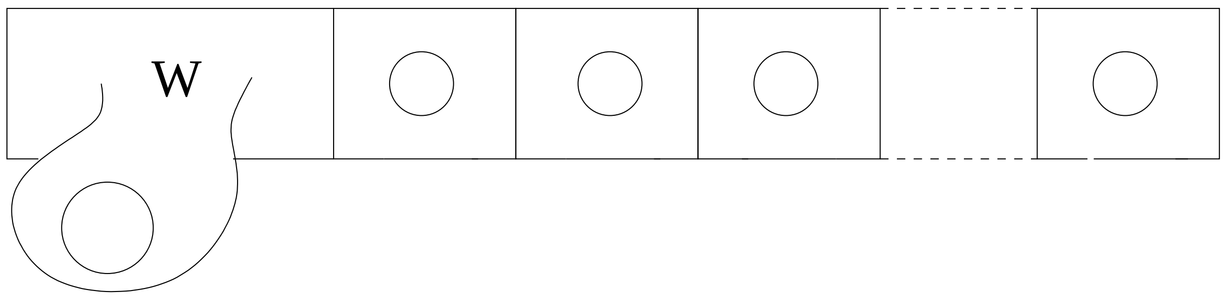
Let W be a manifold of dimension d with non-empty boundary $\partial W \supset \partial_0 W$.

Let $W_n := W \setminus n$ pts and consider the maps $W_n \to W_{n+1}$ induced by boundary connected sum.



Theorem A:
$H_*(B\mathrm{Diff}(W_n \operatorname{rel} \partial_0)) \to H_*(B\mathrm{Diff}(W_{n+1} \operatorname{rel} \partial_0))$
is split injective and an isomorphisms for $* \leq n/2$.

Let $W_n D := W \setminus \bigcup_n D$ be W without n disks, and $\mathrm{Diff}(W_n D)$ be the Σ_n extension of $\mathrm{Diff}(W_n D \,\mathrm{rel}\, \partial_0 \bigcup_n \partial D)$.



Theorem B:

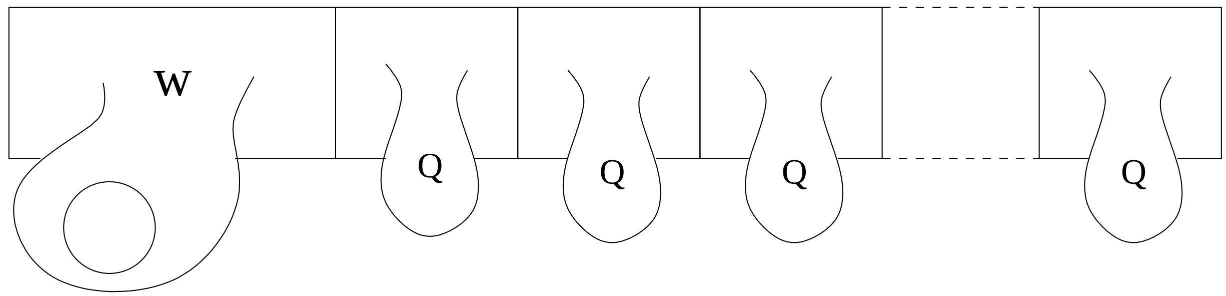
$H_*(B\mathrm{Diff}(W_n D)) \to H_*(B\mathrm{Diff}(W_{n+1} D))$

is split injective and an isomorphism for $* \leq n/2$.

Remark:

- Hatcher-Wahl proved Theorem A for m.c.g.s;
- Theorem B is new even for surfaces.

Let Q be another manifold of dimension d containing ∂D as a boundary component, and let $W\sharp_n Q$ be the manifold obtained by gluing n copies of Q to $W_n D$.



Define the symmetric diffeomorphism group as

$$\Sigma\mathrm{Diff}(W\sharp_n Q) := \mathrm{Diff}(W\sharp_n Q, \bigcup_n Q)$$

Theorem C:
$H_*(\Sigma\mathrm{Diff}(W\sharp_n Q)) \to H_*(\Sigma\mathrm{Diff}(W\sharp_{n+1} Q))$ is split injective and an isomorphisms for $* \leq n/2$.

Remarks:

- Theorems A, B, and C also hold for m.c.g.s;

- the split injections are induced by stable split injections;

- different versions by considering subgroups of the diffeomorphism groups;

- the role of the symmetric group may be played by the alternating group (using results of Martin Palmer).

Sketch of proof for Theorem A (B and C)

Configurations: $\tilde{\mathcal{C}}_n(W) := \mathrm{Emb}(\{1, \ldots, n\}; W) \subset W^n$

Unordered configurations: $\mathcal{C}_n(W) := \tilde{\mathcal{C}}_n(W)/\Sigma_n$

Example: If $W = \mathbb{R}^\infty$ then $\mathcal{C}_n(W) = \mathcal{M}(n\text{pts}) \simeq B\Sigma_n$.

Well-known:
$H_*(\mathcal{C}_n(W)) \to H_*(\mathcal{C}_{n+1}(W))$ is split injective and an isomorphism for $* \leq n/2$.

Fibration: $W_n = W$ with n marked points

$$\mathcal{C}_n(W) \longrightarrow \mathrm{Emb}(W_n;\mathbb{R}^\infty)/\mathrm{Diff}(W_n) \longrightarrow \mathrm{Emb}(W;\mathbb{R}^\infty)/\mathrm{Diff}(W)$$

Spectral sequence:

$$E^2_{pq} = H_p(B\mathrm{Diff}(W); H_q(\mathcal{C}_n(W)) \Rightarrow H_{p+q}(B\mathrm{Diff}(W_n)))$$

$\Rightarrow$ Theorem A.

For Theorem B: Use configurations with twisted(!) labels in the space of neighborhoods of a point.

For Theorem C: Extend Theorem B to allow for twisted labels.

Thank you!