

Computational Algebraic Topology Topic B:
Sheaf cohomology and applications to quantum
non-locality and contextuality
Lecture 1

Samson Abramsky

Department of Computer Science
The University of Oxford

Background Material

See the syllabus and reading material at

<http://people.maths.ox.ac.uk/tillmann/CAT.html>

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Papers (all available on arxiv):

1. Essential

- 1 S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(2011):113036, 2011.
- 2 S. Abramsky, R. S. Barbosa and S. Mansfield, The Cohomology of Non-Locality and Contextuality, in *Proceedings of QPL 2011*, EPTCS 2011.
- 3 S. Abramsky, R. S. Barbosa, K. Kishida, R. Lal and S. Mansfield, Contextuality, Cohomology and Paradox (submitted).

2. Useful additional reading

- S. Abramsky and L. Hardy. Logical Bell Inequalities. *Phys. Rev. A* 85, 062114 (2012).
- S. Abramsky, G. Gottlob and P. Kolaitis, Robust Constraint Satisfaction and Local Hidden Variables in Quantum Mechanics, Proceedings IJCAI 2013.
- S. Abramsky, Relational Databases and Bell's Theorem, In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.

Beginnings ...

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The first axiom I learnt in Computer Science:

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It is no longer safe to assume this!

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- Beyond algorithms and complexity it offers new challenges and opportunities across the range of Computer Science: in programming languages and methods, logic and semantics.
- There is a fascinating two-way interplay developing between Computer Science and Physics, extending to the foundations of both, as well as to more practical matters. Quantum technology — “hacking matter” — will be a huge feature of 21st Century science and engineering, and a lot of it will be to do with information.

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- This is an exciting emerging area, attracting students with backgrounds in CS, Physics, Mathematics, Philosophy, . . .

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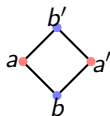
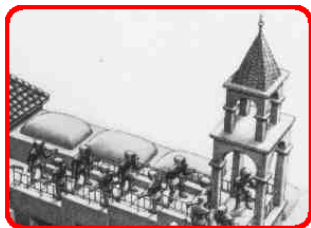
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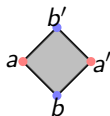
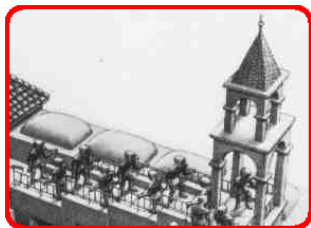
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In a nutshell: where we have a family of data which is **locally consistent**, but **globally inconsistent**.

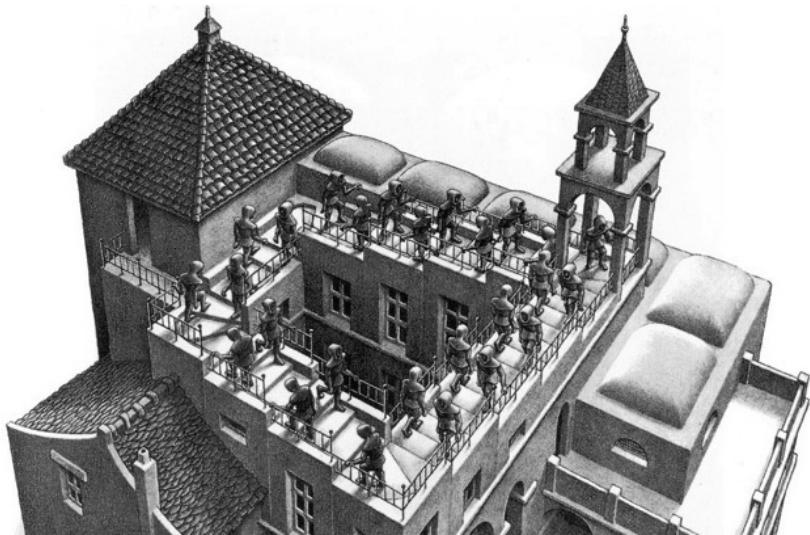
Contextuality Analogy: Local Consistency



Contextuality Analogy: Local Consistency



Contextuality Analogy: Global Inconsistency



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A “transcendental deduction” of the **incompatibility** (in general) of observables.

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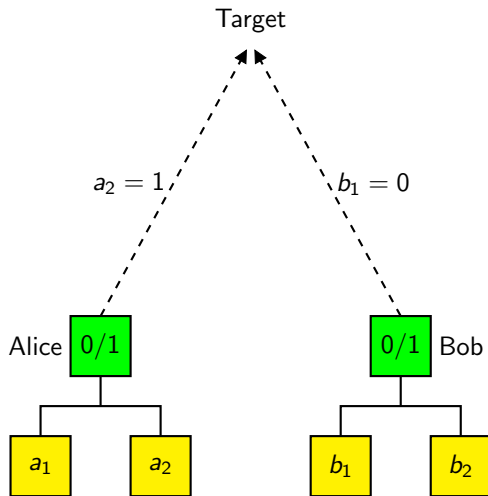
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Our results show that it does apply, in a very direct way, to the analysis of contextuality.

Gives **positive, constructive witnesses** for contextuality.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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The entry in row 2 column 3 says:

If Alice looks at a_1 and Bob looks at b_2 , then $1/8$ th of the time, Alice sees a 0 and Bob sees a 1.

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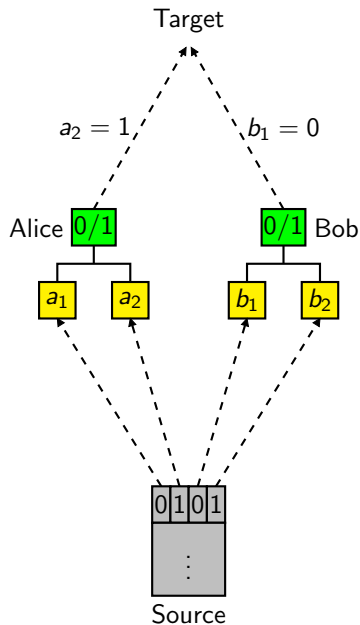
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How can we explain this behaviour?

Classical Correlations



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Using elementary probability theory, we can calculate:

$$p_N \leq \text{Prob}\left(\bigvee_{i=1}^{N-1} \neg\phi_i\right) \leq \sum_{i=1}^{N-1} \text{Prob}(\neg\phi_i) = \sum_{i=1}^{N-1} (1 - p_i) = (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^N p_i \leq N - 1.$$

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If we read 0 as true and 1 as false, the highlighted positions in each row of the table are represented by the following propositions:

$$\varphi_1 = a \wedge b \vee \neg a \wedge \neg b = a \leftrightarrow b$$

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The violation of the logical Bell inequality is 1/4.

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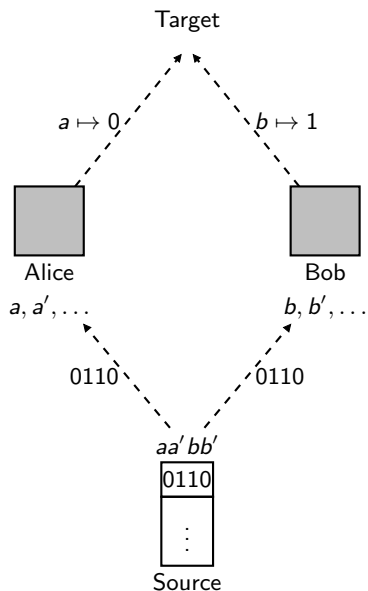
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However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



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Hardy models: those whose **support** satisfies

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(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

Which 'instruction set' λ could the outcomes (0, 0) for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

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The 'Hardy Paradox': Bell's theorem without inequalities

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Thus Hardy models are **contextual**. They cannot be explained by a classical source.

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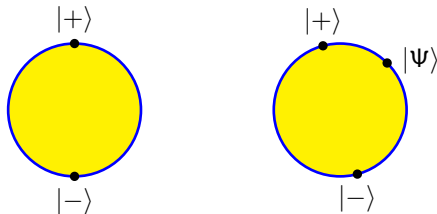
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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than “which register to load”.

The Quantum Case: Spin Measurements

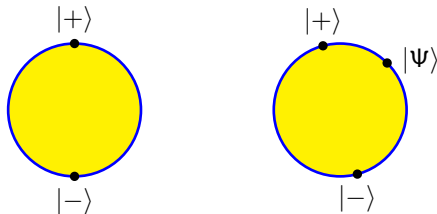
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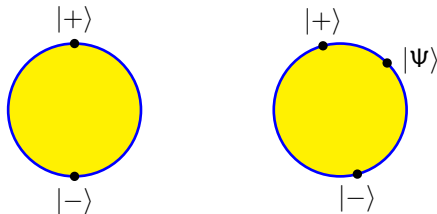
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Spin can be measured in any direction; so there are a continuum of possible measurements. There are **two possible outcomes** for each such measurement; spin in the specified direction, or in the opposite direction. These two directions are represented by a pair of orthogonal vectors. They are represented on the sphere as a pair of **antipodal points**.

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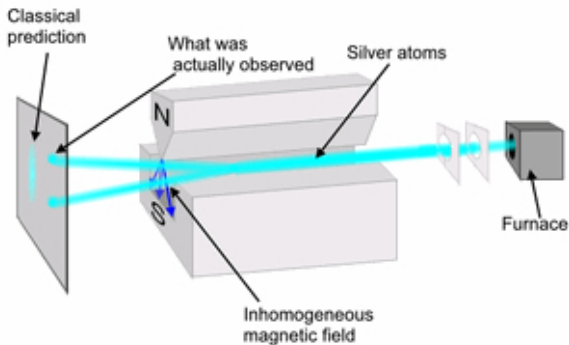
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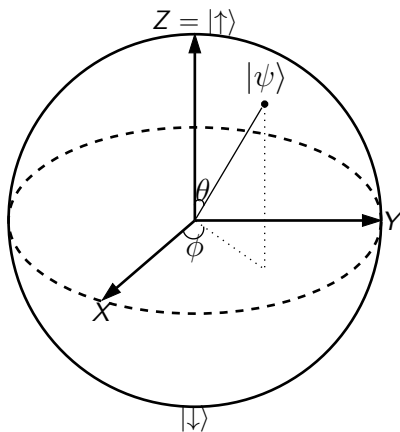
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Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



The Bloch sphere representation of qubits



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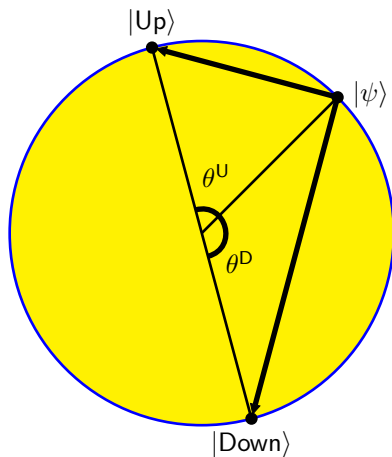
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- When we subject a qubit to a measurement (Up, Down), the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points ($|\text{Up}\rangle, |\text{Down}\rangle$) which specify the measurement. In algebraic terms, $|\psi\rangle, |\text{Up}\rangle$ and $|\text{Down}\rangle$ are unit vectors in the complex vector space \mathbb{C}^2 , and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

$$|\langle\psi|\text{Up}\rangle|^2.$$

This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

Truth makes an angle with reality



Qubits vs. Bits

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But in addition, we have the important feature that there are a continuum of possible questions we can ask. However, note that on each run of the system, we can only ask **one** of these questions. We cannot simultaneously observe Up or Down in two different directions. Note that this corresponds to the feature of the scenario we discussed, that Alice and Bob could only look at one their local registers on each round.

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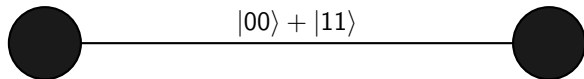
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Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

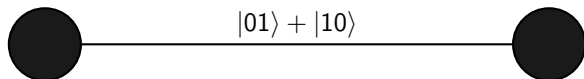
Quantum Entanglement

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Bell state:

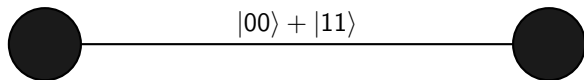


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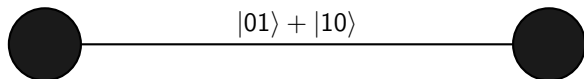


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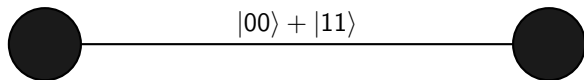
Compound systems are represented by **tensor product**: $\mathcal{H}_1 \otimes \mathcal{H}_2$. Typical element:

$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

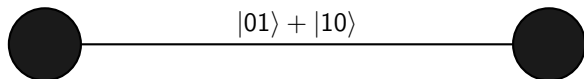
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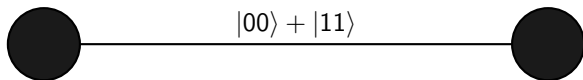
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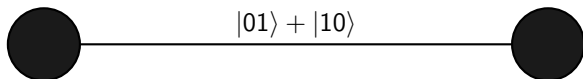
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Bell's theorem: QM is **essentially non-local**.

A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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Important note: this is **physically realizable!**

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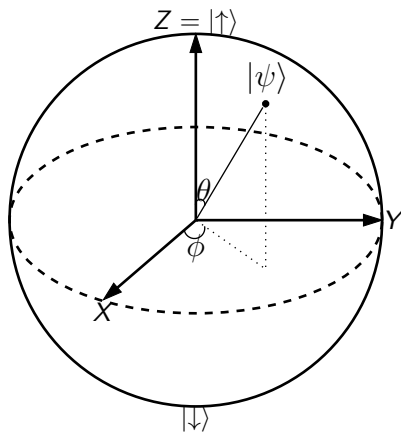
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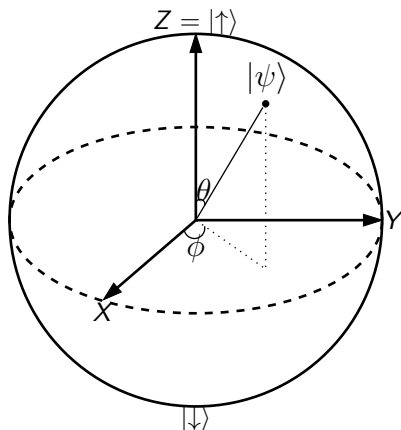
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Extensively tested experimentally.

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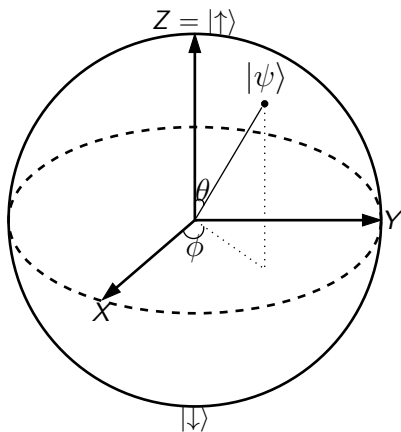


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Spin measurements lying in the equatorial plane of the Bloch sphere
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X itself, $\phi = 0$:

Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

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The other entries can be computed similarly.